MAT 141: Foundational Discrete Math - Notes

Charles Rocca

August 17, 2026

Contents

Preface **5**

1 Symbolic Logic and Predicates **7**
- 1.1 Logical Statements . . . 7
- 1.2 Implications . . . 11
- 1.3 Using Logical Laws and Arguments . . . 15
- 1.4 Predicates . . . 19

2 Sequences and Sums **27**
- 2.1 Sequences . . . 27
- 2.2 Sums . . . 35
- 2.3 Using and Finding Formulas . . . 43

3 Sets, Relations, and Functions **55**
- 3.1 Sets . . . 55
- 3.2 Relations . . . 63
- 3.3 Functions . . . 75

4 Combinatorics **83**
- 4.1 Basic Counting . . . 83
- 4.2 Permutations and Combinations . . . 89
- 4.3 Pigeon Hole Principle . . . 100

5 Graphs and Trees **103**
- 5.1 Basic Definitions for Graphs . . . 103
- 5.2 Walks in Graphs . . . 106
- 5.3 Adjacency Matrices . . . 110
- 5.4 Basic Properties of Trees . . . 115

A Logic Reference Table **117**

B Sets Reference Table **121**

Bibliography **125**

Index **127**

Preface

These notes are meant to add structure to a first course in discrete mathematics; these do not represent a full coverage of the content. The class these are used in has a prerequisite of Precalculus partly for content but also for mathematical maturity. The prerequisites mean that the course difficulty is on par with Calculus 1 however, the nature of the material in the course is much different. While algebra is used in and needed for this course the thinking used is more frequently logical or qualitative rather than quantitative or algebraic. For this reason students will need to be particularly diligent since this way of thinking will likely be new to them. The material in this set of notes is based largely, but not exclusively on material in *Discrete Mathematics with Applications* by Epp[1]. There will on occasion also be material pulled from discrete mathematics textbooks by Rosen [5] and Grimaldi [2].

Chapter 1

Symbolic Logic and Predicates

1.1 Logical Statements

Definition 1.1 (Statements). A ***statement*** (or ***proposition***) is a sentence which is true or false but not both.

Definition 1.2 (New Statements From Old). Given statements P and Q:

- The ***negation*** of a statement, $\sim P$ or *not* P, always has the opposite truth value.
- The ***conjunction*** of two statements, $P \wedge Q$ or P *and* Q, is true when both statements are true.
- The ***disjunction*** of two statements, $P \vee Q$ or P *or* Q, is true when either statement is true.

Practice Problem 1.1 (Statements and Their Negations). Fill in the negation for the statements: $R \equiv$ "It is raining."; $G \equiv$ "The grass is wet."; and $U \equiv$ "I have an umbrella."

- $\sim R \equiv$
- $\sim G \equiv$
- $\sim U \equiv$

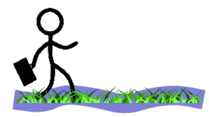
Practice Problem 1.2 (Visualizing Truth Values)**.** For each picture indicate all the statements from practice problem 1.1 that apply to it by writing the appropriate letter or its negation.

(1)

(2)

(3)

(4)

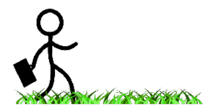


(5)

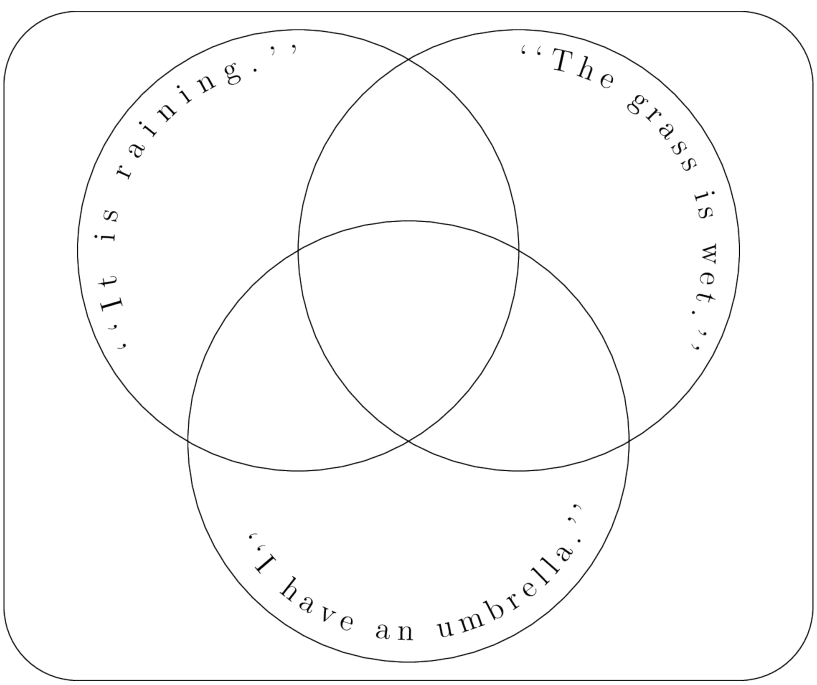
(6)

(7)

(8)



Practice Problem 1.3 (Relating Statements to Sets)**.** For each region of the ***Venn Diagram***, write down which picture from practice problem 1.2 belongs in that region.



Practice Problem 1.4 (Combinations and Modifications of Statements)**.** For each combination of statements, indicate which set of pictures from practice problem 1.2 make the statement true.

1. $U \wedge G$:
2. $\sim (U \wedge G)$:
3. $\sim U \wedge \sim G$:
4. $U \vee G$:
5. $\sim (U \vee G)$:
6. $\sim U \vee \sim G$:

Practice Problem 1.5 (Negating Compound Statements)**.** Using $U \equiv$ "I have an umbrella." and $G \equiv$ "The grass is wet." fill in the ***truth table*** below. You can refer to the previous practice problems and the images above to help you.

U	G	$\sim U$	$\sim G$	$U \wedge G$	$U \vee G$	$\sim (U \wedge G)$	$\sim (U \vee G)$	$U \wedge \sim G$
T	T							
T	F							
F	T							
F	F							

Definition 1.3 (Tautology and Contradiction)**.** A ***tautology*** is a statement that is always true and a ***contradiction*** is a statement that is always false. For example, in the figures in practice problem 1.2, "the grass is green" is a tautology since it is always true, "there is a zombie" is a contradiction since it is always false.

Practice Problem 1.6 (True or False)**.** For which pictures in practice problem 1.2 are the following sentences true?

1. "The grass is green" $\vee$ "I have an umbrella."
2. "The grass is green" $\wedge$ "I have an umbrella."
3. "There is a zombie" $\vee$ "I have an umbrella."
4. "There is a zombie" $\wedge$ "I have an umbrella."
5. "There is a zombie" $\vee$ "the grass is green."
6. "There is a zombie" $\wedge$ "the grass is green."

Exposition 1.1. Some "Basic" Logical Rules So far we have identified the following ways to put together statements using $\wedge$, $\vee$, and $\sim$:

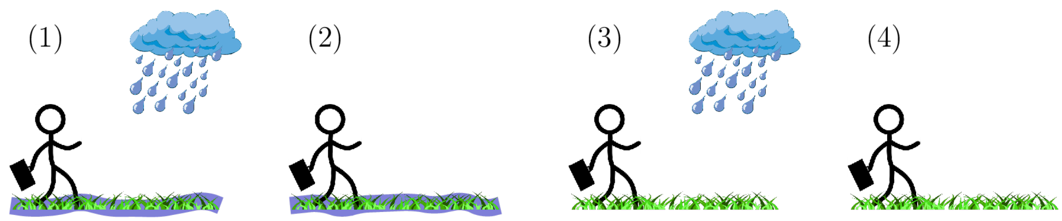
- ***Negation (not)***: $\sim P$
- ***Conjunction (and)***: $P \wedge Q$
- ***Disjunction (or)***: $P \vee Q$
- ***Exclusive Or (xor)***: $P \oplus Q \equiv (P \vee Q) \wedge \sim (P \wedge Q)$
- ***Double Negation***: $\sim (\sim P) \equiv P$
- ***DeMorgan's Laws***:
  - $\sim (P \wedge Q) \equiv \sim P \vee \sim Q$
  - $\sim (P \vee Q) \equiv \sim P \wedge \sim Q$
- ***Commutative Law***:
  - $P \vee Q \equiv Q \vee P$
  - $P \wedge Q \equiv Q \wedge P$
- ***Distributive Law***:
  - $P \wedge (Q \vee R) \equiv P \wedge Q \vee P \wedge R$
  - $P \vee (Q \wedge R) \equiv P \vee Q \wedge P \vee R$
- ***Associative Law***:
  - $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
  - $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
- ***Identity Laws***:
  - $t \wedge P \equiv P$
  - $c \vee P \equiv P$
- ***Inverse Laws***:
  - $P \vee \sim P \equiv t$
  - $P \wedge \sim P \equiv c$
- ***Domination Laws***:
  - $t \vee P \equiv t$
  - $c \wedge P \equiv c$

Practice Problem 1.7. Which statements in practice problem 1.6 represent which laws in exposition 1.1?

1.2 Implications

Definition 1.4 (Implications/Conditional Statements)**.** A ***conditional statement***, or ***implication***, is a compound statement of the form *"if P, then Q."* The first statement, P, is called the *hypothesis* or *premise* and the second is the *conclusion*. Conditionals can also be read *"P implies Q," "P only if Q,", "Q, if P."* Notationally, we write "if P, then Q" as $P \to Q$.

Practice Problem 1.8 (When are Implications False)**.** Looking at the following images,



1. For which images is the statement ***"If it is raining, then the grass is wet"*** *($R \to G$)* false?

2. Why is the statement false in those cases?

3. In what situations must it then be true?

Use the same sort of reasoning to try and fill in the following ***truth table***.

R	G	$\sim R$	$\sim G$	$R \to G$	$G \to R$	$\sim R \to \sim G$	$\sim G \to \sim R$	$R \wedge \sim G$
T	T							
T	F							
F	T							
F	F							

Practice Problem 1.9 (True or False: The Return)**.** In practice problem 1.8, for which images are these true?

1. If "there is a zombie," then "the grass is wet."

2. If "there is a zombie," then "it is raining."

3. If "it is raining," then "the grass is green."

4. If "the grass is wet," then "the grass is green."

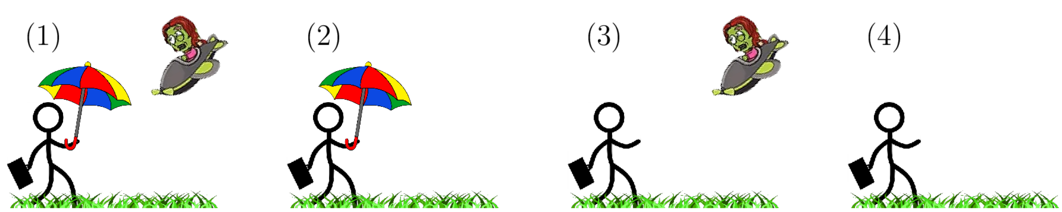
Exposition 1.2 (Conditionals and Their Relations)**.** Starting with a conditional $P \to Q$ we can write the following:

1. ***Converse***: $Q \to P$
2. ***Inverse***: $\sim P \to \sim Q$
3. ***Contrapositive***: $\sim Q \to \sim P$
4. ***Negation***: $P \wedge \sim Q$

Where $(P \to Q) \equiv (\sim Q \to \sim P)$, $(Q \to P) \equiv (\sim P \to \sim Q)$, and $\sim (P \to Q) \equiv (P \wedge \sim Q)$. We also say $P \to t$ and $c \to Q$ are ***vacuously true*** because their negations are always false:

$$\sim (P \to t) \equiv (P \wedge c) \equiv c \text{ and } \sim (c \to Q) \equiv (c \wedge \sim Q) \equiv c.$$

Practice Problem 1.10 (Making Arguments)**.** If we assume it is true that "If I carry an umbrella, then zombies dive bomb me," then which pictures should we look at? ____________



What if, together with the original statement, we assume that it is true that "I am carrying an umbrella," which picture(s) do we look at then ____________ , are there zombies dive bombing me ____________ ?

What if we also assume that it is true that "zombies dive bomb me," which picture(s) do we look at then ____________ , am I carrying an umbrella ____________ ?

Exposition 1.3 (More Rules and Arguments)**.** Given statements P and Q the following are some common valid and invalid argument forms.

Modus Ponens:

$$\begin{array}{l} P \to Q \\ P \\ \hline \therefore\ Q \end{array}$$

Modus Tollens:

$$\begin{array}{l} P \to Q \\ \sim Q \\ \hline \therefore\ \sim P \end{array}$$

Converse Error:

$$\begin{array}{l} P \to Q \\ Q \\ \hline \therefore\ P \end{array}$$

Inverse Error:

$$\begin{array}{l} P \to Q \\ \sim P \\ \hline \therefore\ \sim Q \end{array}$$

Disjunctive Syllogism:

$$\begin{array}{l} P \vee Q \\ \sim P \\ \hline \therefore\ Q \end{array}$$

Law of Simplification:

$$\begin{array}{l} P \wedge Q \\ \hline \therefore\ P \text{ and } Q \end{array}$$

Transitivity:

$$\begin{array}{l} P \to Q \\ Q \to R \\ \hline \therefore\ P \to R \end{array}$$

Contradiction:

$$\begin{array}{l} P \wedge \sim Q \to c \\ \hline \therefore\ P \to Q \end{array}$$

We can demonstrate the validity or invalidity of an argument using a truth table and by looking at ***critical rows*** which are the rows of the tables in which all the premises for the argument are true. For example this table shows the validity of Modus Ponens since the conclusion is true in the only row in which the premises are true:

P	Q	P→Q	
T	T	T	← Critical Row
T	F	F	
F	T	T	
F	F	F	

Where as, this table shows that the Converse Error is an invalid argument since in one of the critical rows the premises are true while the conclusion is false.

P	Q	P→Q	
T	T	T	← Critical Row
T	F	F	
F	T	T	← Critical Row
F	F	F	

Practice Problem 1.11. Fill in the truth table below and identify the critical rows in order to demonstrate the validity of Modus Tollens.

P	Q	~P	~Q	P→Q

Practice Problem 1.12. Fill in the truth table below and identify the critical rows in order to demonstrate that the inverse error is not a valid argument form.

P	Q	~P	~Q	P→Q

Exposition 1.4. Finally, we can understand implications and arguments in terms of ***Venn Diagrams***. For example, how does figure 1.1 demonstrate that

"If you are a zombie, then you are green"

is true, while

"If you are green, then you are a zombie"

is false?

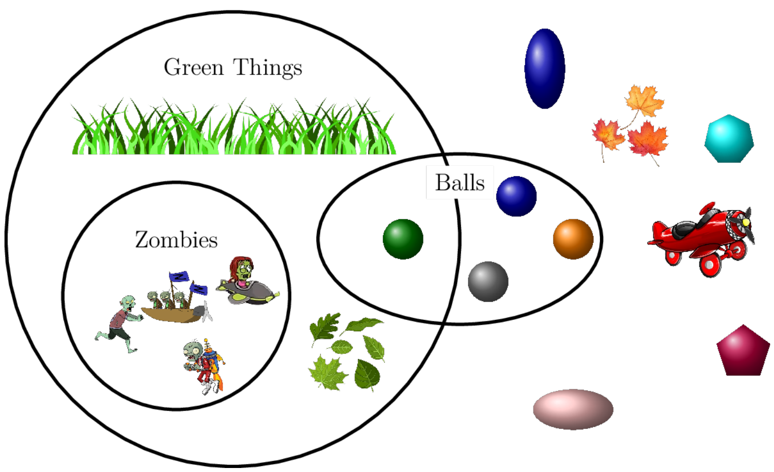


Figure 1.1: Diagramming an Implication and Arguments

Further, how can we use figure 1.1 to explain the validity or invalidity of all the arguments in Exposition 1.3 on page 13?

1.3 Using Logical Laws and Arguments

Exposition 1.5. [a] Consider the following argument:

$$p \to r \tag{1.1}$$
$$\sim p \to q \tag{1.2}$$
$$q \to s \tag{1.3}$$
$$\therefore \sim r \to s \tag{1.4}$$

Expressions 1.1, 1.2, and 1.3 are called ***premises*** (or ***hypotheses***), they are what we are assuming to be true. Expression 1.4 is the ***conclusion***, we are claiming that it is a logical necessity given the premises. Our goal now is to explain why the conclusion necessarily follows from the premise. One way to do this by laying out our proof in two columns.

Steps	Justifications	
$p \to r$	Premise	(1.5)
$\sim r \to \sim p$	Contrapositive and 1.5	(1.6)
$\sim p \to q$	Premise	(1.7)
$\sim r \to q$	Transitivity, 1.6, & 1.7	(1.8)
$q \to s$	Premise	(1.9)
$\therefore \sim r \to s$	Transitivity, 1.8, & 1.9	(1.10)

Thus we can see that using ideas previously discussed we can assemble and remix statements to arrive at a conclusion. Note, this could have been done using a truth table and looking at critical rows or using Venn diagrams; however, with four statements a truth table would have had sixteen rows or we'd need a Venn diagram with four overlapping sets with up to sixteen regions to consider.

[a] This example and the others in this section are borrowed or adapted from Grimaldi [2] or Rosen [5].

Practice Problem 1.13. Below is a proof that this is a valid argument:

$$(\sim p \vee \sim q) \rightarrow (r \wedge s) \tag{1.11}$$
$$r \rightarrow t \tag{1.12}$$
$$\sim t \tag{1.13}$$

$$\therefore p \tag{1.14}$$

Before looking at the proof, what are the premises in this argument?

And, what is the conclusion in the argument?

Now, fill in the missing justifications in the proof, for each justification you need to include one of the valid argument forms from Exposition 1.3 on page 13.
Proof:

Steps	Justifications	
$r \rightarrow t$	________________	(1.15)
$\sim t$	________________	(1.16)
$\sim r$	________________	(1.17)
$\sim r \vee \sim s$	________________	(1.18)
$\sim (r \wedge s)$	________________	(1.19)
$(\sim p \vee \sim q) \rightarrow (r \wedge s)$	________________	(1.20)
$\sim (\sim p \vee \sim q)$	________________	(1.21)
$p \wedge q$		(1.22)
$\therefore p$		(1.23)

Practice Problem 1.14. Here is another valid argument:

$$\sim p \wedge q \tag{1.24}$$
$$r \rightarrow p \tag{1.25}$$
$$\sim r \rightarrow s \tag{1.26}$$
$$s \rightarrow t \tag{1.27}$$
$$\therefore t \tag{1.28}$$

Below you need to try and fill in both the steps and justifications for a proof. Before looking at the proof, what are the premises in this argument?

And, what is the conclusion in the argument?

Now, fill in the steps and justifications for the proof, for each justification you need to include one of the valid argument forms from Exposition 1.3 on page 13.
Proof:

Steps	Justifications	
		(1.29)
		(1.30)
		(1.31)
		(1.32)
		(1.33)
		(1.34)
		(1.35)
$\therefore t$		(1.36)

Practice Problem 1.15. For this last example the argument is given in sentences:

1. "If you send me an e-mail message, then I will finish writing the program."

2. "If you do not send me an e-mail message, then I will go to sleep early."

3. "If I go to sleep early, then I will wake up feeling refreshed."

4. ∴ "If I do not finish writing the program, then I will wake up feeling refreshed."

Below you need to try and fill in both the steps and justifications for a proof. But first identify the individual component statements which make up each implication and give them a name like p or q (try to make it meaningful). Then identify the premises and conclusion in this argument?

Finally, fill in the steps and justifications for the proof, for each justification you need to include one of the valid argument forms from Exposition 1.3 on page 13.
Proof:

Steps	Justifications	
		(1.37)
		(1.38)
		(1.39)
		(1.40)
		(1.41)
		(1.42)
∴		(1.43)

1.4 Predicates

Definition 1.5 (Predicates and Quantifiers)**.** A ***predicate*** is a statement containing a finite number of variables. The predicate becomes true or false based on the value of the variables. The variables may be specific values or elements of a set. If a predicate is always true, or always false, this may be indicated with the ***universal quantifier,*** $\forall$, which is read "for all". If it is true for selected values we use the ***existential quantifier,*** $\exists$, which is read as "there exists".

Practice Problem 1.16 (Sample Predicates)**.** Find values for the variables which make the predicates true or false.

1. $P(x) : x \in \mathbb{Z} \wedge x > 5$
2. $Q(x) : x \in \mathbb{R} \wedge x > 5$
3. $G(a, b) : a > b$
4. $\forall a, b \in \mathbb{Z} : G(a, b) \to G(a^2, b^2)$
5. $\forall a, b \in \mathbb{Z} : G(a, b) \to G(a^3, b^3)$
6. $S(n, m) : n^2 > m$
7. $\exists n \in \mathbb{Z} : S(n, 25)$
8. $\forall n \in \mathbb{Z} : S(n, 0)$
9. $\exists n \in \mathbb{Z} :\sim S(n, 0)$
10. $\exists n \in \mathbb{Z} : S(n, m)$
11. $\forall m \in \mathbb{Z}\ \exists n \in \mathbb{Z} : S(n, m)$
12. $\exists n \in \mathbb{Z} :\sim S(n, 0)$
13. $\forall a, b \in \mathbb{R} : G(a, b) \leftrightarrow S(a, b)$
14. $\forall n \in \mathbb{N}\ \exists a, b, c \in \mathbb{N} : a^n + b^n = c^n$

Practice Problem 1.17 (Shapes, Colors, and Zombies, Oh My!!)**.** Use figure 1.2 on page 21 to identify when these predicates are true or false.

1. $C(x)$: x is a consonant
2. $V(x)$: x is a vowel
3. $Z(x)$: x is a zombie
4. $S(x)$: x is a stick figure
5. $N(x)$: x is a brain
6. $R(x)$: x is ***red***

1. $B(x)$: x is ***blue***
2. $G(x)$: x is ***green***
3. $P(x)$: x is a pentagon (5-sides)
4. $H(x)$: x is a heptagon (7-sides)
5. $W(x, y)$: x is in the same row as y
6. $L(x, y)$: x is in the same column as y

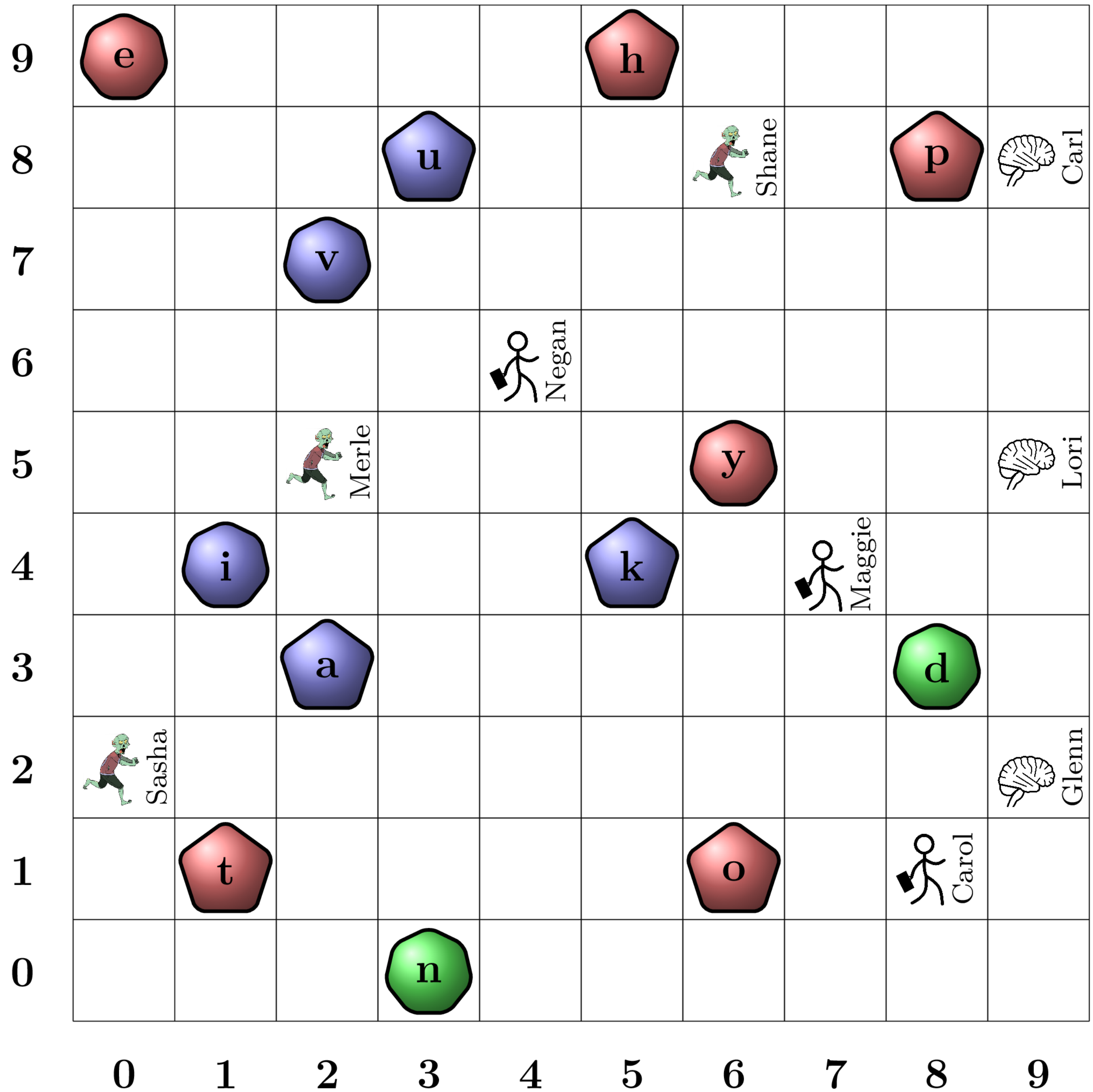


Figure 1.2: Tarski's World

Exposition 1.6 (Truth Sets)**.** Given a predicate, the ***truth set*** for that predicate is the collection of all elements where it is true. For example for $V(x) : x$ is a vowel, the truth set is

$$T_V = \{a, e, i, o, u, y\}$$

and for $P(x) : x$ is a pentagon the truth set is

$$T_P = \{t, a, u, h, k, o, p\}.$$

Figure 1.3 shows how these two sets are related, in particular we can see that

$$V(x) \wedge P(x) \text{ and } V(x) \wedge \sim P(x) \text{ and } \sim V(x) \wedge P(x)$$

are all true if we pick the right value of x.

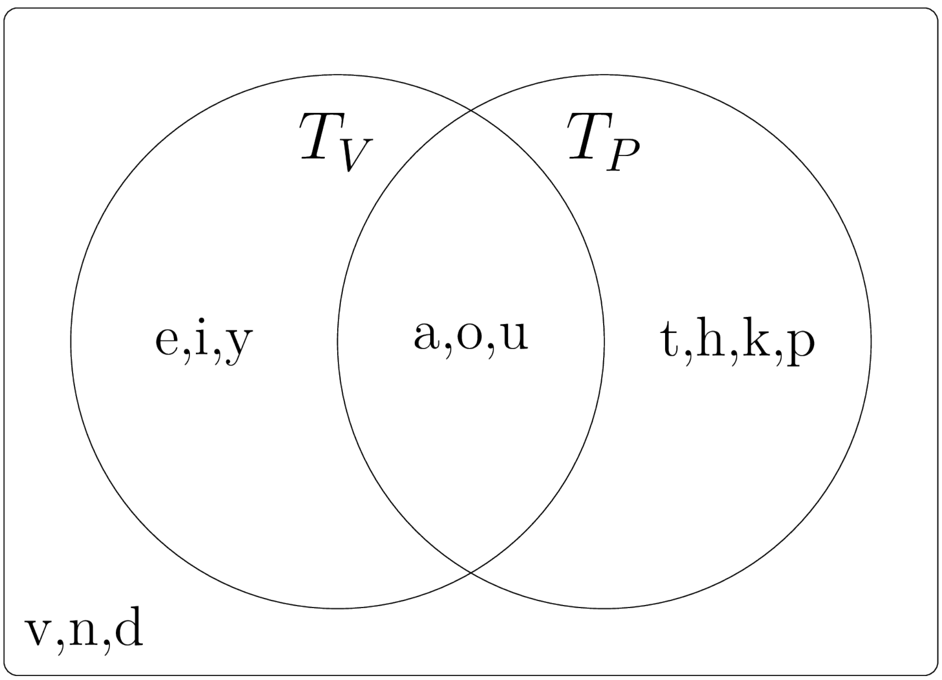


Figure 1.3: Venn Diagram of T_V and T_P

The predicates like

$$W(x, y) : x \text{ is in the same row as } y$$

would have the truth set like

$$T_W = \{(e, h), (h, e), (u, p), (p, u), (i, k), (k, i), (a, d), (d, a), (t, o), (o, t)\},$$

consisting of ordered pairs.

Practice Problem 1.18 (Truth Zombies)**.** Using figure 1.2, fill in the ***truth set*** for each of the predicates.

1. The truth set for $G(x)$ is

$$T_G = \left\{ \qquad \right\}$$

2. The truth set for $H(x)$ is

$$T_H = \left\{ \qquad \right\}$$

3. Fill in the Venn Diagram in figure 1.4 for each of the previous truth sets.

4. Where would the set T_Z fit in figure 1.4? What does that tell us about the statement

$$\forall x : Z(x) \rightarrow G(x)?$$

5. The truth set for $L(x, y)$ is

$$T_L = \left\{ \qquad \right\}$$

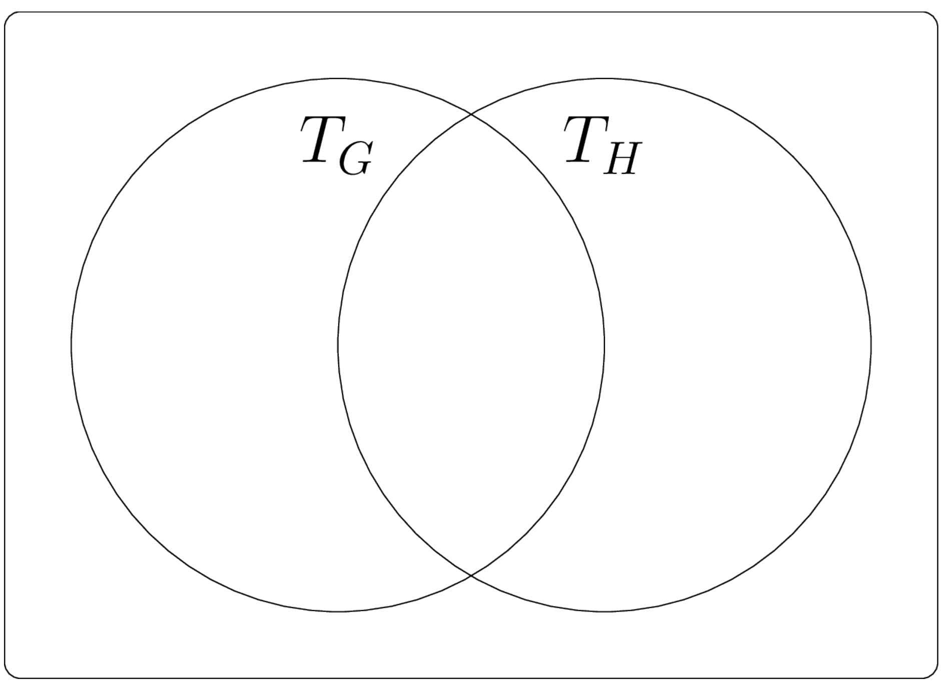


Figure 1.4: Venn Diagram of T_G and T_H

Exposition 1.7 (Quantified Predicates)**.** Recall $\exists$ is the ***existential quantifier*** and $\forall$ the ***universal quantifier***. Now, if we write

$$\exists x : V(x) \wedge P(x)$$

we can read that as

"there exists a vowel which is also a pentagon"

or in plainer English

"some vowels are pentagons."

Is the previous statement true or false? How can we see it in the Venn Diagram in figure 1.3?

And, if we write

$$\forall x : V(x) \vee P(x)$$

we read it as

"all the shapes are vowels or pentagons."

Is this true or false? How can we see it in the Venn diagram in figure 1.3?

In addition to using them one at a time we can combine quantifiers like so

$$\forall x\, \exists y : C(x) \rightarrow L(x, y),$$

which we read as

"every consonant shares a column with some letter,"

or like this

$$\exists y\, \forall x : C(x) \rightarrow L(x, y)$$

which we read as

"some letter shares a column with every consonant."

Which of the previous two statements is true and which is false? How do we know?

Practice Problem 1.19 (Tarski's Zombies)**.** Referring to figure 1.2, which of the following predicates are true and which are false? Try to write each in plain English and negate the false predicates.

1. $\forall x : H(x) \to C(x)$

2. $\exists x : V(x) \land R(x)$

3. $\forall x\, \exists y : C(x) \to V(y) \land L(x, y)$

4. $\forall x\, \forall y : C(x) \land V(y) \to L(x, y) \lor W(x, y)$

5. $\forall x \in T_Z, \exists y \in T_N : W(x, y)$

6. $\exists y \in T_N, \forall x \in T_Z : W(x, y)$

Finally, look at figure 1.2 and construct three quantified statements of your own using the given predicates. Make two of them true and one a lie. Make at least one of them doubly quantified.

- Truth:

- Truth:

- Lie:

Exposition 1.8 (A Couple Comments on Quantifiers)**.** As we have seen in the examples above the negation of a universal quantifier is an existential and vice versa:

$$\sim (\forall x : P(x) \to Q(x)) \equiv \exists x : P(x) \wedge \sim Q(x).$$

With that in mind, what is the negation of "There exists a green vowel which is also a consonant?"

How would we write each of these symbolically?

Referring to figure 1.2 on 21, why is the original statement always false and the negation ***vacuously true***?

The ways in which we represent ideas with quantifiers are not unique. Here are some examples of equivalent statements from practice problem 1.19 which use the predicates for figure 1.2 on page 21.

- $\forall x : H(x) \to C(x)$ can be written
$$\forall x \in T_H : C(x).$$
- $\exists x : V(x) \wedge R(x)$ can be written
$$\exists x \in T_V : R(x).$$
- $\forall x\, \exists y : C(x) \to V(y) \wedge L(x, y)$ is equivalent to
$$\forall x : C(x) \to \exists y : V(y) \wedge L(x, y).$$
- $\exists y \in T_N, \forall x \in T_Z : W(x, y)$ is the same as
$$\exists y : N(y) \wedge \forall x : Z(x) \to W(x, y)$$
and
$$\exists y \forall x : N(y) \wedge Z(x) \to W(x, y).$$

When looking at these and similar examples it is helpful to think of ***for all*** statements as frequently relating to inclusion or equivalence and ***there exists*** statements as overlap. Try making some of these types of statements either with you examples from practice problem 1.19 or using the Venn Diagram in figure 1.1 on page 14.

Chapter 2

Sequences and Sums

2.1 Sequences

2.1.1 Working with Sequences and their Formula

Definition 2.1 (Sequence)**.** A ***sequence*** is an ordered list of numbers or objects such as

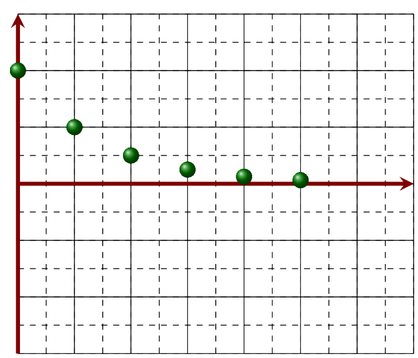
$$s_0 = 0, s_1 = 2, s_2 = 4, s_3 = 6, \ldots.$$

In this sequence we say that s_0 is the ***initial term***, each subscript is called an ***index***, and $s_n = 2n$ would be the ***explicit*** or ***general*** formula.

Exposition 2.1. Consider the sequence with explicit formula $a_n = 1/2^n$:

- $a_0 = 1/2^0 = 1$
- $a_1 = 1/2^1 = 1/2$
- $a_2 = 1/2^2 = 1/4$
- $a_3 = 1/2^3 = 1/8$
- $a_4 = 1/2^4 = 1/16$
- $a_5 = 1/2^5 = 1/32$

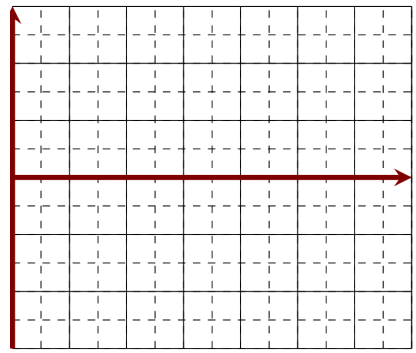
Description: Bounded and decreasing



Practice Problem 2.1. Consider the sequence $b_n = 2^n$:

- $b_0 =$
- $b_1 =$
- $b_2 =$
- $b_3 =$
- $b_4 =$

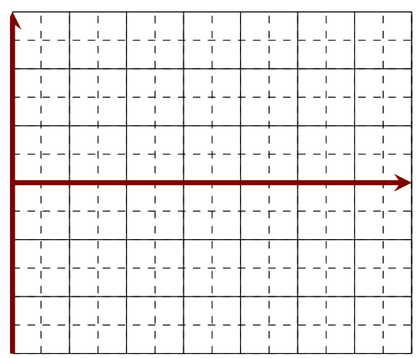
Description:



Practice Problem 2.2. Consider $c_n = (-2/3)^n$:

- $c_0 =$
- $c_1 =$
- $c_2 =$
- $c_3 =$
- $c_4 =$

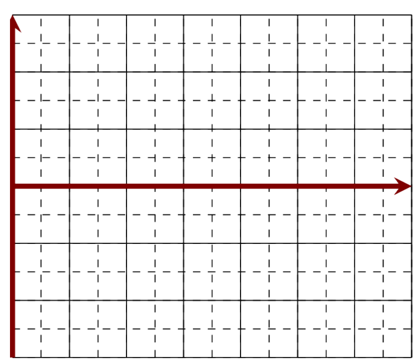
Description:



Practice Problem 2.3. Consider $d_n = 1 - 1/3^n$:

- $d_0 =$
- $d_1 =$
- $d_2 =$
- $d_3 =$
- $d_4 =$

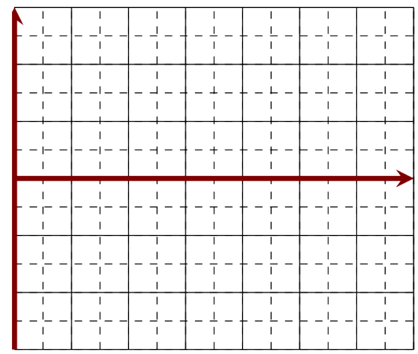
Description:



Practice Problem 2.4. Consider $f_n = 1 + (-1/2)^n$:

- $f_0 =$
- $f_1 =$
- $f_2 =$
- $f_3 =$
- $f_4 =$

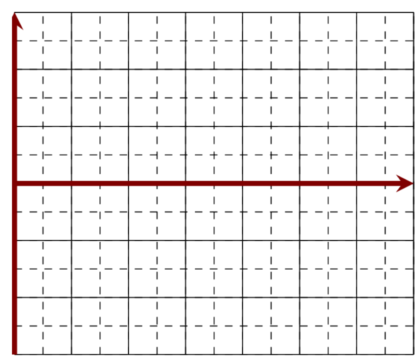
Description:



Practice Problem 2.5. Consider $h_n = (-1)^n + (-1/2)^n$:

- $h_0 =$
- $h_1 =$
- $h_2 =$
- $h_3 =$
- $h_4 =$

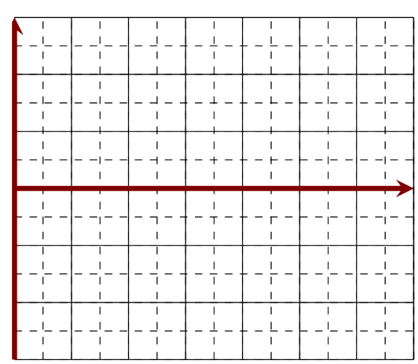
Description:



Practice Problem 2.6. Consider $j_n = (-1)^n - (-1/2)^n$:

- $j_0 =$
- $j_1 =$
- $j_2 =$
- $j_3 =$
- $j_4 =$

Description:



Theorem 2.1. *A bounded sequence has a* **convergent subsequence**. *(The converse is not true.)*

Practice Problem 2.7. Consider $k_n = 0$ for even n and 2^n for odd n:

- $k_0 =$
- $k_1 =$
- $k_2 =$
- $k_3 =$
- $k_4 =$

Description:

Definition 2.2 (Recursive Sequence)**.** A sequence is a ***recursive sequence*** when the initial value is given explicitly, but subsequent values depend on previous values in the sequence.

Practice Problem 2.8. Consider $r_0 = 3$ and $r_n = r_{n-1}/2$:

- $r_0 =$
- $r_1 =$
- $r_2 =$
- $r_3 =$
- $r_4 =$

Description:

Practice Problem 2.9. Consider $s_0 = 0$ and $s_n = 2s_{n-1} + 1$:

- $s_0 =$
- $s_1 =$
- $s_2 =$
- $s_3 =$
- $s_4 =$

Description:

Practice Problem 2.10 (Fibonacci Sequence)**.** Consider $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$:

- $F_0 =$
- $F_1 =$
- $F_2 =$
- $F_3 =$
- $F_4 =$

Description:

Practice Problem 2.11. Let's consider

$$a_0 = 1 \text{ and } a_n = \frac{1}{3}a_{n-1},$$

but this time we will work down from the value we want to the initial term (or ***base case***).

Find a_5:

$a_5 = \frac{1}{3}a_4$	$a_n = \frac{1}{3}a_{n-1}$
$=$	$a_n = \frac{1}{3}a_{n-1}$
$=$	$a_n = \frac{1}{3}a_{n-1}$
$=$	$a_n = \frac{1}{3}a_{n-1}$
$=$	$a_n = \frac{1}{3}a_{n-1}$
$=$	$a_0 = 1$

Find a_{10}:

$a_{10} = \frac{1}{3}a_9$	$a_n = \frac{1}{3}a_{n-1}$
$=$	$a_n = \frac{1}{3}a_{n-1}$
$=$	$a_n = \frac{1}{3}a_{n-1}$
$=$	$a_n = \frac{1}{3}a_{n-1}$
$=$	$a_n = \frac{1}{3}a_{n-1}$
$=$	$a_5 =$

Practice Problem 2.12. Using

$$b_0 = 1 \ \& \ b_1 = 2 \text{ and } b_n = b_{n-1} + 2b_{n-2},$$

find b_5:

$$\begin{aligned}
b_5 &= b_4 + 2b_3 && b_n = b_{n-1} + 2b_{n-2} \\
&= && b_n = b_{n-1} + 2b_{n-2} \\
&= && b_n = b_{n-1} + 2b_{n-2} \\
&= && b_n = b_{n-1} + 2b_{n-2} \ \& \ b_1 = 2 \\
&= && b_0 = 1 \ \& \ b_1 = 2
\end{aligned}$$

2.1.2 Finding Sequence Formulas

Practice Problem 2.13. Given the following values determine a formula for the sequence:

$$a_0 = 1, a_1 = 5, a_2 = 9, a_3 = 13, a_4 = 17, \ldots$$

Practice Problem 2.14. Given the following values determine a formula for the sequence:

$$b_0 = 3, b_1 = 12, b_2 = 48, b_3 = 192, a_4 = 768, \ldots$$

Practice Problem 2.15. Given the following values determine a formula for the sequence:

$$c_0 = -\frac{1}{2}, c_1 = \frac{1}{6}, c_2 = -\frac{1}{12}, c_3 = \frac{1}{20}, c_4 = -\frac{1}{30}, \ldots$$

Practice Problem 2.16. Given the following values determine a formula for the sequence:

$$d_0 = 1, d_1 = 5, d_2 = 13, d_3 = 29, d_4 = 61, \ldots$$

Practice Problem 2.17. Given the following values determine a formula for the sequence:

$$e_0 = 2, e_1 = 3, e_2 = 7, e_3 = 13, e_4 = 27, e_5 = 53, e_6 = 107, \ldots$$

Practice Problem 2.18. Given the following values determine a formula for the sequence:

$$f_1 = 1, f_2 = 7, f_3 = 25, f_4 = 79, f_5 = 241, \ldots$$

Practice Problem 2.19. Given the following values try to describe the sequence, then use the ***On-Line Encyclopedia of Integer Sequences*** (OEIS) (https://oeis.org/) to get the formula.

$$g_1 = 1, g_2 = 22, g_3 = 333, g_4 = 4444, g_5 = 55555, \ldots$$

2.2 Sums

2.2.1 Summation & Product Notation

Definition 2.3 (Summation)**.** A ***summation*** of terms in a sequence a_k can can be written

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + \cdots + a_n$$

and is read "the sum from i equals zero to n of a_i." For example in figure 2.1 we are adding terms in the sequence $a_k = k^2 + 1$ and we would read it as "the sum from i equals zero to 10 of $a_i = i^2 + 1$."

The 10 is the upper bound

The Σ (sigma) means sum or add

Ellipses, $\cdots$, mean and so on

$$\sum_{i=0}^{10} (i^2 + 1) = (0^2 + 1) + (1^2 + 1) + (2^2 + 1) + \cdots + (10^2 + 1)$$

The 0 is the lower bound

The i is the index

Each time we iterate we add another term by evaluating the expression for the new i

Figure 2.1: Summation Notation

Definition 2.4 (Product)**.** A ***product*** of terms in a sequence a_k can can be written

$$\prod_{i=0}^{n} a_i = a_0 \times a_1 \times a_2 \times \cdots \times a_n$$

and is read "the product from i equals zero to n of a_i." For example in figure 2.2 we are multiplying terms in the sequence $a_k = k^2 + 1$ and we would read it as "the product from i equals zero to 10 of $a_i = i^2 + 1$."

The 10 is the upper bound

The Π (pi) means product or multiply

Ellipses, $\cdots$, mean and so on

$$\prod_{i=0}^{10} (i^2 + 1) = (0^2 + 1) \times (1^2 + 1) \times (2^2 + 1) \times \cdots \times (10^2 + 1)$$

The 0 is the lower bound

The i is the index

Each time we iterate we multiply by another factor by evaluating the expression for the new i

Figure 2.2: Product Notation

2.2.2 Arithmetic Sums

Exposition 2.2 (Arithmetic Sum)**.** Let's look at the ***arithmetic sum***:

$$S_n = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

- $S_1 =$
- $S_2 =$
- $S_3 =$
- $S_4 =$

For $n = 100$ what happens when we add the 100^{th} sum to its self in reverse order?

$$\begin{aligned} S_{100} &= 1 + 2 + 3 + \cdots + 98 + 99 + 100 \\ S_{100} &= 100 + 99 + 98 + \cdots + 3 + 2 + 1 \end{aligned}$$

$$2 \cdot S_{100} =$$

$$S_{100} =$$

For $n = k$ what happens when we add the k^{th} sum to its self in reverse order?

$$\begin{aligned} S_k &= 1 + 2 + 3 + \cdots + (k-2) + (k-1) + k \\ S_k &= k + (k-1) + (k-2) + \cdots + 3 + 2 + 1 \end{aligned}$$

$$2 \cdot S_k =$$

$$S_k =$$

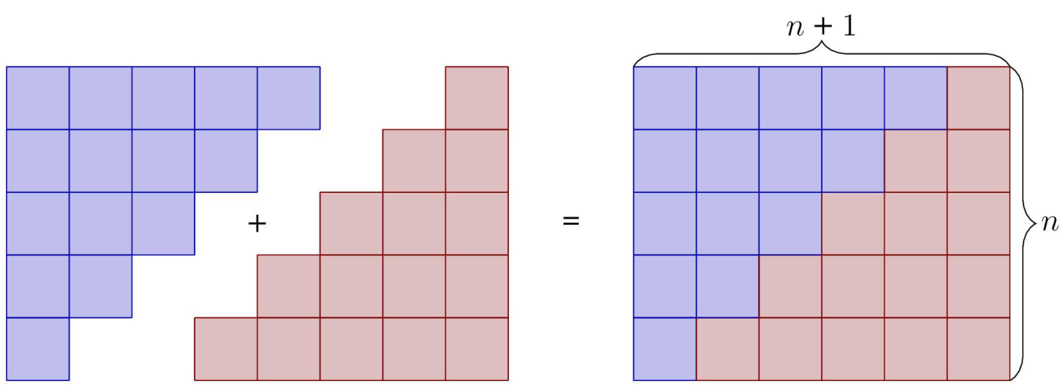


Figure 2.3: Visualizing an Arithmetic Sum

Definition 2.5 (Arithmetic Sum)**.** An ***arithmetic sum*** is a summation of the form

$$\sum_{i=1}^{n} ai + b$$

which has the propery that consecutive terms have a constant difference, a.

Practice Problem 2.20. Now consider $S_n = \sum_{i=1}^{n} 2 \cdot i$:

$$S_n = \sum_{i=1}^{n} 2 \cdot i = 2 + 4 + 6 + \cdots + 2n$$

$$S_5 =$$

For $n = k$ use the same strategy as above in Exposition 2.2 to find a formula for the sum:

$$S_k = 2 + 4 + 6 + \cdots + 2(k-2) + 2(k-1) + 2k$$
$$S_k = 2k + 2(k-1) + 2(k-2) + \cdots + 6 + 4 + 2$$

$$2 \cdot S_k =$$

$$S_k =$$

Practice Problem 2.21. Consider $S_n = \sum_{i=1}^{n} i + 3$:

$$S_n = \sum_{i=1}^{n} i + 3 = 4 + 5 + 6 + \cdots + (n+3) = \sum_{j=4}^{n+3} j$$

$$S_5 =$$

For $n = k$ use the same strategy as above in Exposition 2.2 to find a formula for the sum:

$$S_k = 4 + 5 + 6 + \cdots + (k+1) + (k+2) + (k+3)$$
$$S_k = (k+3) + (k+2) + (k+1) + \cdots + 6 + 5 + 4$$

$$2 \cdot S_k =$$

$$S_k =$$

Exposition 2.3 (General Arithmetic Sum)**.** Finally consider the general arithmetic summation

$$S_n = \sum_{i=1}^{n} ai + b = (a+b) + (2a+b) + (3a+b) + \cdots + (n \cdot a + b).$$

For $n = k$ we can again use the same strategy as Exposition 2.2 to find a formula for this sum.

$$S_k = (a+b) + (2a+b) + (3a+b) + \cdots + (k \cdot a + b)$$
$$S_k = (k \cdot a + b) + \cdots + (3a+b) + (2a+b) + (a+b)$$

$$2 \cdot S_k =$$

$$S_k =$$

$$S_k =$$

2.2.3 Geometric Sums

Exposition 2.4. Let's look at what happens if we sum powers of two.

$$S_n = \sum_{i=0}^{n} 2^i = 1 + 2 + 4 + 8 + \cdots + 2^n$$

$$S_n = \sum_{i=0}^{3} 2^i =$$

Figure 2.4: Summing Powers of Two

For $n = 10$:

$$S_{10} = 1 + 2 + 4 + 8 + \cdots + 2^9 + 2^{10}$$
$$2 \cdot S_{10} = 2 + 4 + 8 + 16 + \cdots + 2^{10} + 2^{11}$$

$$2 \cdot S_{10} - S_{10} =$$

$$S_{10} =$$

For $n = k$:

$$S_k = 1 + 2 + 4 + \cdots + 2^{k-1} + 2^k$$
$$2 \cdot S_k = 2 + 4 + 8 + \cdots + 2^k + 2^{k+1}$$

$$2 \cdot S_k - S_k = \underline{\hspace{6cm}}$$

$$S_k =$$

Definition 2.6 (Geometric Sum)**.** A ***geometric sum*** is a summation of the form

$$\sum_{i=0}^{n} ar^n$$

which has the property that the ratio of any two consecutive terms is a constant ratio, r.

Practice Problem 2.22. Now consider the sum of powers of 1/2.

$$S_n = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^n$$

$$S_4 = \sum_{i=0}^{4} \left(\frac{1}{2}\right)^i =$$

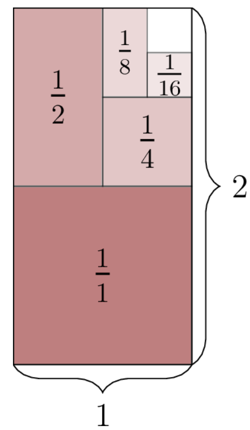


Figure 2.5: Summing Powers of One Half

For $n = k$:

$$S_k = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{k-1} + \left(\frac{1}{2}\right)^k$$

$$\left(\frac{1}{2}\right) S_k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^{k+1}$$

$$\left(\frac{1}{2}\right) S_k - S_k = \left(\frac{1}{2} - 1\right) S_k =$$

$$S_k =$$

Practice Problem 2.23. Next consider the sum of powers of 1/4.

$$S_n = \sum_{i=0}^{n} \left(\frac{1}{4}\right)^i = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots + \left(\frac{1}{4}\right)^n$$

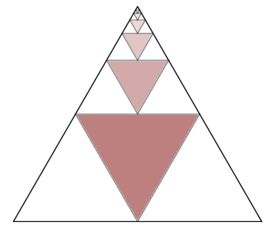


Figure 2.6: Summing Powers of One Quarter

$$S_4 = \sum_{i=0}^{4} \left(\frac{1}{4}\right)^i =$$

For $n = k$:

$$S_k = \frac{1}{1} + \frac{1}{4} + \frac{1}{16} + \cdots + \left(\frac{1}{4}\right)^{k-1} + \left(\frac{1}{4}\right)^k$$

$$\left(\frac{1}{4}\right) S_k = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots + \left(\frac{1}{4}\right)^k + \left(\frac{1}{4}\right)^{k+1}$$

$$\left(\frac{1}{4}\right) S_k - S_k = \left(\frac{1}{4} - 1\right) S_k =$$

$$S_k =$$

Exposition 2.5 (General Geometric Summation)**.** Finally, what happens if we use the same strategy as before to sum multiples of powers of a common ratio.

$$S_n = \sum_{i=0}^{n} a \cdot r^i = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots + a \cdot r^n$$

$$S_4 = \sum_{i=0}^{4} a \cdot r^i =$$

For $n = k$:

$$S_k = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots + a \cdot r^n$$

$$r \cdot S_k = a \cdot r + a \cdot r^2 + a \cdot r^3 + a \cdot r^4 + \cdots + a \cdot r^{n+1}$$

$$r \cdot S_k - S_k =$$

$$S_k =$$

2.3 Using and Finding Formulas

Note 2.1 (Sum Formulas). Here are some common summation formulas.

- Consecutive integers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{2.1}$$

- General Arithmetic Sum:

$$\sum_{i=1}^{n} a \cdot i + b = n \cdot b + a\left(\frac{n(n+1)}{2}\right) \tag{2.2}$$

- Consecutive squares:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{2.3}$$

- Consecutive cubes:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \tag{2.4}$$

- Consecutive odds:

$$\sum_{i=1}^{n} (2i-1) = n^2 \tag{2.5}$$

- Consecutive evens:

$$\sum_{i=1}^{n} 2i = n(n+1) \tag{2.6}$$

- Geometric Sum:

$$\sum_{i=0}^{n} a \cdot r^i = a\left(\frac{r^{n+1}-1}{r-1}\right) \tag{2.7}$$

Exposition 2.6. Once we have a formula for a type of summation we can use it to find other sums. For example lets find the following sum in a couple different ways.

$$\sum_{i=5}^{31} i = 5 + 6 + 7 + \cdots + 30 + 31$$

Solution 1 (Re-Indexing):

$$\begin{aligned}
\sum_{i=5}^{31} i &= 5 + 6 + \cdots + 30 + 31 \\
&= (1+4) + (2+4) + \cdots + (26+4) + (27+4) \\
&= \sum_{j=1}^{27} (j+4) && \text{let } j = i - 4 \\
&= 27 \cdot 4 + \frac{27 \cdot 28}{2} && \text{equation 2.2 p.43} \\
&= 108 + 378 \\
&= 486
\end{aligned}$$

Solution 2 (Adding Zero):

$$\begin{aligned}
\sum_{i=5}^{31} i &= 5 + 6 + \cdots + 30 + 31 \\
&= (1 + 2 + 3 + 4 + 5 + 6 + \cdots + 30 + 31) - (1 + 2 + 3 + 4) \\
&= \left(\sum_{i=1}^{31} i\right) - \left(\sum_{i=1}^{4} i\right) && \text{sum notation} \\
&= \left(\frac{31 \cdot 32}{2}\right) - \left(\frac{4 \cdot 5}{2}\right) && \text{equation 2.1 p.43} \\
&= 496 - 10 \\
&= 486
\end{aligned}$$

Practice Problem 2.24. Use the techniques in Exposition 2.6 and equation 2.4, p.43, to find the sum of:

$$\sum_{i=17}^{236} i^3 = 17^3 + 18^3 + 19^3 + \cdots + 235^3 + 236^3$$

Solution 1 (Re-Indexing):

$$\sum_{i=17}^{236} i^3 = 17^3 + 18^3 + 19^3 + \cdots + 235^3 + 236^3$$

$$=$$

$$=$$

$$=$$

$$=$$

Solution 2 (Adding Zero):

$$\sum_{i=17}^{236} i^3 = 17^3 + 18^3 + \cdots + 235^3 + 236^3$$

$$=$$

$$=$$

$$=$$

$$=$$

Practice Problem 2.25. Use the techniques in Exposition 2.6 and equation 2.7, p.43, to find the sum of:

$$\sum_{i=10}^{106}\left(\frac{3}{7}\right)^i = \left(\frac{3}{7}\right)^{10} + \left(\frac{3}{7}\right)^{11} + \cdots + \left(\frac{3}{7}\right)^{106}$$

Solution 1 (Re-Indexing):

$$\sum_{i=10}^{106}\left(\frac{3}{7}\right)^i = \left(\frac{3}{7}\right)^{10} + \left(\frac{3}{7}\right)^{11} + \cdots + \left(\frac{3}{7}\right)^{106}$$

$$=$$

$$=$$

$$=$$

$$=$$

Solution 2 (Adding Zero):

$$\sum_{i=10}^{106}\left(\frac{3}{7}\right)^i = \left(\frac{3}{7}\right)^{10} + \left(\frac{3}{7}\right)^{11} + \cdots + \left(\frac{3}{7}\right)^{106}$$

$$=$$

$$=$$

$$=$$

$$=$$

Exposition 2.7 (Iteration Algorithm)**.** Given $a_0 = b$ and an expression $a_k = c \cdot a_{k-1} + d$:

1. While $k > 0$:
   (a) Replace a_k with $(c \cdot a_{k-1} + d)$
   (b) Collect like terms
   (c) Decrement k
2. When $k = 0$ replace a_0 with b
3. Simplify the final expression with appropriate formula
4. Generalize

Apply ***Iteration*** to $a_0 = 2$, $a_n = a_{n-1} + 2$, starting at a_3 $(k = 3)$:

$$\begin{aligned}
a_3 &= (a_2 + 2) && \text{apply steps 1a - 1bc} \\
&= ((a_1 + 2) + 2) && \text{apply step 1a} \\
&= (a_1 + 2 \cdot 2) && \text{apply step 1b \& 1c} \\
&= ((a_0 + 2) + 2 \cdot 2) && \text{apply step 1a} \\
&= (a_0 + 3 \cdot 2) && \text{apply step 1b \& 1c} \\
&= (2 + 3 \cdot 2) && \text{apply step 2} \\
&= 4 \cdot (2) && \text{apply step 3} \\
a_n &= (n + 1) \cdot (2) && \text{apply step 4}
\end{aligned}$$

Note. The objective of this example and those that follow is to demonstrate an ability to understand and apply a specific ***algorithm***, not just to get an answer. In fact, as we shall see, for these basic recursive sequences (first order linear recurrence relations) the solution is essentially the same every time. In contrast, algorithms play a crucial role in mathematics and computer science.

Definition 2.7 (Algorithm)**.** An ***algorithm*** is a specific set of instructions for carrying out a procedure or solving a problem, usually with the requirement that the procedure terminate at some point. Specific algorithms sometimes also go by the name method, procedure, or technique. The word “algorithm” is a distortion of al-Khwārizmī, a Persian mathematician who wrote an influential treatise about algebraic methods. The process of applying an algorithm to an input to obtain an output is called a computation. [7]

Exposition 2.8. Apply the iteration algorithm (exposition 2.7, p. 47) to the recusance relation defined by $a_0 = 4$ and $a_n = 6 \cdot a_{n-1} + 4$.

$$
\begin{aligned}
a_4 &= (6 \cdot a_3 + 4) && \text{step 1a: } a_k = 6 \cdot a_{k-1} + 4 \\
&= 6 \cdot (6 \cdot a_2 + 4) + 4 && \text{step 1a: } a_k = 6 \cdot a_{k-1} + 4 \\
&= 6^2 \cdot a_2 + 6 \cdot 4 + 4 && \text{step 1b \& 1c} \\
&= 6^2 \cdot (6 \cdot a_1 + 4) + 6 \cdot 4 + 4 && \text{step 1a: } a_k = 6 \cdot a_{k-1} + 4 \\
&= 6^3 \cdot a_1 + 6^2 \cdot 4 + 6 \cdot 4 + 4 && \text{step 1b \& 1c} \\
&= 6^3 \cdot (6 \cdot a_0 + 4) + 6^2 \cdot 4 + 6 \cdot 4 + 4 && \text{step 1a: } a_k = 6 \cdot a_{k-1} + 4 \\
&= 6^4 \cdot a_0 + 6^3 \cdot 4 + 6^2 \cdot 4 + 6 \cdot 4 + 4 && \text{step 1b \& 1c} \\
&= 6^4 \cdot 4 + 6^3 \cdot 4 + 6^2 \cdot 4 + 6 \cdot 4 + 4 && \text{step 2: } a_0 = 4 \\
&= \sum_{i=0}^{4} 4 \cdot 6^i = 4\left(\frac{6^5 - 1}{6 - 1}\right) && \text{step 3 and formula 2.7 p.43} \\
a_k &= \sum_{i=0}^{k} 4 \cdot 6^i = 4\left(\frac{6^{k+1} - 1}{6 - 1}\right) && \text{step 4}
\end{aligned}
$$

Note. If you look at the work above you see that we never replaced something like $6^2 \cdot 4$ with 144; arithmetic can be your enemy when looking for patterns. Collect similar terms/factors, but try to not do so much arithmetic that you lose the pattern. This takes practice.

Practice Problem 2.26. Apply the iteration algorithm (exposition 2.7, p. 47) to the recusance relation defined by $b_0 = 5$ and $b_n = 3 \cdot b_{n-1} + 5$ to show that

$$b_n = 5\left(\frac{3^{n+1} - 1}{3 - 1}\right).$$

Remember to always replace b_n with $3 \cdot b_{n-1} + 5$ if $k \neq 0$ and 5 if it does.

$b_4 =$

$=$

$=$

$=$

$=$

$=$

$=$

$b_n =$

Practice Problem 2.27. Apply the iteration algorithm (exposition 2.7, p. 47) to the recusance relation defined by $c_0 = 2$ and $c_n = -2 \cdot c_{n-1} + 2$ to show that

$$c_n = 2\left(\frac{(-2)^{n+1} - 1}{-2 - 1}\right).$$

Remember to always replace c_n with $-2 \cdot c_{n-1} + 2$ if $k \neq 0$ and 2 if it does.

$c_4 =$

$=$

$=$

$=$

$=$

$=$

$=$

$c_n =$

Exposition 2.9. If we try the iteration algorithm with $F_1 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ we get the following:

$$\begin{aligned}
F_5 &= F_4 + F_3 && F_n = F_{n-1} + F_{n-2} \\
&= (F_3 + F_2) + (F_2 + F_1) && F_n = F_{n-1} + F_{n-2} \\
&= F_3 + 2 \cdot F_2 + F_1 \\
&= (F_2 + F_1) + 2 \cdot (F_1 + F_0) + F_1 && F_n = F_{n-1} + F_{n-2} \\
&= F_2 + 4 \cdot F_1 + 2 \cdot F_0 \\
&= (F_1 + F_0) + 4 \cdot F_1 + 2 \cdot F_0 && F_n = F_{n-1} + F_{n-2} \\
&= 5 \cdot F_1 + 3 \cdot F_0.
\end{aligned}$$

But $F_4 = 5$ and $F_3 = 3$ so this doesn't really tell us anything.

Definition 2.8 (Second Oder Linear Recurrence Relation)**.** Since the next term in the sequence depends on two prior terms, sequences like the Fibonacci sequence are called ***second order linear recurrence relations***. Where as the other recursive sequences we looked at were ***first order linear recurrence relations*** since the next term only depended on the one previous term.

Theorem 2.2 (Distinct Roots Version)**.** *Given a second order linear recurrence relation with* r_0, r_1, *and* $r_n = A \cdot r_{n-1} + B \cdot r_{n-2}$, *if the roots of the* **characteristic polynomial**,

$$x^2 - Ax - B = 0,$$

are distinct values s_0 *and* s_1, *then* $r_n = C \cdot s_0^n + D \cdot s_1^n$ *for appropriate values of* C *and* D.

Exposition 2.10. In order to apply theorem 2.2 to the Fibonacci Sequence in which $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, we first find the characteristic polynomial. Since $F_n = F_{n-1} + F_{n-2}$, $A = B = 1$ and so we need the roots of $x^2 - x - 1$ which are

$$x = \frac{1 \pm \sqrt{5}}{2}$$

and by theorem 2.2

$$F_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for some C and D. We can use F_0 and F_1 to set up a system of two equations and two unknowns in order to solve for C and D,

$$F_0 = C + D = 1 \text{ and } F_1 = C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

Using these we get

$$C = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) \text{ and } D = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right).$$

Therefore the closed formula for the Fibonacci sequence is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

Note 2.2 (Golden Ratio). The value

$$\phi = \frac{1+\sqrt{5}}{2}$$

is called the ***golden ratio*** and appears in many places in mathematics, art, and nature. It was studied by the ancient Greeks as the solution to the expression $a/b = b/(a-b)$. We can understand its relation to the Fibonacci sequence by looking at the limit

$$L = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}.$$

For large values of n,

$$L \approx \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} \approx 1 + \frac{1}{L}.$$

But then, $L = 1 + 1/L$ or $L^2 - L - 1 = 0$, which, as before. gives $L = \left(1 \pm \sqrt{5}\right)/2$. Finally, this line of thinking gives us a very rough justification for the characteristic polynomial in theorem 2.2.

Practice Problem 2.28. Use theorem 2.2 and the characteristic polynomial to find a closed formula for

$$G_0 = 2, \; G_1 = 3, \; \text{and } G_n = 4 \cdot G_{n-1} + 5 \cdot G_{n-2}.$$

Theorem 2.3 (Single Root Version)**.** *Given r_0, r_1, and $r_n = A \cdot r_{n-1} + B \cdot r_{n-2}$, if the only root of*

$$x^2 - Ax - B = 0,$$

is the value s_0, then $r_n = C \cdot s_0^n + D \cdot n \cdot s_0^n$ for appropriate values of C and D.

Practice Problem 2.29. Using theorem 2.3 and the characteristic polynomial find a closed formula for

$$H_0 = 5,\ H_1 = 4,\ \text{and } H_n = 4 \cdot H_{n-1} - 4 \cdot H_{n-2}.$$

Chapter 3

Sets, Relations, and Functions

3.1 Sets

3.1.1 Set Notation and Common Sets

Definition 3.1 (Set)**.** A set is a finite or infinite collection of objects in which order has no significance, and multiplicity is generally also ignored (unlike a list or multiset). Members of a set are often referred to as elements and the notation a in A is used to denote that a is an element of a set A. The study of sets and their properties is the object of set theory.[6]

Note 3.1 (Common Sets)*.* Here are six sets that we will frequently use:

1. ***Empty Set***: $\varnothing = \{\}$,
2. ***Natural Numbers***:
$$\mathbb{N} = \{1, 2, 3, \ldots\},$$
3. ***Integers***:
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\},$$
4. ***Rational Numbers***:
$$\mathbb{Q} = \{a/b \big| a, b \in \mathbb{Z} \wedge b \neq 0\},$$
5. ***Real Numbers***:
$$\mathbb{R} = \mathbb{Q} \cup \{irrationals\}$$
or all the numbers used to denote or measure distances on a line, and
6. ***Complex Numbers***:
$$\mathbb{C} = \left\{a + bi \middle| a, b \in \mathbb{R} \wedge i^2 = -1\right\}.$$

Definition 3.2 (Set Notations)**.** Describing a set by listing its elements is called ***set roster*** notation:

$$S = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}\,. \tag{3.1}$$

Describing a set by setting conditions for membership is called ***set builder*** notation:

$$S = \{2n | n \text{ is an integer and } 0 < n < 10\}\,. \tag{3.2}$$

Either way this could be read as

> The set of all integers of the form $2n$ with n from 1 to 9,

or in plainer language

> All the even integers from two to eighteen.

To read the sets in set builder notation we need to note some new-ish notation:

"such that" "and"

$$\mathbb{Q} = \{a/b | a, b \in \mathbb{Z} \wedge b \neq 0\}$$

"the set containing" or "the set of" "element of"

Figure 3.1: Set Builder Notation

So we can translate the description of the rationals as

> The rational numbers are ***the set of*** all ratios a/b ***such that*** a and b are ***elements of*** the integers ***and*** b does not equal 0.

Practice Problem 3.1. Given the set $A = \{1/2, 2/2, 3/2, 4/2, 5/2, \ldots\}$ write the set as a sentence and then in set builder notation.

Verbal/Written Description:

> The set A is the set of

Set Builder Description:

$$A = \left\{ \qquad \middle| \qquad \right\}$$

Practice Problem 3.2. Given the set $B = \left\{ n^2/5 \middle| n \in \mathbb{Z} \right\}$ write the set as a sentence and then in set roster notation.

Verbal/Written Description:

The set B is the set of

Set Roster Description:

$$B = \left\{ \qquad\qquad\qquad\qquad \right\}$$

Practice Problem 3.3. The set C is the set of all odd natural numbers less than 30, write the set in set roster notation and then in set builder notation.

Set Roster Description:

$$C = \left\{ \qquad\qquad\qquad\qquad \right\}$$

Set Builder Description:

$$C = \left\{ \qquad \middle| \qquad\qquad \right\}$$

3.1.2 New Sets From Old

Definition 3.3 (Basic Set Combinations)**.** Given sets $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7\}$:

- ***Union of A and B***:

$$A \cup B = \{x | x \in A \ \vee x \in B\} = \{1, 2, 3, 5, 7, 9\} \tag{3.3}$$

- ***Intersection of A and B***:

$$A \cap B = \{x | x \in A \ \wedge x \in B\} = \{3, 5, 7\} \tag{3.4}$$

- ***Difference between A and B***:

$$A \setminus B = \{x | x \in A \ \wedge x \notin B\} = \{1, 9\} \tag{3.5}$$

Practice Problem 3.4. Given $C = \{n | n \in \mathbb{Z} \wedge -5 \le n \le 5\}$ and $D = \{-2, -3, -5, -7, -11, -13, -17, -19\}$ find the following:

- $C \cup D = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$
- $C \cap D = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$
- $C \setminus D = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$
- $D \setminus C = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$

Practice Problem 3.5. Given $V = \{a, e, i, o, u, y\}$ of vowels and $C = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$ of consonants find the following:

- $V \cup C = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$
- $V \cap C = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$
- $C \setminus V = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$
- $V \setminus C = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$

Exposition 3.1 (Universal Sets, Compliments, and Visualization)**.** Again letting $A = \{1,3,5,7,9\}$ and $B = \{2,3,5,7\}$ we could let

$$\mathscr{U} = \{n|n \in \mathbb{N} \wedge n \leq 10\} = \{1,2,3,4,5,6,7,8,9,10\}$$

be the ***universal set*** from which A and B draw elements so we can now write

$$A = \{x|x \in \mathscr{U} \wedge x \text{ is odd}\} \text{ and } B = \{x|x \in \mathscr{U} \wedge x \text{ is prime}\}.$$

And we say A and B are ***subsets***[a] of $\mathscr{U}$. We can also now define the ***compliment*** of a set A as

$$A^c = \{x|x \in \mathscr{U} \wedge x \notin A\} = \{2,4,6,8,10\}.^b \tag{3.6}$$

Finally we can picture all of this with a ***Venn Diagram***:

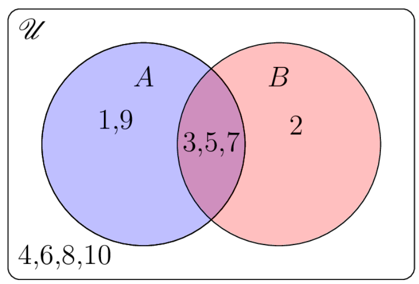


Figure 3.2: Venn Diagram for Sets A and B

[a]Or that $\mathscr{U}$ is a superset of A and B.
[b]Some texts us the notation $\overline{A}$ for the compliment of A.

Practice Problem 3.6. Use the Venn diagram to help you fill in the following sets.

$$\mathscr{U} = \Big\{ \qquad\qquad\qquad\qquad \Big\}$$
$$C = \Big\{ \qquad\qquad\qquad\qquad \Big\}$$
$$C^c = \Big\{ \qquad\qquad\qquad\qquad \Big\}$$
$$A \cap C^c = \Big\{ \qquad\qquad\qquad\qquad \Big\}$$
$$A^c \cap B \cap C = \Big\{ \qquad\qquad\qquad\qquad \Big\}$$
$$A \cup B^c \cup C^c = \Big\{ \qquad\qquad\qquad\qquad \Big\}$$

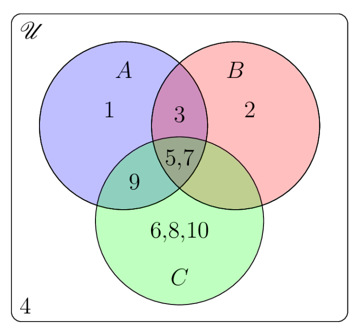


Figure 3.3: Venn Diagram with Sets A, B, and C

Practice Problem 3.7. Use the Venn diagram to help you match up the following sets.

___ (a) A^c	(1) $A^c \cap B$
___ (b) $B \setminus A$	(2) $(A^c \cup B^c)^c$
___ (c) $A \cup B^c$	(3) $\mathscr{U} \setminus A$
___ (d) $A \cap B$	(4) $(A^c \cap B)^c$
___ (e) $(A \cup B) \setminus (A \cap B)$	(5) $(B \setminus A) \cup (A \setminus B)$

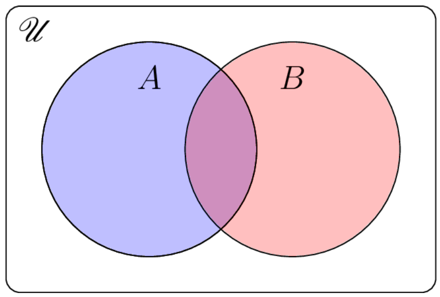


Figure 3.4: Blank Two Set Venn Diagram

Definition 3.4 (Products of Sets)**.** Again letting $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7\}$ we can define the ***Cartesian Product*** of A and B as

$$\begin{aligned} A \times B &= \{(a, b) | a \in A \wedge b \in B\}, \\ &= \{(1,2),(1,3),(1,5),(1,7),(3,2),(3,3),(3,5),(3,7),(5,2),(5,3), \\ &\quad (5,5),(5,7),(7,2),(7,3),(7,5),(7,7),(9,2),(9,3),(9,5),(9,7)\} \end{aligned} \tag{3.7}$$

a new set with ***ordered pairs*** of numbers instead of just individual numbers as elements.

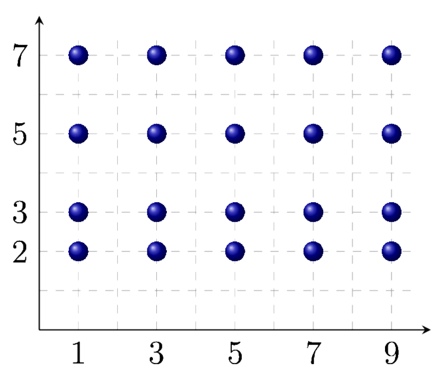


Figure 3.5: Cartesian Product, $A \times B$

Definition 3.5 (Subsets and Power Sets)**.** A set X is a ***subset*** of a set B if every element in X is also in B, and we write

$$X \subseteq B \text{ if and only if } \forall x \in X : x \in B.$$

With $B = \{2, 3, 5, 7\}$ we can use this to define the ***power set*** of a set

$$\begin{aligned}\mathscr{P}(B) &= \{\text{All the subsets of } B\} \\ &= \{\varnothing, \{2\}, \{3\}, \{5\}, \{7\}, \{2,3\}, \{2,5\}, \{2,7\}, \{3,5\}, \{3,7\}, \{5,7\} \\ &\qquad \{2,3,5\}, \{2,3,7\}, \{2,5,7\}, \{3,5,7\}, \{2,3,5,7\}\},\end{aligned} \tag{3.8}$$

which is a new set with sets as elements.

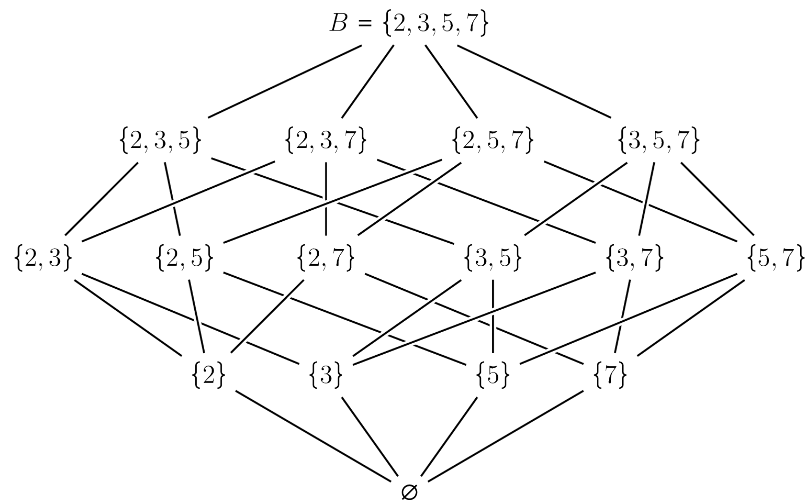


Figure 3.6: Set Inclusions in a Power Set

Practice Problem 3.8. Use $C = \{a, b, c\}$, $D = \{a, b, c, d\}$, and $N = \{1, 2, 3\}$ to fill in each of the following:

$C \times N = \{ \qquad \}$
$N \times C = \{ \qquad \}$
$\mathscr{P}(C) = \{ \qquad \}$
$\mathscr{P}(D) = \{ \qquad \}$

On a side note, what is the distinction between $C = \{a, b, c\}$ and the set $\{\{a\}, \{b\}, \{c\}\}$?

Exposition 3.2. Look again at the power sets for $C = \{a, b, c\}$ and $D = \{a, b, c, d\}$. How can we derive the power set for D starting with the power set for C? How do they relate? What does this tell us about the size of one compared to the other?

Theorem 3.1 (Sizes of Power Sets)**.** *Given a finite set X with n elements, the size (**cardinality**) of the power set of X, $\mathscr{P}(X)$, is equal to 2^n, i.e. $|\mathscr{P}(X)| = 2^n$.*

3.2 Relations

Basics of Relations

Definition 3.6 (Relations)**.** A ***relation*** between to sets A and B is a subset of their Cartesian product $A \times B$. For example given $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7\}$,

$$L = \{(1, 2), (1, 3), (1, 5), (1, 7), (2, 3), (2, 5), (2, 7), (3, 5), (3, 7), (5, 7)\}$$

is a relation between A and B. We normally prefer to describe the relation, for example we can describe L by

$$\forall x \in A\, \forall y \in B : x\, L\, y \text{ if and only if } x < y,$$

or equivalently

$$L = \{(x, y) | x \in A \wedge y \in B \wedge (x < y)\}.$$

Practice Problem 3.9. Let $E = \{2n | n \in \mathbb{N}\}$ and $O = \{2n + 1 | n \in \mathbb{N}\}$. Define a relation by

$$\forall x \in E\, \forall y \in O : x\, D\, y \text{ if and only if } x = 2y.$$

List five additional ordered pairs in the relation D:

$$D = \{(2, 1), (14, 7), (26, 13), \qquad\qquad\qquad\qquad \}$$

Write the set of ordered pairs for D in set builder notation:

Practice Problem 3.10. Define a relation on the set $\mathbb{Z}$ by

$$M_5 = \{(x, y) | x, y \in \mathbb{Z} \wedge \exists q \in \mathbb{Z} : x - y = 5q\} \tag{3.9}$$

List five additional ordered pairs in the relation M_5:

$$M_5 = \{(17, 12), (30, 5), (-20, -45), \qquad\qquad\qquad\qquad \}$$

Write the a description of M_5 using quantifiers:

Properties of Relations

Definition 3.7 (Properties of Relations)**.** Of special interest are relations between a set and its self, such as M_5 described in equation (3.9). In particular when we have a relation R between a set A and its self we look for the following properties:

1. ***Reflexive:*** $\forall a \in A : a\,R\,a$,
2. ***Symmetric:*** $\forall a, b \in A :$ if $a\,R\,b$, then $b\,R\,a$,
3. ***Transitive:*** $\forall a, b, c \in A :$ if $a\,R\,b$ and $b\,R\,c$, then $a\,R\,c$, and
4. ***Antisymmetric:*** $\forall a, b \in A :$ if $a\,R\,b$ and $b\,R\,a$, then $a = b$.

Recall that in equation (3.9) a pair of integers (x, y) were in the relation M_5 if and only if their difference, $x - y$, was a multiple of 5. For example $(17, 12) \in M_5$ since $17 - 12 = 5$, similarly $(12, -28) \in M_5$ since $12 - (-28) = 40 = 8 \times 5$. But we can note that:

$$\begin{aligned}(12-17) &= -1\times(17-12)\\ &= -1\times 5, \qquad (3.10)\\ (-28-12) &= -1\times(12-(-28))\\ &= -8\times 5, \text{ and} \qquad (3.11)\\ (17-(-28)) &= (17-12)+(12-(-28))\\ &= 5+8\times 5\\ &= 9\times 5. \qquad (3.12)\end{aligned}$$

Therefore we get that $(12, 17), (-28, -12)$, and $(17, -28)$ are in M_5 as well.
Further, the calculations in equations (3.10) and (3.11) can be generalized to show that *if* $(a, b) \in M_5$*, then* $(b, a) \in M_5$, i.e. M_5 is a ***symmetric*** relation.
The calculation in equation (3.12) can likewise be generalized to show that *if* (a, b) *and* (b, c) *are in* M_5*, then* (a, c) *is also in* M_5, i.e. M_5 is a ***transitive*** relation.
Finally, since

$$n - n = 0 = 0 \times 5 \qquad (3.13)$$

for all integers n, we know that *for all integers* n*,* (n, n) *is in* M_5, i.e. M_5 is ***reflexive***.
Since, M_5 is reflexive, symmetric, and transitive we say that it is an ***equivalence relation***.
If a relation is reflexive, *antisymmetric*, and transitive, like the less than or equal to relation, then we say it is a ***partial-ordering***.

Practice Problem 3.11. Consider the relation M_7 defined by:

$$\forall x, y \in \mathbb{Z} : x\, M_7\, y \text{ if and only if } \exists q \in \mathbb{Z} : (x - y) = 7q.$$

Rewrite M_7 in set builder notation and list five ordered pairs of integers in M_7. Then try to show that M_7 is reflexive, symmetric and transitive, i.e. it is an equivalence relation.

Practice Problem 3.12. Consider the relation L defined by:

$$\forall x, y \in \mathbb{Z} : x\, L\, y \text{ if and only if } x < y$$

Rewrite L in set builder notation and list five ordered pairs of integers in L. Try to show that L is antisymmetric and transitive, then give examples showing that it is not reflexive or symmetric and so is not an equivalence relation.

Practice Problem 3.13. Consider the relation L' defined by:

$$\forall x, y \in \mathbb{Z} : x\, L'\, y \text{ if and only if } x \leq y$$

Rewrite L' in set builder notation and list five ordered pairs of integers in L'. (Try to list some that were not in L.) Try to show that L' is reflexive, antisymmetric, and transitive, then give examples showing that it is not symmetric and so is not an equivalence relation. Relations like L' are called ***partial-ordering***.

Practice Problem 3.14. Consider the relation D defined by:

$$\forall x, y \in \mathbb{Z} : x\, D\, y \text{ if and only if } x = 2y$$

Rewrite D in set builder notation and list five ordered pairs of integers in D. Try to show that D is antisymmetric, then give examples showing that it is not reflexive, symmetric, or transitive and so is not an equivalence relation.

Practice Problem 3.15. Consider the relation B_ϵ defined by:

$$\forall x, y \in \mathbb{R} : x \, B_\epsilon \, y \text{ if and only if } |x - y| < \epsilon$$

Rewrite B_ϵ in set builder notation and list five ordered pairs of real number in B_ϵ in the case when $\epsilon = 0.5$, i.e. $B_{0.5}$. Try to show that $B_{0.5}$ is reflexive and symmetric, then give examples showing that it is not transitive and so is not an equivalence relation.

Practice Problem 3.16. Consider the relation Q defined by:

$$\forall x, y \in \mathbb{R} : x \, Q \, y \text{ if and only if } x - y \in \mathbb{Q}$$

Rewrite Q in set builder notation and list five ordered pairs of real number in Q. Try to show that Q is an equivalence relation.

Practice Problem 3.17. Consider the relation W defined by:

> Two words, x and y, are in W if and only if they begin and end with the same letters

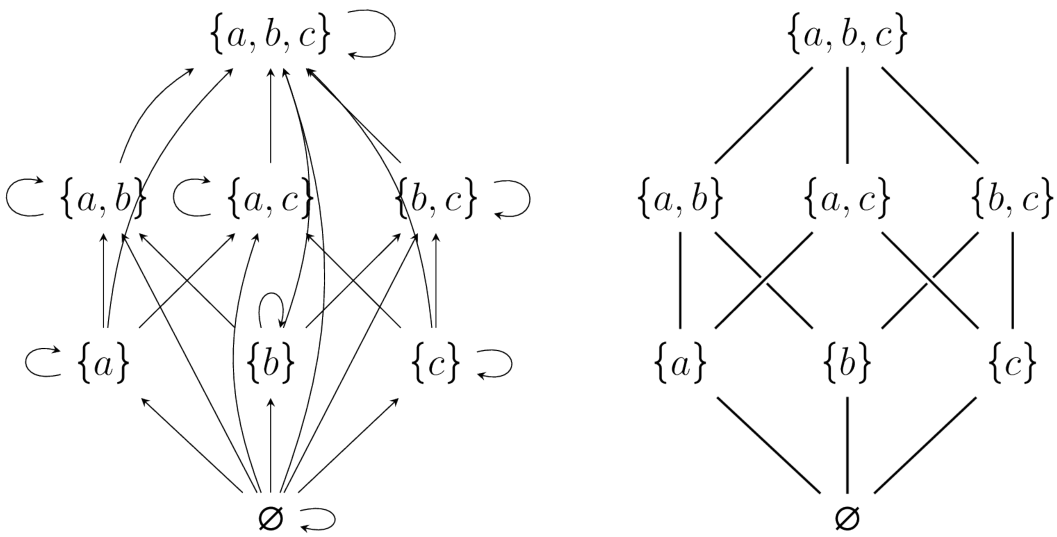
For example $(their, tear) \in W$, but $(there, tear) \notin W$. Try to write W in set builder notation and list five ordered pairs of words in W and five pairs not in W. Try to show that W is reflexive, symmetric, and transitive, i.e. that it is an equivalence relation.

Visualizing Relations

Exposition 3.3. Subset Relation Given a set $X = \{a, b, c\}$ we can define a partial-ordering on $\mathscr{P}(X)$ using the subset or equal to relation:

$$A \sim B \text{ if and only if } A \subseteq B.$$

We can then visualize this as a graph where all the subsets are vertices and two vertices are connected if there is a relation between them.

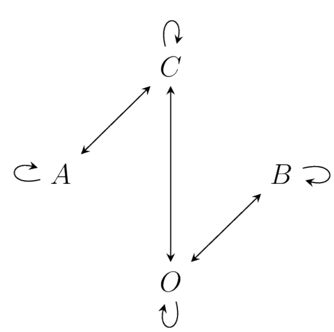


(a) Diagram of Subset Relation (b) Hasse Diagram of Subset Relation

In the first diagram each time an element x relates to an element y we draw an arrow from x to y. The second diagram is drawn assuming we have a partial-ordering and the physically lowest value in the diagram is the ***minimal value*** in the partial-ordering, the loops are implied, and so are arrows going from lower values to higher.

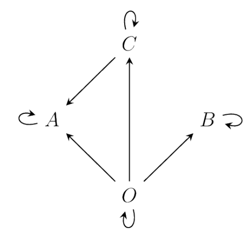
Practice Problem 3.18. Looking at the graph of a relation on the right:

1. How do we know this relation is *reflexive*?
2. How do we know this relation is *symmetric*?
3. How do we know this relation is not *transitive*?



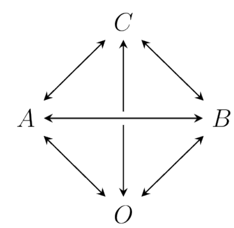
Practice Problem 3.19. Looking at the graph of a relation on the right:

1. How do we know this relation is *reflexive*?
2. How do we know this relation is *antisymmetric*?
3. How do we know this relation is *transitive*?



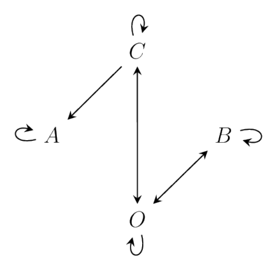
Practice Problem 3.20. Looking at the graph of a relation on the right:

1. How do we know this relation is not *reflexive*?
2. How do we know this relation is *symmetric*?
3. How do we know this relation is not *transitive*?



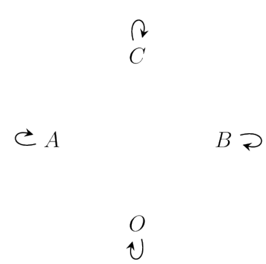
Practice Problem 3.21. Looking at the graph of a relation on the right:

1. How do we know this relation is *reflexive*?
2. Why is this relation not *symmetric*, *antisymmetric*, or *transitive*?



Practice Problem 3.22. Looking at the graph of a relation on the right:

1. How do we know this relation is *reflexive*?
2. Why is this relation *symmetric*, *antisymmetric*, and *transitive*?



Equivalence Classes

Exposition 3.4. Earlier we saw that the relation M_5 (equation (3.9) p.63) was an equivalence relation, i.e. it is reflexive, symmetric, and transitive. We also saw that the values 17, 12, and -28 are all ***"equivalent"***. If we group all the integers together which are equivalent using M_5 we get five sets:

- $[0] = \{0, \pm 5, \pm 10, \pm 15, \ldots\} = \{5k | k \in \mathbb{Z}\}$
- $[1] = \{1, 6, -4, 11, -9, 16, -14, \ldots\} = \{1 + 5k | k \in \mathbb{Z}\}$
- $[2] = \{2, 7, -3, 12, -8, 17, -13, \ldots\} = \{2 + 5k | k \in \mathbb{Z}\}$
- $[3] = \{3, 8, -2, 13, -7, 18, -12, \ldots\} = \{3 + 5k | k \in \mathbb{Z}\}$
- $[4] = \{4, 9, -1, 14, -6, 19, -11, \ldots\} = \{4 + 5k | k \in \mathbb{Z}\}$

And, because of the symmetry and transitivity of the relation $[n] \cap [m] = \varnothing$ whenever $n \neq m$. We call these sets ***equivalence classes*** and the values 0, 1, 2, 3, and 4 are ***equivalence class representatives***, we let them stand in for all the numbers they are equal to.

Definition 3.8 (Equivalence Class)**.** Given an equivalence relation $\mathcal{R}$ defined on a set S we say that two elements of $a, b \in S$ are in the same ***equivalence class*** of $\mathcal{R}$ if and only if a is related to b by $\mathcal{R}$, $a\mathcal{R}b$. Alternately, we can say that the equivalence class for a is the set of all b such that $a\mathcal{R}b$, which is written:

$$[a]_{\mathcal{R}} = \{b \in S | a\mathcal{R}b\}$$

If a is the element of the equivalence class we typically work with we call it the ***equivalence class representative***. Further, we can show that:

- every element of S is in exactly one equivalence class,
- any two equivalence classes are either disjoint or equal, and
- $S = \bigcup_{a \in S} [a]_{\mathcal{R}}$.

These three conditions give us a ***partition*** of S.

[4]	···	9	24	4	19	-1	14	29	···
[3]	···	23	3	18	-2	13	28	8	···
[2]	···	2	17	-3	12	27	7	22	···
[1]	···	16	-4	11	26	6	21	1	···
[0]	···	-5	10	25	5	20	0	15	···

All of $\mathbb{Z}$

Figure 3.8: Relation M_5 Partitioning $\mathbb{Z}$

Practice Problem 3.23. Earlier we defined the relation M_7 as:

$$\forall x, y \in \mathbb{Z} : x\, M_7\, y \text{ if and only if } \exists q \in \mathbb{Z} : (x - y) = 7q.$$

We then *"showed"* that it was an equivalence relation. This relation will have seven equivalence classes with representatives 0, 1, 2, 3, 4, 5, and 6. Try to write down/describe the elements of the equivalence classes:

- $[0] = \Big\{ \qquad\qquad\qquad \Big\} = \Big\{ \qquad\qquad \Big| \qquad\qquad \Big\}$
- $[1] = \Big\{ \qquad\qquad\qquad \Big\} = \Big\{ \qquad\qquad \Big| \qquad\qquad \Big\}$
- $[2] = \Big\{ \qquad\qquad\qquad \Big\} = \Big\{ \qquad\qquad \Big| \qquad\qquad \Big\}$
- $[3] = \Big\{ \qquad\qquad\qquad \Big\} = \Big\{ \qquad\qquad \Big| \qquad\qquad \Big\}$
- $[4] = \Big\{ \qquad\qquad\qquad \Big\} = \Big\{ \qquad\qquad \Big| \qquad\qquad \Big\}$
- $[5] = \Big\{ \qquad\qquad\qquad \Big\} = \Big\{ \qquad\qquad \Big| \qquad\qquad \Big\}$
- $[6] = \Big\{ \qquad\qquad\qquad \Big\} = \Big\{ \qquad\qquad \Big| \qquad\qquad \Big\}$

Practice Problem 3.24. The relations M_5 and M_7 are specific examples of ***equivalence modulo*** n:

$$\forall x, y \in \mathbb{Z} : x \equiv y \pmod{n} \text{ if and only if } \exists q \in \mathbb{Z} : (x - y) = qn. \tag{3.14}$$

As with M_5 and M_7 this defines and equivalence relation. This relation will have n equivalence classes with representatives $0 \leq k \leq (n-1)$. Try to write down/describe the elements of a general equivalence class:

$$[k] = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\} \tag{3.15}$$

$$= \left\{ \qquad\qquad\qquad \middle| \qquad\qquad\qquad \right\} \tag{3.16}$$

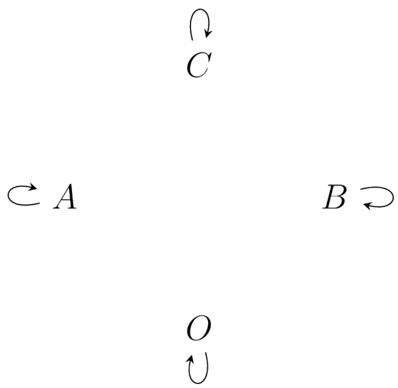
The idea of equivalence modulo n plays an important role in many areas of Math and Computer Science. Modular calculations are built into most programming languages with commands such as `%`, `mod`, `fmod`, `rem`, and `remainder`.

Practice Problem 3.25. Earlier in Problem 3.16 on page 67, we said x and y are related by Q if and only if $x - y \in \mathbb{Q}$ and showed that it was an equivalence relation. List five distinct equivalence classes in Q.

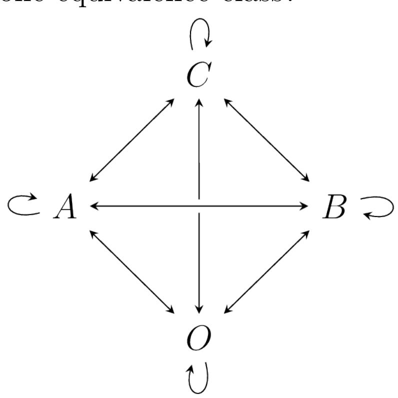
- $C_1 =$
- $C_2 =$
- $C_3 =$
- $C_4 =$
- $C_5 =$

Do you think it is possible to list all the equivalence classes for this relation? Why or Why Not?

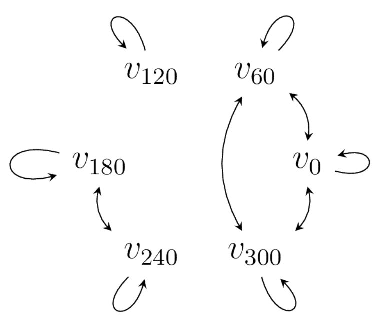
Practice Problem 3.26. The graph represents an, admittedly boring, equivalence relation. What are the four equivalence classes?



Practice Problem 3.27. The graph represents an equivalence relation. Why is there only one equivalence class?



Practice Problem 3.28. The graph represents an equivalence relation. What are the equivalence classes?



3.3 Functions

Functions as Relations

Definition 3.9 (Function)**.** A ***function*** is a relation between two sets, the first called the ***domain*** and the second the ***codomain***, such that for each x in the domain there is *exactly* one y in the codomain associated with x. Further, the set of all y from the codomain which are associated with at least one x in the domain is called the ***range***.

Exposition 3.5.

Define a function f from the Domain

$$D = \{a, b, c, d\}$$

to the Codomain

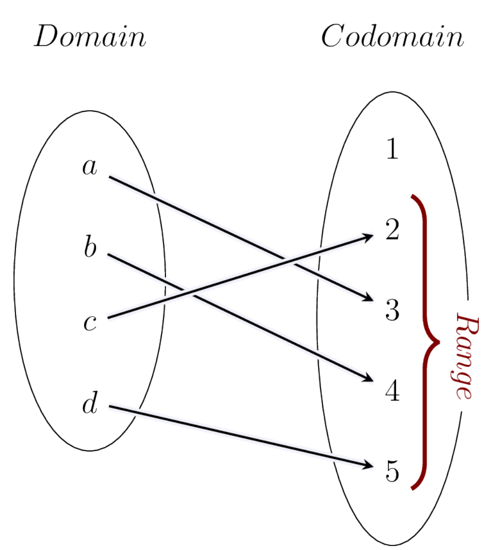
$$C = \{1, 2, 3, 4, 5\}$$

with the following set of pairs:

$$f = \{(a, 3), (b, 4), (c, 2), (d, 5)\}.$$

Since there are no letters associated with the $1 \in C$, the Range is

$$\begin{aligned} R &= \{y \in C \mid \exists x \in D : f(x) = y\} \\ &= \{2, 3, 4, 5\}. \end{aligned}$$



Practice Problem 3.29.

Define a function S from the Primes

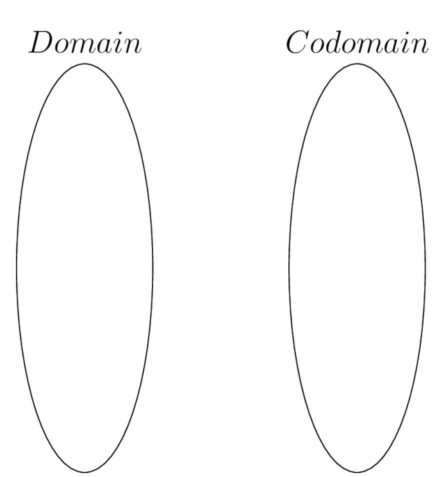
$$P = \{2, 3, 5, 7, 11, \dots\}$$

to the Alphabet

$$A = \{a, b, c, d, e, \dots, w, x, y, z\}$$

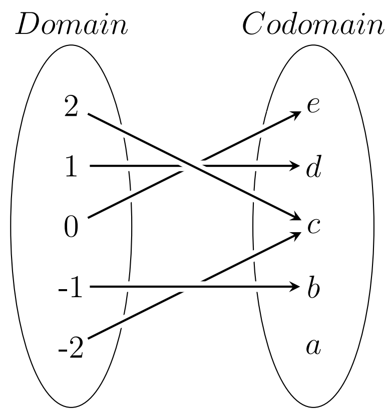
with the following set of pairs:

$$S = \{(2, w), (3, h), (5, i), (7, e), (11, l), \dots\}.$$



Practice Problem 3.30.

What ordered pairs are in the function illustrated on the right?



What is the range of the illustrated function? test

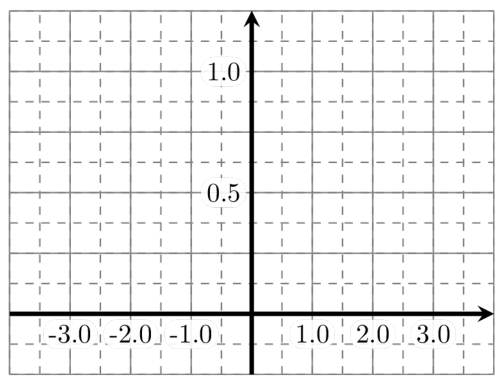
Practice Problem 3.31. Recall that ***function notation*** like $S(x)$ is read "S of x" and whatever we replace x with on the left, we do the same on the right.

Define

$$S(x) = 1/\sqrt{x^2+1}$$

Evaluate the function at selection of points in the domain

$$D = (-4,4) \subset \mathbb{R}.$$



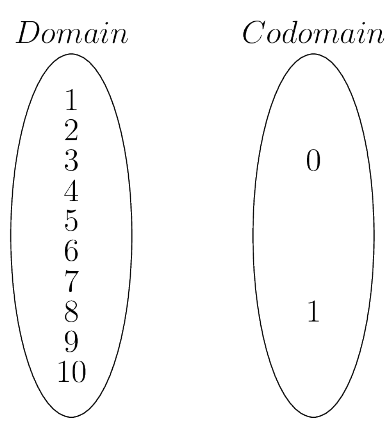
Practice Problem 3.32. This is an example of a ***piecewise defined*** function.

Define

$$P(x) = \begin{cases} 0 & if\ x\ is\ not\ prime \\ 1 & if\ x\ is\ prime \end{cases}$$

Evaluate the function at each point in the domain

$$D = \{1,2,3,5,6,7,8,9,10\}.$$

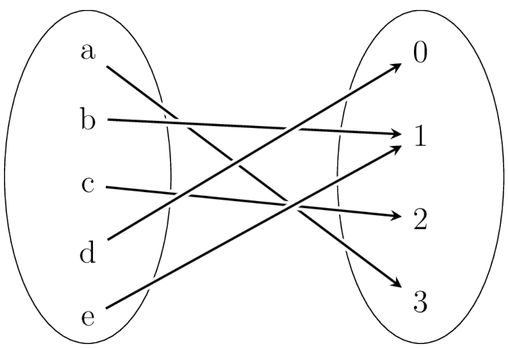


Properties of Functions

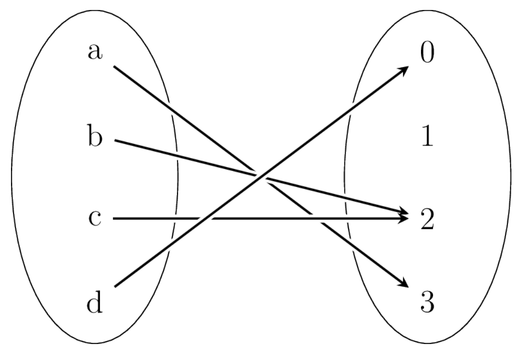
Definition 3.10 (Properties of Functions)**.** A function $f : A \longrightarrow B$ is said to be ***one-to-one*** $(1-1)$ if for all y in the range there is exactly one x in the domain such that $f(x) = y$. The function is ***onto*** if for all y in the codomain there is at least one x in the domain such that $f(x) = y$. If a function is both one-to-one and onto we say that it is a ***bijection***.

- One-to-One: $\forall x_0, x_1 \in A : f(x_0) = f(x_1) \Longrightarrow x_0 = x_1$
- Onto: $\forall y \in B\, \exists x \in A : f(x) = y$

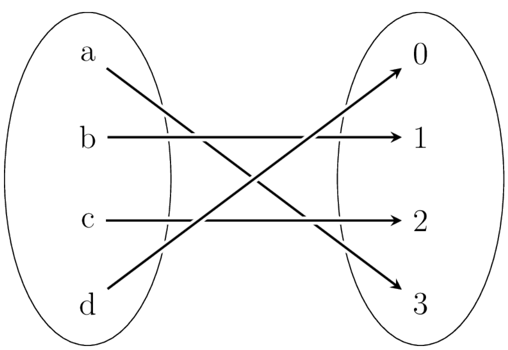
Practice Problem 3.33. For each function circle the correct description: 1-1, onto, both, or neither.



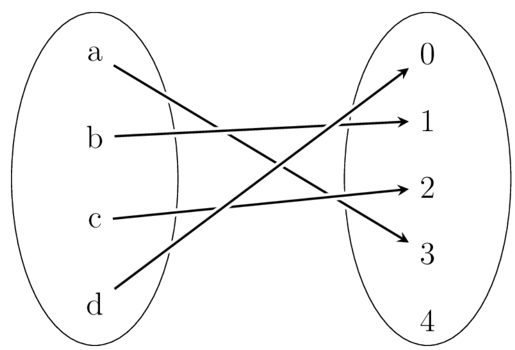
(a) 1-1, onto, both, or neither



(b) 1-1, onto, both, or neither



(c) 1-1, onto, both, or neither



(d) 1-1, onto, both, or neither

Figure 3.9: Function Properties Practice

Theorem 3.2. *A function between two finite sets of the same size, cardinality, is either both one-to-one and onto, or neither one-to-one nor onto.*

Practice Problem 3.34. Is the below function 1-1, onto, both, or neither? The answer depends on the *domain* and *codomain*. For each description, let S be the set of points on the spiral in figure 3.10.

1. *Domain:* $\mathbb{R}$, *Codomain* $\mathbb{R}$: $f(x) = y$, if $(x, y) \in S$

2. *Domain:* $\mathbb{R}$, *Codomain* $\mathscr{P}(\mathbb{R})$: $f(x) = \{y | (x, y) \in S\}$

3. *Domain:* $\mathbb{R}^+$, *Codomain* $\mathbb{R} \times \mathbb{R}$: $f(t) = \left\langle \frac{t}{2\pi} \cdot \cos(t), \frac{t}{2\pi} \cdot \sin(t) \right\rangle$

4. *Domain:* S, *Codomain* $\mathbb{R}^+$: for $s \in S$, $f(s) =$ length of the spiral up to s

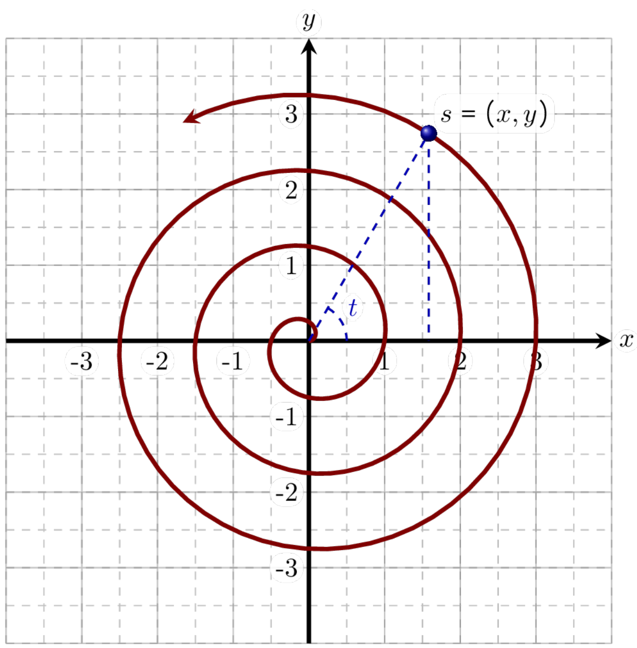


Figure 3.10: Points on a Spiral

Definition 3.11 (Inverses)**.** Given a relation R define the ***inverse*** of R to be the set

$$R^{-1} = \{(y, x) | (x, y) \in R\}.$$

If R and R^{-1} are both functions, then we say R is ***invertible***.

Practice Problem 3.35. For each function in practice block 3.33 on 77, write the function and its "inverse" as sets of ordered pairs R and R^{-1}. Which functions are invertible? What other property do all the invewrtible functions have?

Example 1:

Example 2:

Example 3:

Example 4:

Theorem 3.3 (Invertible Functions)**.** *A function will be invertible if and only if it is* ____

Definition 3.12 (Function Composition)**.** Given a function $f : A \to B$ and $g : B \to C$, the ***composition*** of f and g is a function from the domain of f to the codomain of g defined by $g \circ f(x) = g(f(x))$ and we read $g \circ f$ as "g compose f."

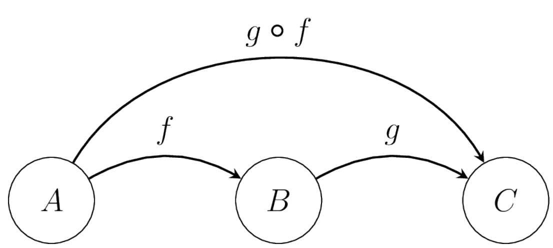
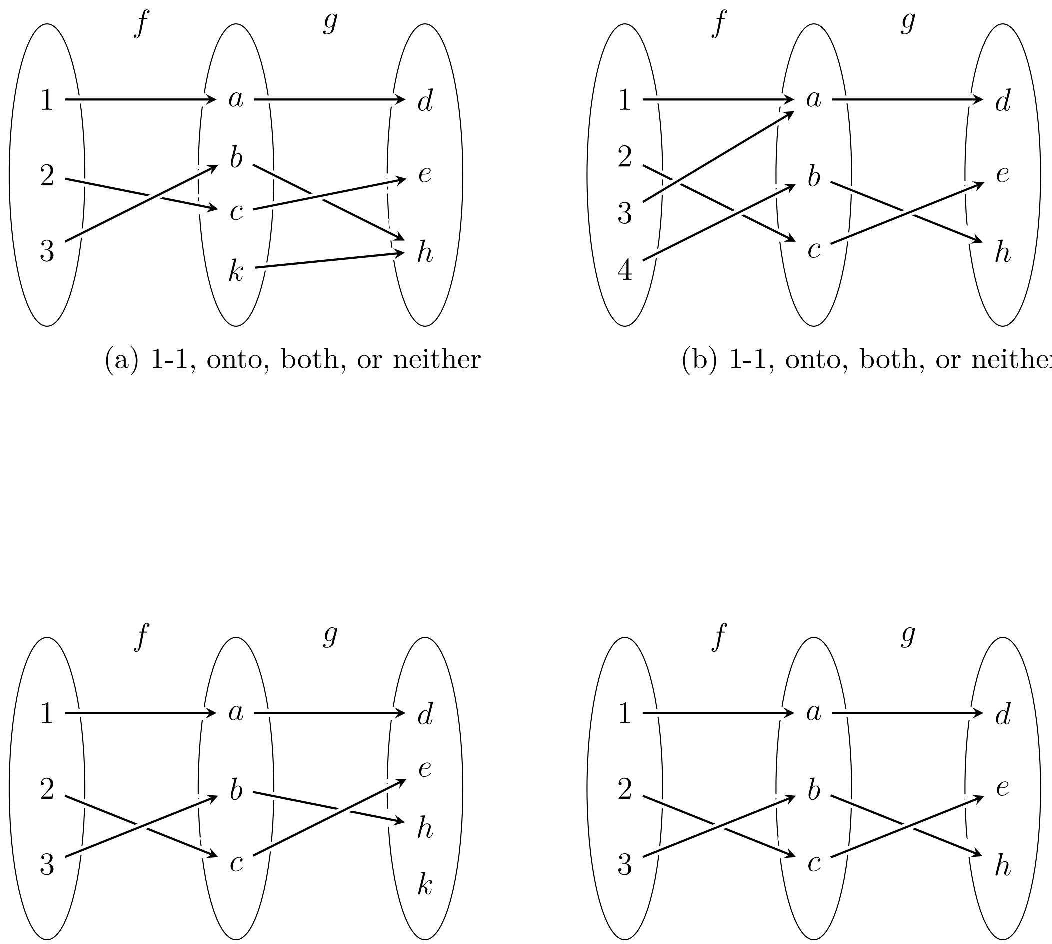


Figure 3.11: Schematic of Function Composition

Note, this is well defined as long as the range of f is a subset of the domain of g.

Practice Problem 3.36. For each image decide if $g \circ f(x)$ 1-1, onto, both, or neither? How does this compare to f and g individually?



(a) 1-1, onto, both, or neither

(b) 1-1, onto, both, or neither

(c) 1-1, onto, both, or neither

(d) 1-1, onto, both, or neither

Figure 3.12: Composed Functions

Theorem 3.4. *The composition of functions $g \circ f$ is one-to-one only if f is one-to-one and it is onto only if g is onto.*

Practice Problem 3.37. Using $f(x) = x^2$, $g(x) = \sqrt{x}$, and $h(x) = x + 1$, find each of the following:

1. $f \circ h(x) =$

2. $h \circ f(x) =$

3. $f \circ g(x) =$

4. $g \circ f(x) =$

Chapter 4

Combinatorics

4.1 Basic Counting

Definition 4.1 (Independent Events)**.** When one action does not influence the outcome of another action, such as rolling two different dice, we say they are ***independent events***.

Definition 4.2 (Disjoint Events)**.** When two outcomes can not happen at the same time, such as a number being both even and odd, we say that they are ***disjoint events***.

Definition 4.3 (Multiplication Principle)**.** If we have k ***independent*** choices to make and n_i options for each choice, then the total number of possible choices is

$$\prod_{i=1}^{k} n_i = n_1 \times n_2 \times \cdots \times n_k.$$

It is helpful here to think in terms of events that can be described by saying "and then."

Definition 4.4 (Addition Principle)**.** If we have k ***disjoint*** options and each option can be chosen in n_i different ways, then the total number of options is

$$\sum_{i=1}^{k} n_i = n_1 + n_2 + \cdots + n_k.$$

It is helpful here to think in terms of events that can be described by saying "or."

Exposition 4.1. As you enter Sandwich-Way you see the following directions for making a sandwich:

1. Choose one of 12 bread options.
2. Choose one of 16 "Protein" options.
3. Choose up to 1 of 7 cheese options.
4. Choose up to 3 of 10 vegetable options.
5. Choose up to 3 of 13 condiments.

You then find yourself wondering how many different sandwiches you can make.
What are the number of sandwiches if we use the ***minimum*** number of ingredients?

$$\begin{aligned}&Bread \times Protein \times Cheese \times Vegetables \times Condiments\\ &= 12 \times 16 \times 1 \times 1 \times 1\\ &= 192\ Sandwiches\end{aligned}$$

What are the number of sandwiches if we use the ***maximum*** number of ingredients (without repetition)?[a]

$$\begin{aligned}&Bread \times Protein \times Cheese \times Vegetables \times Condiments\\ &= 12 \times 16 \times 7 \times (10 \cdot 9 \cdot 8) \times (13 \cdot 12 \cdot 11)\\ &= 1,660,538,880\ Sandwiches\end{aligned}$$

How many sandwiches are there if we either minimize or maximize the number of ingredients?

$$192\ Sandwiches + 1,660,538,880\ Sandwiches = 1,660,539,072\ Sandwiches$$

[a] For simplicity we are assuming we care what order cheeses and condiments are added.

Practice Problem 4.1. Returning for Another Sandwich How many different sandwiches are there with no vegetables?

What if you really like dry sandwiches with no condiments?

Exposition 4.2. We can map out and compile a sequence of options using a ***decision tree***. For example here is how we can create a sandwich with one or no choice of protein and one or no choice of cheese.

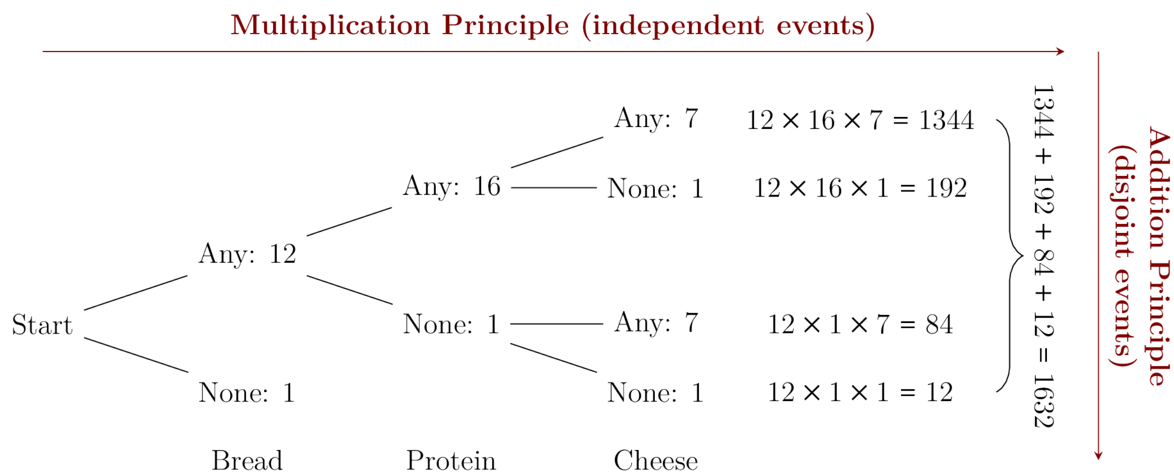


Figure 4.1: Sandwich Decision Tree

Note that for small examples/simple situations we can actually draw a decision tree, for anything particularly large it is a useful organizational tool but not practical to draw.

Practice Problem 4.2. License Plates Suppose that all the license plates in a state must consist of a digit from 1 to 9, followed by four different letters, followed by a different digit from 1-9. How many different license plates could we create? What if you could repeat letters or numbers? What if you could also choose two of four colors?

Practice Problem 4.3. Preparing for a party you go to the store to get soda, chips, and cookies. For cookies and soda there are regular and sugar free varieties, and for chips there is regular or low sodium.

	Soda		Chips		Cookies	
	Regular	Sugar Free	Regular	Low Salt	Regular	Sugar Free
Varieties	10	7	15	12	14	8

How many buying options do you have if you just get one type of each product? (Use a decision tree.)

Definition 4.5 (Inclusion-Exclusion Principle)**.** For any set X let $|X|$ be the ***cardinality***, or size, of X. Given sets A, B, and C,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

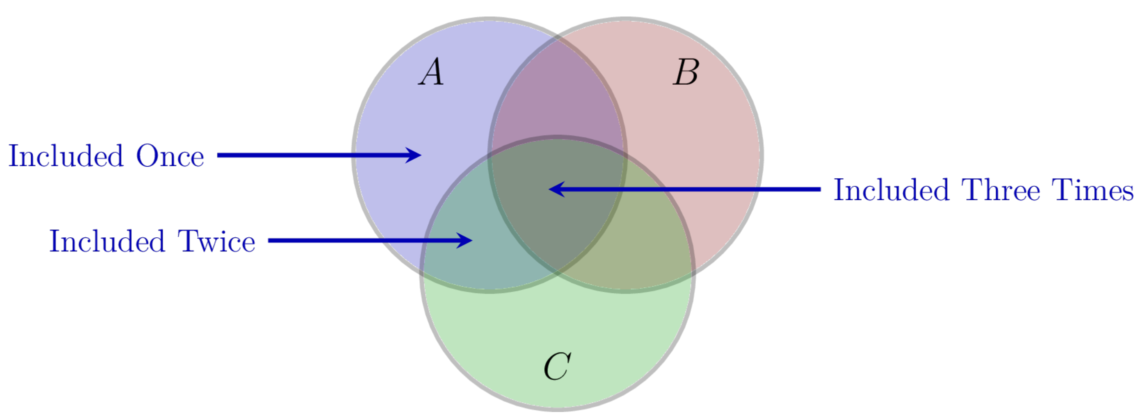


Figure 4.2: Inclusion-Exclusion Principle for Three Sets

Exposition 4.3. Let $\overline{C}$ be the set of all sandwiches with no cheese and $\overline{V}$ be the set of all sandwiches with no vegetables. How many sandwiches have no cheese or no vegetables?

- $\overline{C}$ is sandwiches with no cheese.
- $\overline{V}$ is sandwiches with no vegetables.
- $\overline{C} \cup \overline{V}$ is sandwiches with no cheese ***OR*** no vegetables.
- $\overline{C} \cap \overline{V}$ is sandwiches with no cheese ***AND*** no vegetables.

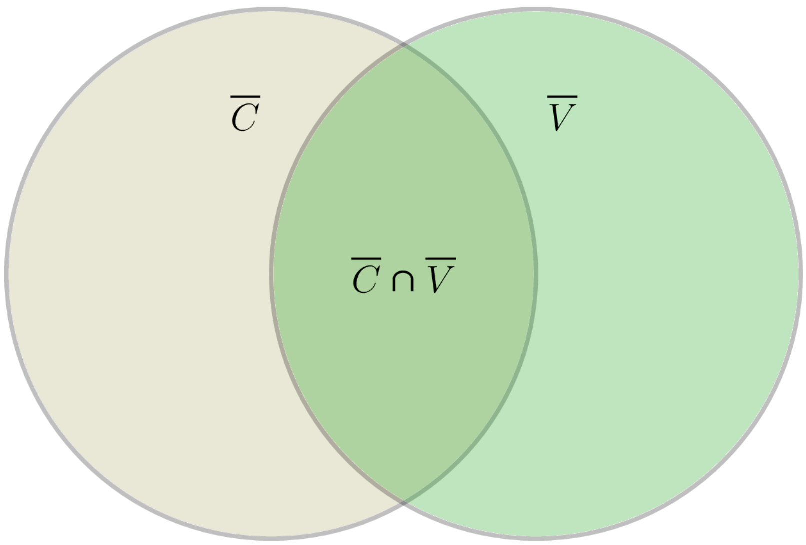


Figure 4.3: No Cheese and/or No Vegetables

Using the multiplication and addition principles we can get:

$$\begin{aligned} |\overline{C}| &= 12,773,376 \\ |\overline{V}| &= 580,608 \\ |\overline{C} \cap \overline{V}| &= 72,576 \end{aligned}$$

Therefore, the number of sandwiches with no cheese or no vegetables is:

$$\begin{aligned} |\overline{C} \cup \overline{V}| &= |\overline{C}| + |\overline{V}| - |\overline{C} \cap \overline{V}| \\ &= 12773376 + 580608 - 72576 \\ &= 13,281,408 \end{aligned}$$

Practice Problem 4.4. Your license plates consisting of a digit from 1 to 9, followed by four different letters, followed by a different digit from 1-9, have been getting complaints because people confuse the B with the 8. To solve this problem you need to exclude one of both of these characters, in how many ways can you do that?

Practice Problem 4.5. Suppose you need to get sugar free soda or sugar free cookies, in how many ways could you do that?

	Soda		Chips		Cookies	
	Regular	Sugar Free	Regular	Low Salt	Regular	Sugar Free
Varieties	10	7	15	12	14	8

4.2 Permutations and Combinations

Definition 4.6 (Factorials)**.** The number of ways to arrange n objects is called ***n-factorial*** and is calculated like so:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

Definition 4.7 (Permutations)**.** ***Permutations*** represent the number of ways to select k object from n options when ***order does matter*** and are calculated like so:

$${}_nP_k = \frac{n!}{(n-k)!} = n \cdot (n-1) \cdot (n-2) \cdots (n-k+1).$$

We can derive this like so:

Definition 4.8 (Combinations)**.** ***Combinations*** represent the number of ways to select k object from n options when ***order does not matter*** and are calculated like so:

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdots 2 \cdot 1}.$$

This is sometimes also called a ***binomial coefficients*** because we either choose an option or not. We can derive this like so:

Practice Problem 4.6. Basic Calculations Translate each of the following into a mathematical expression and then complete the calculation.

1. Pick 5 out of 7 distinct objects when order matters:

2. Arrange 6 distinct objects:

3. Pick 3 out of 5 distinct objects when order doesn't matter:

4. Pick 10 out of 15 distinct objects when order matters:

5. Arrange 17 distinct objects:

6. Pick 7 out of 10 distinct objects when order doesn't matter:

Exposition 4.4. Below is a standard deck of cards with four suits and thirteen ranks per suit.

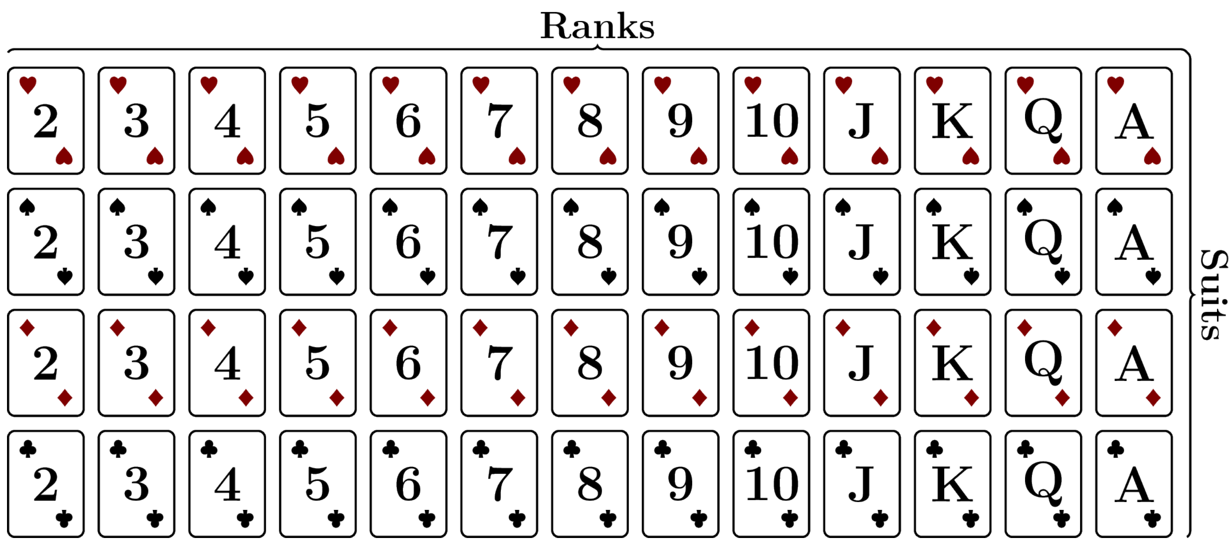


Figure 4.4: Standard Deck of Cards

If we deal 5 cards from a standard deck, there are 3744 full houses like:

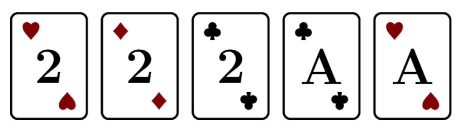


Figure 4.5: Full House

We can use combinations to see why. First ***make a plan*** and second calculate.

1. Pick 1 of 13 ranks for the 3 of a kind
2. Pick 3 of 4 from that rank
3. Pick 1 of 12 ranks for the pair
4. Pick 2 of 4 from that rank

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6$$
$$= 3744 \checkmark$$

Practice Problem 4.7. If we deal 5 cards from a standard deck, there are 624 four of a kinds like:

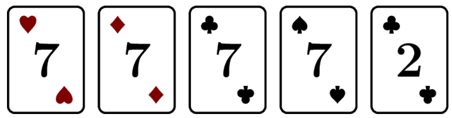


Figure 4.6: Four of a Kind

We can use combinations to see why. First ***make a plan*** and second calculate.

1.

2.

3.

Exposition 4.5 (Three of a Kind Done the Wrong)**.** Now we want to know how many ways we can make a three of a kind, looking it up the answer is 54912, but why?

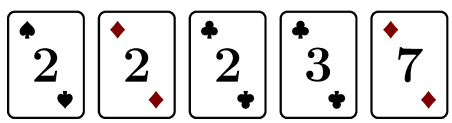


Figure 4.7: Three of a Kind

We can use combinations to see why. First ***make a plan*** and second calculate.

1. Pick 1 of 13 ranks for the 3 of a kind
2. Pick 3 of 4 from that rank
3. Pick 2 of 49 other cards

$$\binom{13}{1}\binom{4}{3}\binom{49}{2} = 13 \cdot 4 \cdot \frac{49 \cdot 48}{2}$$
$$= 61152 \text{ ✗}$$

So, what did we do wrong?

Practice Problem 4.8 (Three of a Kind Done the Right)**.** Let's look at figure 4.7 again, and lets make a better plan to show there are 54912 possible three of a kinds. First ***make a plan*** and second calculate.

1.

2.

3.

4.

5.

Practice Problem 4.9. There are 123552 different ways to be dealt two pair.

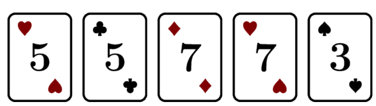


Figure 4.8: Two Pair

We can use combinations to see why. First ***make a plan*** and second calculate.

1.

2.

3.

4.

5.

Practice Problem 4.10. There are 2533180 different ways to be dealt a hand with at least one red card.

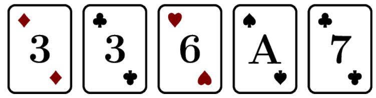


Figure 4.9: One or More Red Cards

We can use combinations to see why. First ***make a plan*** and second calculate. (This one is tricky.)

1.
2.
3.

Theorem 4.1 (Pascal's Formula)**.** *Given positive integers* $0 < k < n$*,*

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}. \tag{4.1}$$

Exposition 4.6 (Pascal's Triangle)**.** Using Pascal's Formula (equation 4.1) we can create Pascal's Arithmetic Triangle:

$$\begin{array}{l ccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 1 & & 1 & & & & \\
n=2 & & & & 1 & & 2 & & 1 & & & \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & \\
n=4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n=5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Figure 4.10: Pascal's Triangle

which lists the ***binomial coefficients*** for each value of n.

Theorem 4.2 (Binomial Theorem)**.** *Given a positive integer n and variables a and b:*

$$(a+b)^n = (a+b)(a+b)\cdots(a+b) \tag{4.2}$$

$$= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n \tag{4.3}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{4.4}$$

Practice Problem 4.11 (Sum of Pascal Rows)**.** Show that the sum of the coefficients in row n of Pascal's Triangle is 2^n. (Hint: $(1+1) = 2$)

Practice Problem 4.12. In the product

$$(x+y+z)^5 = (x+y+z)(x+y+z)(x+y+z)(x+y+z)(x+y+z)$$

the coefficient for the term x^2yz^2 is 30. We can use combinations to see why. First ***make a plan*** and second calculate.

1.

2.

3.

Practice Problem 4.13. There are 34650 distinct ways to rearrange the 11 letters in the word Mississippi. We can use combinations to see why. First ***make a plan*** and second calculate.

1.

2.

3.

4.

Definition 4.9 (Multinomial Coefficients)**.** There are two ways to think of a ***multinomial coefficient***. Given k different types of objects with n_i of each type, a ***multinomial coefficient*** where $\sum n_i = n$, represents the number of ways to arrange all the objects.

OR

Given n copies of a set each with the same k distinct objects, a ***multinomial coefficient*** where $\sum n_i = n$, represents the number of ways to select n_i copies of object type i.

Either way it is calculated with:

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

Theorem 4.3 (Combinations with Repetition)**.** *If we have to make r choices from n options, order does not matter, and we are allowed to select the same option more than once, then this is a* **combination with repetition**. *the number of ways we can do this is:*

$$\binom{r+n-1}{n-1} = \binom{r+n-1}{r}.$$

Exposition 4.7. Suppose you program a machine that scoops ice cream. It can only follow two commands, scoop to take a scoop of whatever ice cream it is above and move to move to the next flavor. Assuming you get three scoops of ice cream and there are five flavors, the commands for the robot could look like:

```
def serving([m,m,s,m,s,m,s]):
    move arm
    move arm
    scoop ice cream
    move arm
    scoop ice cream
    move arm
    scoop ice cream
    return ice cream
```

- Why are there only four move commands if there are five flavors?
- If $r = 3$ is the number of scoops and $n = 5$ is the number of flavors, why does it make sense to use $r + n - 1 = 7$ like in theorem 4.3?

One way of looking at this is that

$$\binom{r+n-1}{r} = \binom{7}{3} = 35$$

is the number of ways we can place the three scoop commands in the seven lines of code, and then put the move commands on the remaining empty lines:

```
def serving([s,s,_,_,s,_,_]):
    scoop ice cream
    scoop ice cream
    _______________
    _______________
    scoop ice cream
    _______________
    _______________
    return ice cream
```

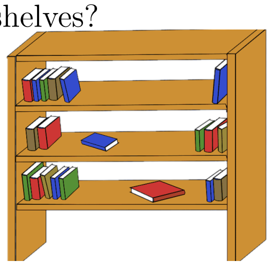
Exposition 4.8. Another way to think of theorem 4.3 is to ask in how many ways could you divide a row of identical objects into individual groups (we allow some groups to be empty). For example we can split a group of 10 zombies into 4 smaller groups like so

where none of the groups are empty; $1+2+3+4=10$. Which we can also think of as 1 zombie before the first wall, 3 total before the second, 6 total before the third, and 10 altogether. Or, like so

where the first and last groups are empty; $0+4+6+0=10$. Which again we can think of as 0 zombies before the first wall, 4 total before the second, and 10 total before the third, and also 10 altogether.
Either way we need to arrange our 10 zombies and 3 walls in a row to get our 4 groupings. Thus the total number of possible choices is

$$\binom{r+n-1}{r} = \binom{10+4-1}{10} = \binom{13}{10} = 286.$$

Practice Problem 4.14. Book Shelves In how many ways can you place 23 books onto 3 shelves?



Practice Problem 4.15. Buying Soda You go to a store to buy soda for a party, if there are 5 types of soda and you need to get 48 sodas total, in how many ways can you pick out the sodas?

Practice Problem 4.16. You realize that you should really make sure to have at least 6 cans of each type of soda. With this in mind, how many ways can you now pick out sodas?

Practice Problem 4.17. You will also need chips. There are 3 types of chips and you want 7 bags. However, there are only 3 bags of one of the types. With this limitation, in how many ways can you select chips?

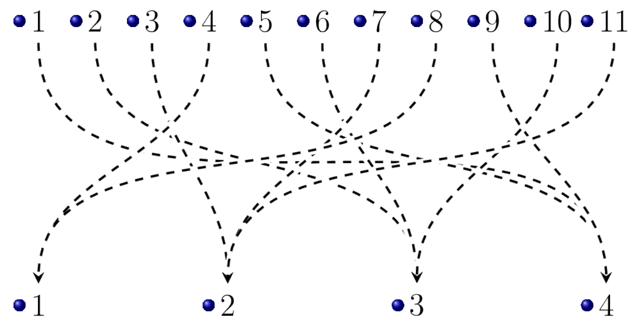
Practice Problem 4.18. Look at the code below and note that $0 \leq k \leq j \leq i \leq 9$, why does this code print "Discrete Rules!!!" 220 times? (Hint: There are ten numbers to move through and three variables to assign.)

```
count=1
for i in range(10):
    for j in range(i+1):
        for k in range(j+1):
            print("(%d) Discrete Rules!!!"%count)
            count+=1
```

4.3 Pigeon Hole Principle

Theorem 4.4 (The Pigeonhole Principle)**.** *Given finite sets X and Y with $|X| > |Y|$, a function from $f : X \longrightarrow Y$ can not be one-to-one. Moreover, if $k < |X|/|Y|$, then there exists some $y \in Y$ such that $|f^{-1}(y)| \geq k + 1$.*

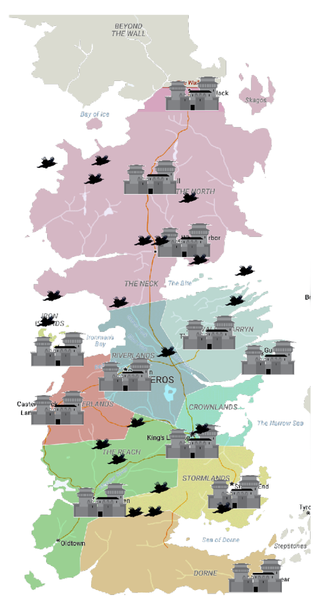
For example given $X = \{1, \ldots, 11\}$ and $Y = \{1, \ldots, 4\}$, $|X| = 11 > |Y| = 4$ and $11/4 > 2$; for any function there will be some y such that $|f^{-1}(y)| \geq 2 + 1 = 3$.



$$|f^{-1}(4)| = |\{1, 5, 9\}| = 3$$

Figure 4.11: Pigeonhole Demonstration

Practice Problem 4.19. If the Citadel in Old Town sends out 21 ravens to 12 castles around Westeros announcing that "Winter iſ Coming!!!" why is there at least one castle that receives 2 ravens?



(Ravens and Castles not to scale.)

Figure 4.12: Westros

Practice Problem 4.20. Given the set of integers $X = \{1, ..., 9\}$ any set of six numbers selected at random will have at least two which sum to 10.

Practice Problem 4.21. If an integer n is divided by 7, and n is not a multiple of 7, then the number of repeating digits is at most 6. (Try $37 \div 7$.)

Practice Problem 4.22. Given 10 lunch tables and 57 students, show that there must be at least one table with six students.

Theorem 4.5. *Given finite sets X and Y, there exists a one-to-one and onto function from X to Y if and only if $|X| = |Y|$.*

Chapter 5

Graphs and Trees

5.1 Basic Definitions for Graphs

Definition 5.1 (Graphs)**.** A ***graph*** consists of two finite sets: a nonempty ***set of vertices*** and a ***set of edges***, where each edge is associated with a set consisting of either one or two vertices called its endpoints. The correspondence from edges to endpoints is called the edge-endpoint function.

Exposition 5.1. Here is some basic terminology:

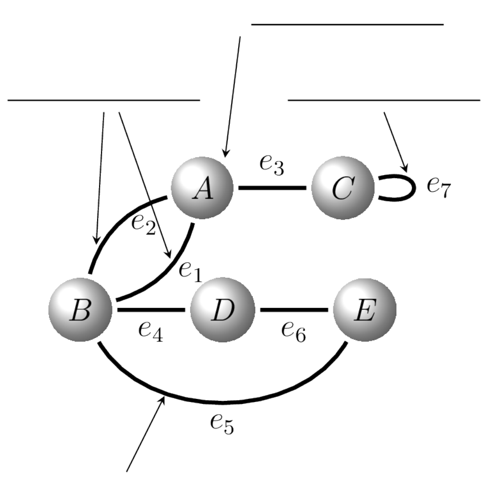


Figure 5.1: Sample Graph

Vertex Set $= \{ \qquad \qquad \qquad \}$

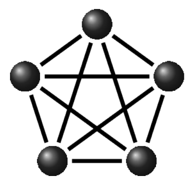
Edge Set $= \{ \qquad \qquad \qquad \qquad \}$

Degrees:

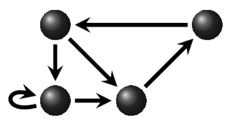
- $deg(A)$ =
- $deg(B)$ =
- $deg(C)$ =
- $deg(D)$ =
- $deg(E)$ =
- $Total\ Degree$ =

Exposition 5.2. Below is a sampling of common types of graphs.

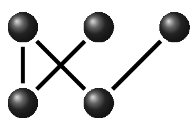
Complete Graph:



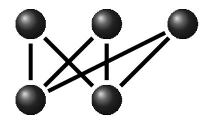
Directed Graph (Digraph):



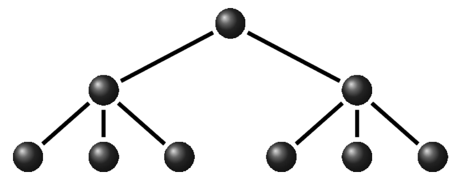
Bipartite Graph:



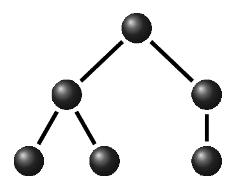
Complete Bipartite Graph:



Tree:



Binary Tree:



Theorem 5.1 (Handshake Theorem)**.** *If G is any graph, then the sum of the degrees of all the vertices of G, the* **total degree***, equals twice the number of edges in G. Specifically, if $V = \{v_1, v_2, \ldots, v_n\}$ is the vertex set of G and $E = \{e_1, e_2, \ldots, e_k\}$ is the edge set then*

$$\textit{total degree of } G = \sum_{v_i \in V} deg(v_i) = 2|E| = 2k.$$

Corollary (Immediate Consequences)**.** *Let G be a graph with vertex set V and edge set E, then*

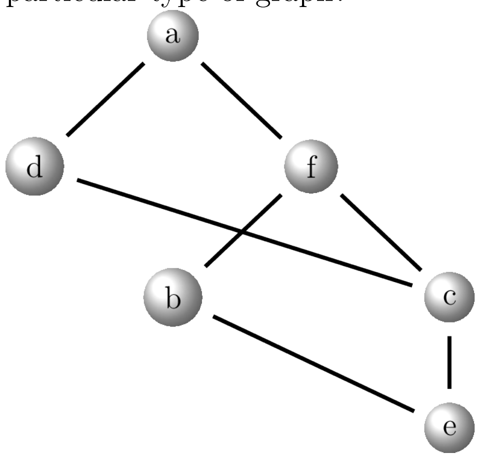
1. *the total degree of G is even, and*
2. *the number of vertices of odd degree is even.*

Theorem 5.2. *If G is a* **complete graph** *with n vertices, then G has ${}_nC_k$ edges.*

Theorem 5.3. *If G is a* **complete bipartite graph** *with n vertices in one set and m in the other, then G has $n \times m$ edges.*

Definition 5.2 (Simple Graph)**.** A ***simple graph*** is a graph with no loops and no parallel edges. Note that, in some settings simple graphs are just called graphs, and graphs with loops or parallel edges are called ***multigraphs***.

Practice Problem 5.1. Find the vertex set, edge set, and degree for each vertex. Is this a particular type of graph?

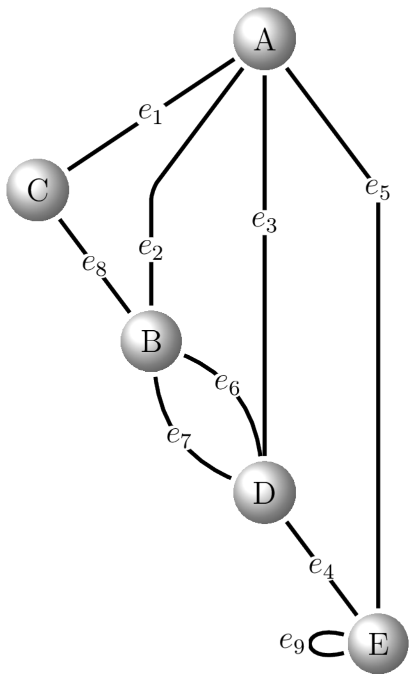


Practice Problem 5.2. Construct a graph from the given information.

Edges	End Points
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_2\}$
e_3	$\{v_1, v_3\}$
e_4	$\{v_2, v_3\}$
e_5	$\{v_3, v_3\}$

5.2 Walks in Graphs

Definition 5.3 (Walks and their Kin)**.** In graph theory we frequently study movement in a graph, for that we need the following definitions.



- ***Walk:*** Finite sequence of vertices and edges.
- ***Trail:*** A walk with no repeated edges.
- ***Euler Trail:*** A trail that traverses every edge.
- ***Circuit:*** A non-trivial closed trail.
- ***Euler Circuit:*** Circuit that traverses every edge.
- ***Simple Circuit:*** Circuit that only repeats the first/last vertex.
- ***Hamilton Circuit:*** A simple circuit that includes every vertex.

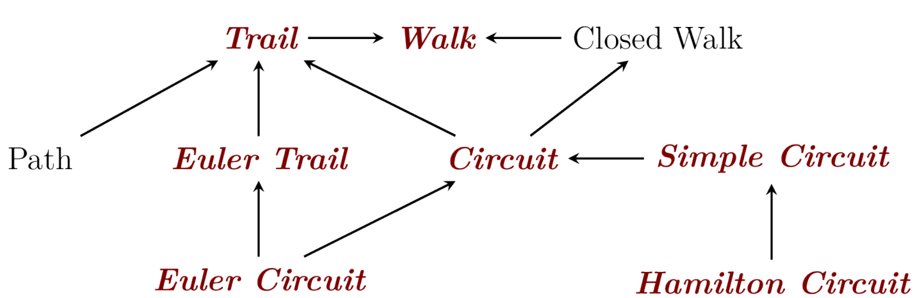


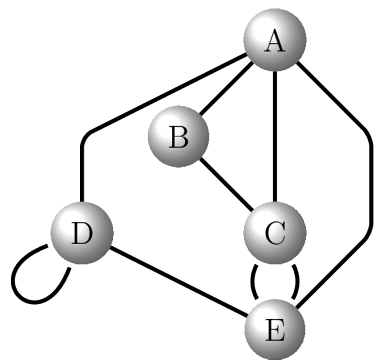
Figure 5.2: Hierarchy of Walks in a Graph

Practice Problem 5.3. Try and determine if the graph below has an:

- ***Euler Trail:*** A trail that traverses every edge.

- ***Euler Circuit:*** Circuit that traverses every edge.

- ***Hamilton Circuit:*** A simple circuit that includes every vertex.

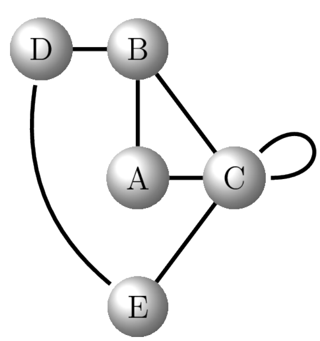


Practice Problem 5.4. Try and determine if the graph below has an:

- ***Euler Trail:*** A trail that traverses every edge.

- ***Euler Circuit:*** Circuit that traverses every edge.

- ***Hamilton Circuit:*** A simple circuit that includes every vertex.



Theorem 5.4. *A connected graph G will have an* **Euler circuit** *if all of its vertices are of even degree and only an* **Euler trail** *if it has exactly two vertices of odd degree.*

Theorem 5.5 (Subgraphs and Hamilton Circuits)**.** *If a connected graph G has a* **Hamilton circuit***, then G has a* **subgraph** *with the following properties:*

1. *H contains every vertex of G.*
2. *H is connected.*
3. *H has an equal number of vertices and edges.*
4. *Every vertex in H has degree 2.*

Practice Problem 5.5. Look at the graph in figure 5.3 and try and find Euler or Hamilton circuits. Do this both with and without the dashed edge.

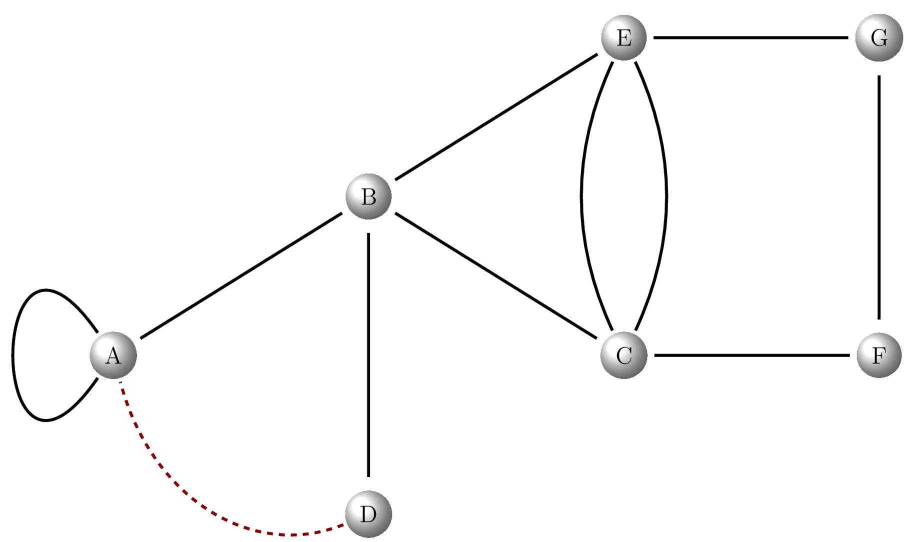
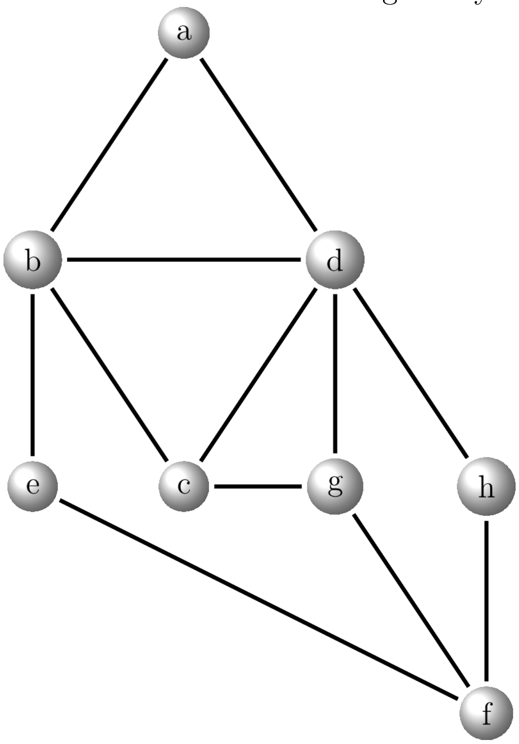


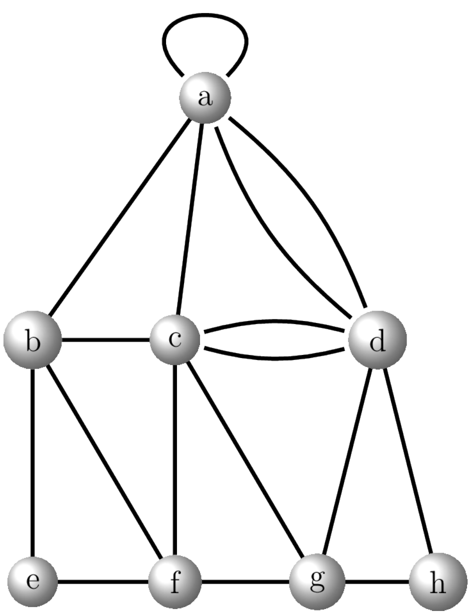
Figure 5.3: Euler and Hamilton Circuits

- Hamilton Circuit w/o Dashed Line: ______
- Euler Circuit w/o Dashed Line: ______
- Euler Trail w/o Dashed Line: ______
- Hamilton Circuit w/ Dashed Line: ______
- Euler Circuit w/ Dashed Line: ______
- Euler Trail w/ Dashed Line: ______

Practice Problem 5.6. Again try to find any Euler or Hamilton Circuits.



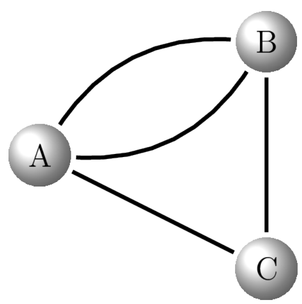
Practice Problem 5.7. Again try to find any Euler or Hamilton Circuits.



5.3 Adjacency Matrices

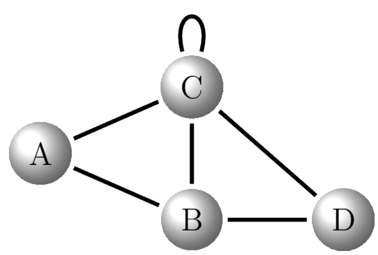
Definition 5.4 (Adjacency Matrix)**.** An ***adjacency matrix*** for a graph is a two dimensional array A in which a_{ij} is the number of edges that we may traverse from vertex i to vertex j.

Practice Problem 5.8. Fill in the entries of the adjacency matrix for the given graph.



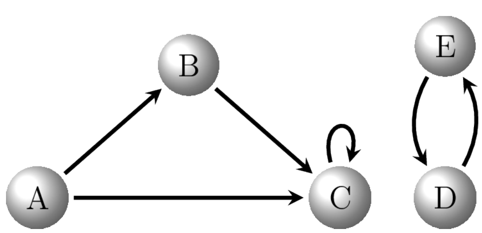
$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} \end{array}$$

Practice Problem 5.9. Fill in the entries of the adjacency matrix for the given graph.



$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

Practice Problem 5.10. Fill in the entries of the adjacency matrix for the given graph.



$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} \quad & \quad & \quad & \quad & \quad \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix} \end{array}$$

Exposition 5.3. Matrix Arithmetic We add and subtract matrices term by term:

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \overset{+}{+} \begin{pmatrix} 1 & 7 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1+1 & 2+7 \\ 0+0 & 3+2 \end{pmatrix}$$

We multiply matrices by combining rows with columns:

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 7 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1+2\cdot 0 & 1\cdot 0+2\cdot -1 & 1\cdot 7+2\cdot 2 \\ 0\cdot 1+3\cdot 0 & 0\cdot 0+3\cdot -1 & 0\cdot 7+3\cdot 2 \end{pmatrix}$$

Practice Problem 5.11. Find the sum, difference, and product of the following matrices where possible.

$$A = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix},\ B = \begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix},\ C = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix},\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1. $A + B =$

2. $B - A =$

3. $AB =$

4. $AI =$

5. $AC =$

6. $CA =$

7. $C + B =$

Practice Problem 5.12. Multiplying an adjacency matrix by itself allows us to count the number of walks from one vertex to another of a given length. Sketch a graph corresponding to the given adjacency matrix.

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

1. How does A on its own give walks of length 1?

2. Find A^2 and compare it to walks of length 2.

3. Find A^3 and compare it to walks of length 3.

Practice Problem 5.13. Sketch a graph corresponding to the given adjacency matrix.

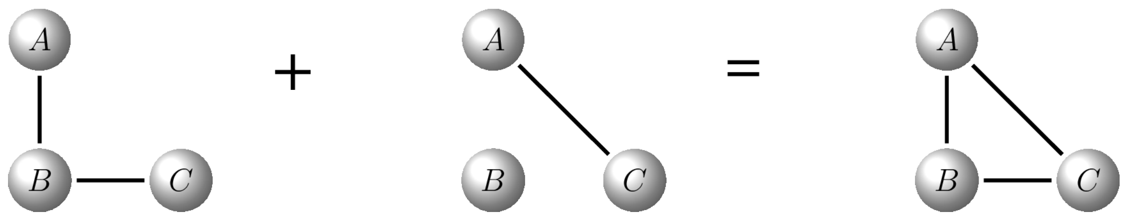
$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

1. Walks of length 2:

2. Walks of length 3:

3. Walks of length 4:

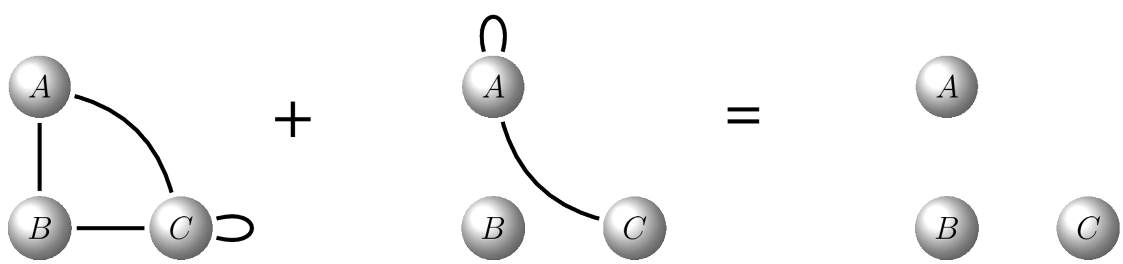
Exposition 5.4. By matching up and "gluing" together corresponding vertices we can ***concatenate*** or add graphs. The adjacency matrix for the new graph is equal to the sum of the matrices for the graphs that were concatenated.



$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Figure 5.4: Concatenating Graphs and Their Adjacency Matrices

Practice Problem 5.14. Write adjacency matrices for each of the initial graphs and then add them to get the matrix for the concatenation of the graphs.



Exposition 5.5. We can use graphs and matrices to model networks and make predictions about movement through the networks.

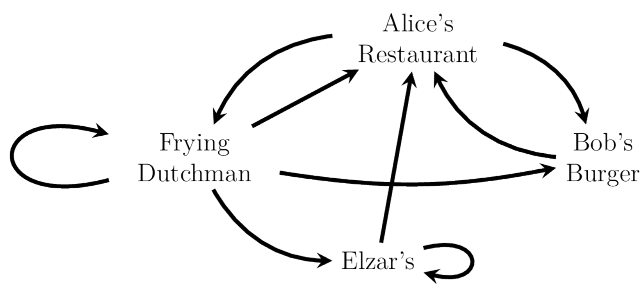


Figure 5.5: A Very Tiny Network

$$L = \begin{array}{c c} & \begin{array}{cccc} A & B & E & F \end{array} \\ \begin{array}{c} A \\ B \\ E \\ F \end{array} & \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix} \end{array} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}$$

```
import numpy as np
from numpy import linalg as LA

# adj. matrix of links/probabilities
L = np.array([[0,0.5,0,0.5],[1,0,0,0],[0.5,0,0.5,0],[0.25,0.25,0.25,0.25]])

P=np.array([1,0,0,0]) # Initial position vector
distance = 1 # Distance from one step to the next
i = 0
while distance>0.001 and i<100:
# Multiply by L to represent making a random click
    temp_P=P@L
# Compare the position vectors
    distance = LA.norm(temp_P-P)
# Update
    P=temp_P
    i+=1
print(P)
```

$$P = \begin{bmatrix} 0.37519702 & 0.24988937 & 0.12502424 & 0.24988937 \end{bmatrix}$$

Starting at the website for Alice's Restaurant and then randomly clicking around there is a 37% chance we get back to Alice's, 25% chance of ending up at Bob's or the Dutchman, and only 13% chance of ending up at Elzar's website. Thus, we rank Alice's site highest and Elzar's lowest.

5.4 Basic Properties of Trees

Definition 5.5 (Basic Tree)**.** A ***tree*** is a *connected* graph with no circuits.

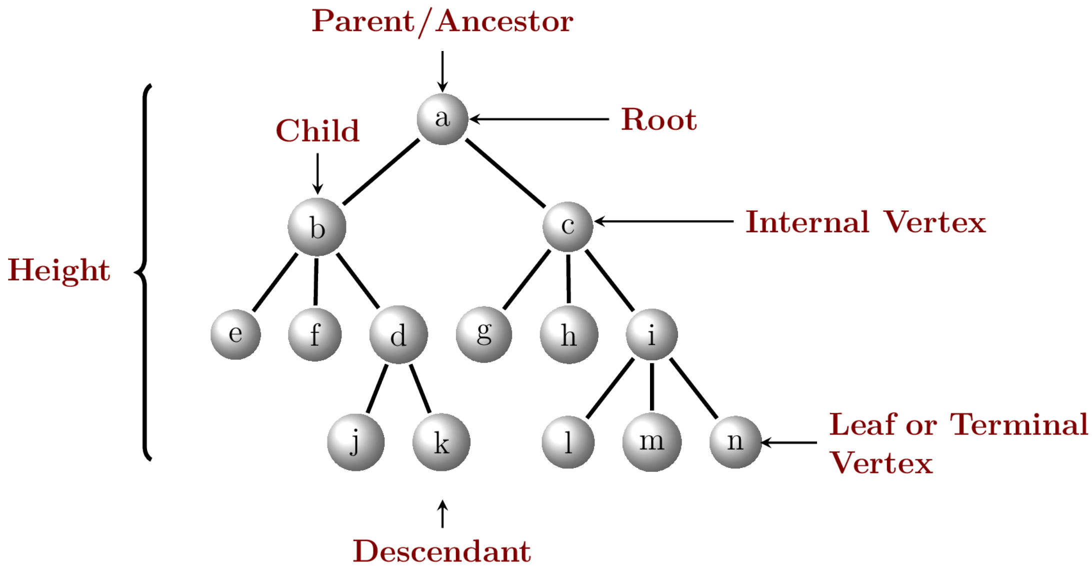


Figure 5.6: Basic Tree Terminology

Theorem 5.6. *Every tree has a vertex of degree 1.*

Theorem 5.7. *A tree with n vertices has $n-1$ edges.*

Definition 5.6 (Binary Trees)**.**

- A ***binary tree*** is a tree in which each parent has ______________________________.
- A ***full binary tree*** is a tree in which each parent has ___________________________.

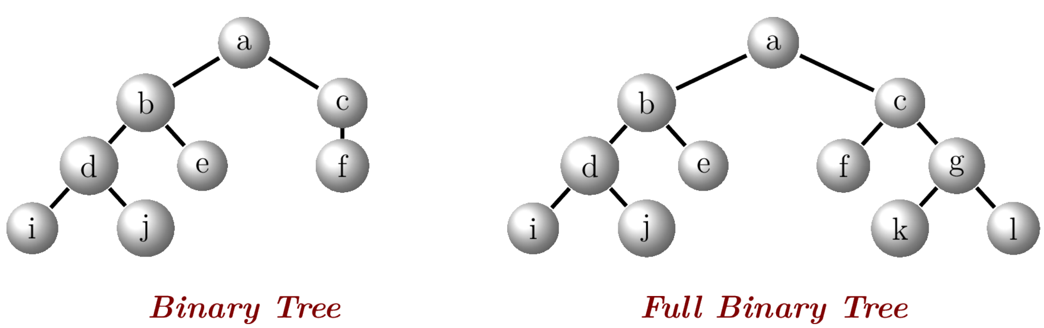


Figure 5.7: Types of Binary Trees

Theorem 5.8. *Given a full binary tree with k internal vertices, there are $n = 2k + 1$ vertices all together and so $k + 1$ leaves.*

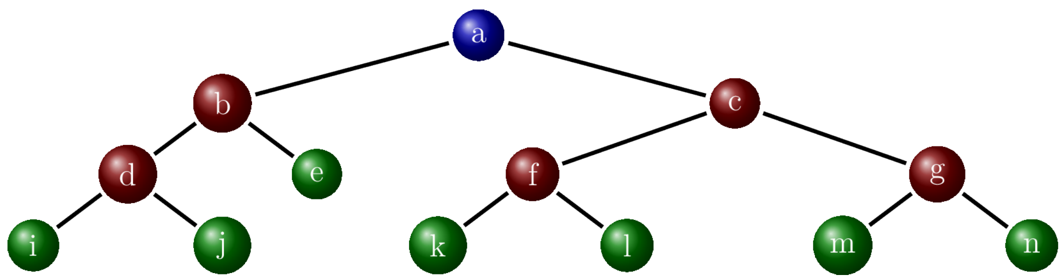


Figure 5.8: Vertices in a Full Binary Tree

Theorem 5.9. *Given a binary tree of height h with t leaves, $t \leq 2^h$ or equivalently $\log_2(t) \leq h$. Equality occurs in full binary trees in which all leaves are the same height.*

Practice Problem 5.15. Label each of the following as true for false and include reasons:

1. The total degree of a tree is equal to $2n - 2$ for some n.

2. There exists a connected graph with 7 vertices and 7 edges.

3. There exists a tree with 7 vertices and 7 edges.

4. There exists a tree with 7 vertices and 6 edges.

5. A graph with 7 vertices and 6 edges is a tree.

6. A connected graph with 10 vertices and total degree 18 is a tree.

Appendix A

Logic Reference Table

TAUTOLOGIES: [1]

Let p, q, r, and s be arbitrary statements, let t be a tautology, and let c be a contradiction, then:

	Tautology	Name	Abbreviation
0.	$p \to q \equiv \sim (p \wedge \sim q)$	Conditional Law	(Def. $\to$)
1.	$p \Rightarrow p \vee q$	Law of Addition	(Add.)
2.	(a) $p \wedge q \Rightarrow p$	Laws of Simplification	(Simp.)
	(b) $p \wedge q \Rightarrow q$		
3.	$(p \vee q) \wedge \sim p \Rightarrow q$	Disjunctive Syllogism	(D.S.)
4.	$\sim (\sim p) \equiv p$	Law of Double Negation	(D.N.)
5.	(a) $p \wedge q \equiv q \wedge p$	Commutative Laws	(Com.)
	(b) $p \vee q \equiv q \vee p$		
6.	(a) $p \wedge p \equiv p$	Laws of Idempotency	(Idemp.)
	(b) $p \vee p \equiv p$		
7.	$(p \to q) \equiv (\sim q \to \sim p)$	Contrapositive Law	(Contrap.)
8.	(a) $\sim (p \wedge q) \equiv \sim p \vee \sim q$	DeMorgan's Laws	(DeM.)
	(b) $\sim (p \vee q) \equiv \sim p \wedge \sim q$		
9.	(a) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative Laws	(Assoc.)
	(b) $(p \vee q) \vee r \equiv p \vee (q \vee r)$		
10.	(a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws	(Dist.)
	(b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$		
11.	$(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$	Transitive Law	(Trans.)

[1] Originally created by Vasily C. Cateforis for Set Theory and Logic at SUNY Postdam based on the text *Set Theory with Applications* by Lin & Lin

	Tautology	Name	Abbreviation
12.	(a) $(p \to q) \wedge (r \to s) \Rightarrow (p \vee r \to q \vee s)$	Constructive Dilemmas	(C.D.)
	(b) $(p \to q) \wedge (r \to s) \Rightarrow (p \wedge r \to q \wedge s)$		
13.	(a) $(p \to q) \wedge (r \to s) \Rightarrow (\sim q \vee \sim s \to \sim p \vee \sim r)$	Destructive Dilemmas	(D.D.)
	(b) $(p \to q) \wedge (r \to s) \Rightarrow (\sim q \wedge \sim s \to \sim p \wedge \sim r)$		
14.	(a) $(p \to q) \wedge p \Rightarrow q$	Modus Ponens	(M.P.)
	(b) $(p \to q) \wedge \sim q \Rightarrow \sim p$	Modus Tollens	(M.T.)
	(c) $(p \to q) \Leftrightarrow (p \wedge \sim q \to c)$	Reductio ad absurdum	(R.A.)
15.	(a) $(p \wedge t) \Leftrightarrow p$	Identity Laws	(I.L.)
	(b) $(p \vee c) \Leftrightarrow p$		
16.	(a) $(p \vee t) \Leftrightarrow t$	Domination Laws	(D.L.)
	(b) $(p \wedge c) \Leftrightarrow c$		
17.	(a) $p \vee \sim p \to t$	Inverse Laws	(Inv.)
	(b) $p \wedge \sim p = c$		
18.	(a) $p \vee (p \wedge q) \Leftrightarrow p$	Absorption Laws	(A.L.)
	(b) $p \wedge (p \vee q) \Leftrightarrow p$		
19.	(a) $c \Rightarrow p$		
	(b) $p \Rightarrow t$		
20.	p and $q \Rightarrow p \wedge q$	Conjunctive inference	(Conj.)
21.	$(p \wedge q \to r) \equiv (p \to (q \to r))$	Exportation Law	(Exp.)

Further Notes:

22. p is sufficient for $q \equiv p \to q \ \equiv\ p$, only if q.

23. p is necessary for $q \equiv q \to p \ \equiv\ p$, if q.

24. p is necessary and sufficient for $q \equiv p \leftrightarrow q \ \equiv\ p$, if and only if q.

25. Quantifier Negation (QN): $\forall \equiv$ *for all* and $\exists \equiv$ *there exists*

    (a) $\sim [(\forall x)(p(x))] \equiv (\exists x)(\sim p(x))$
    (b) $\sim [(\exists x)(p(x))] \equiv (\forall x)(\sim p(x))$

26. Principle of Mathematical Induction (PMI): If $P(n)$ is a statement about or involving the natural number n such that

    (1) $P(1)$ is true, and
    (2) $P(k) \Rightarrow P(k+1)$ for all natural numbers k, then

    $P(n)$ is true for all natural numbers n.

27. Four basic truth tables:

(a) Negation

p	$\sim p$
T	F
F	T

(b) Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(c) Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(d) Conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

28. Converse Error $(P \rightarrow Q) \wedge Q \not\Rightarrow P$:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

29. Inverse Error $(P \rightarrow Q) \wedge \sim P \not\Rightarrow \sim Q$:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Appendix B

Sets Reference Table

Sets: Concepts and Laws[1]

In the tables on the following pages:

(a) A, B, C, X, Γ, and A_γ, for each $\gamma \in \Gamma$ are all sets.

(b) $\mathscr{F}$ is a set of sets.

(c) $\mathscr{U}$ is the universal set (i.e. a set containing all the elements under consideration).

(d) $p(x)$ is a statement about the variable x.

[1]Originally created by Vasily C. Cateforis for Set Theory and Logic at SUNY Postdam based on the text *Set Theory with Applications* by Lin & Lin

Concepts:

Symbols	Meaning (Definition or Statement in English)	Abbreviation
1. $x \in A$	x is an element of the set A	—
2. $x \notin A$	x is not an element of the set A	—
3. $\{x \in A \mid p(x)\}$	the set of all x in A such that $p(x)$ is true	—
4. $\{x \mid p(x)\}$	the set of all x in $\mathscr{U}$ such that $p(x)$ is true	—
5. $\varnothing$	the empty set $\equiv$ the set with no elements	—
6. $A \subseteq B$	A is a subset of $B \equiv \forall x\ (x \in A) \to (x \in B)$	Def. $\subseteq$
7. $A = B$	A is equal to $B \equiv \forall x\ (x \in A) \leftrightarrow (x \in B)$ $\equiv (A \subseteq B) \wedge (B \subseteq A)$	Def. $=$
8. $A \subset B$	A is a proper subset of $B \equiv (A \subseteq B) \wedge (A \neq B)$	Def. $\subset$
9. $\mathscr{P}(A)$	the power set of $A \equiv$ the set of all subsets of A $\equiv \{X \mid X \subseteq A\}$	Def. of Power Set
10. $A \cup B$	A union $B \equiv \{x \mid (x \in A) \vee (x \in B)\}$	Def. $\cup$
11. $A \cap B$	A intersection $B \equiv \{x \mid (x \in A) \wedge (x \in B)\}$	Def. $\cap$
12. $A - B$	A minus $B \equiv \{x \mid (x \in A) \wedge (x \notin B)\}$	Def. $-$
13. $\overline{A}$	the compliment of $A \equiv \{x \mid x \notin A\}$, sometimes denoted A' or A^c	Def. Comp.
14. $\{A_\gamma \mid \gamma \in \Gamma\}$	the family of sets A_γ indexed by the set Γ	—
15. $\bigcup_{\gamma \in \Gamma} A_\gamma$	the union of the sets A_γ indexed by the set Γ $\equiv \{x \mid \exists \gamma \in \Gamma\ (x \in A_\gamma)\}$	—
16. $\bigcap_{\gamma \in \Gamma} A_\gamma$	the intersection of the sets A_γ indexed by the set Γ $\equiv \{x \mid \forall \gamma \in \Gamma\ (x \in A_\gamma)\}$	—
17. $\bigcup_{A \in \mathscr{F}} A$	the union of the sets A in the family $\mathscr{F}$ $\equiv \{x \mid \exists A \in \mathscr{F}\ \ (x \in A)\}$	—
18. $\bigcap_{A \in \mathscr{F}} A$	the intersection of the sets A in the family $\mathscr{F}$ $\equiv \{x \mid \forall A \in \mathscr{F}\ \ (x \in A)\}$	—

Laws:

Symbols	Name
1a. $\forall A\ \varnothing \subseteq A$ and 1b. $\forall A\ A \subseteq A$	—
2. $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$	Transitivity of $\subseteq$
3a. $A \cup \varnothing = A$ and 3b. $A \cap \mathscr{U} = A$	Identity Laws
4a. $A \cap \varnothing = \varnothing$ and 4b. $A \cup \mathscr{U} = \mathscr{U}$	Domination Laws
5a. $A \cap A = A$ and 5b. $A \cup A = A$	Idempotency Laws
6a. $A \cup B = B \cup A$ and 6b. $A \cap B = B \cap A$	Commutativity Laws
7a. $A \cup (B \cup C) = (A \cup B) \cup C$ and 7b. $A \cap (B \cap C) = (A \cap B) \cap C$	Associativity Laws
8. $A \subseteq A \cup B \wedge B \subseteq A \cup B$	Addition Laws
9. $A \cap B \subseteq A \wedge A \cap B \subseteq B$	Simplification Laws
10a. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ 10b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive Laws
11. $A - B = A \cap \overline{B}$	—
12. $\overline{(\overline{A})} = A$	Double Compliment
13a. $\overline{\varnothing} = \mathscr{U}$ and 13b. $\overline{\mathscr{U}} = \varnothing$	—
14a. $A \cap \overline{A} = \varnothing$ and 14b. $A \cup \overline{A} = \mathscr{U}$	Inverse Laws
15a. $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ and 15b. $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$	DeMorgan's Laws
16. $A \subseteq B \Leftrightarrow \overline{B} \subseteq \overline{A}$	Contrapositive Law
17a. $A \cup (A \cap B) = A$ and 17b. $A \cap (A \cup B) = A$	Absorbtion Laws

Bibliography

[1] Susanna Samuels Epp. ***Discrete Mathematics with Applications***. Fifth. Cengage Learning, 2019.

[2] Ralph P. Grimaldi. ***Discrete and Combinatorial Mathematics***. 3rd. Addison-Wesley, 1994.

[3] OEIS Foundation Inc. ***The On-Line Encyclopedia of Integer Sequences***. 2025. URL: https://oeis.org.

[4] S.Y.T. Lin and Y.F. Lin. ***Set Theory with Applications***. Book Publishers, 1985.

[5] K.H. Rosen. ***Discrete Mathematics and Its Applications***. 6th. McGraw-Hill Higher Education, 2006.

[6] Christopher Stover. ***Set***. URL: https://mathworld.wolfram.com/Set.html.

[7] Eric W. Weisstein. ***Algorithm***. URL: https://mathworld.wolfram.com/Algorithm.html.

Index

addition principle, 83
adjacency matrix, 110
algorithm, 47
alternating, 28
antisymmetric, 64
arithmetic sum, 36–38
associative law, 10

base case, 31
bijection, 77
binary trees, 115
binomial theorem, 95
bounded, 27

cardinality, 62, 77, 86
Cartesian product, 60, 63
characteristic polynomial, 51
circuit, 106
closed walk, 106
codomain, 75
combination, 89
combinations with repetition, 96
commutative law, 10
complex numbers, 55
conclusion, 15
conjunction, 7, 10
contradiction, 9, 13
contrapositive, 12
convergent, 30
converse, 12
converse error, 13
critical row, 13

decision tree, 85
deck of cards, 91
decreasing, 27
Demorgan's Laws, 10
disjoint events, 83
disjunction, 7, 10
disjunctive syllogism, 13
distributive law, 10
domain, 75
domination laws, 10
double negation, 10

edge set, 103
empty set, 55
equivalence class, 71
equivalence class representative, 71
equivalence relation, 64, 71
Euler circuit, 106–109
Euler trail, 106–109
exclusive or, 10
    xor, 10
existential quantifier, 24
explicit formula, 27

factorial, 89
first order linear recurrence relations, 47
first order recurrence relation, 51
full binary tree, 115
function, 75
    bijection, 77
    codomain, 75
    composition, 79
    domain, 75
    inverse, 79
    one-to-one, 77
    onto, 77
    piecewise defined, 76
    range, 75
function notation, 76

general formula, 27
geometric sum, 39, 40, 42
golden ratio, 52

graph, 103
  binary tree, 104
  bipartite, 104
  complete, 104
  complete bipartite, 104
  degree, 103
  digraph, 104
  directed, 104
  multigraph, 105
  simple, 105
  total degree, 103, 104
  tree, 104, 115

Hamilton circuit, 106–109
handshake theorem, 104
hypothesis, 15

identity laws, 10
identity matrix, 111
inclusion-exclusion principle, 86
increasing, 28
independent events, 83
index, 27
intersection, 58
inverse, 12
inverse error, 13
inverse laws, 10
inverse matrix, 111
iteration algorithm, 47

law of simplification, 13
laws
  associative, 10
  commutative, 10
  DeMorgan's, 10
  distributive, 10
  domination, 10
  double negation, 10
  identity, 10
  inverse, 10

matrix, 110
  identity, 111
  inverse, 111
matrix arithmetic, 111
modular equivalence, 72
modus ponens, 13
modus tollens, 13
multigraph, 105
multinomial coefficients, 96
multiplication principle, 83

natural numbers, 55
negation, 7, 10, 12

one-to-one, 77
onto, 77
ordered pair, 60

partial-ordering, 64
partition, 71
Pascal's Formula, 94
Pascal's Triangle, 94
path, 106
permutation, 89
piecewise defined, 76
pigeonhole principle, 100
power set, 61
  cardinality of, 62
predicate, 19
premise, 15
product, 35
product notation, 35
proposition, 7

range, 75
rank, 91
rational numbers, 55
re-indexing, 44
real numbers, 55
recurrence relation, 51
recursive sequence, 30
reflexive, 64
relation, 63
  properties, 64
  antisymmetric, 64
  equivalence relation, 64, 65
  inverse, 79
  partial-ordering, 64, 66
  reflexive, 64, 65
  symmetric, 64, 65
  transitive, 64, 65

second order recurrence relation, 51
sequence, 27

set, 55
set builder notation, 56
set complement, 59
set difference, 58
set roster notation, 56
simple circuit, 106
simple graph, 105
statement, 7
subsequence, 30
subset, 59, 61, 63
suit, 91
sum
  arithmetic, 43
  consecutive even integers, 43
  consecutive integers, 43
  consecutive odd integers, 43
  cubes, 43
  geometric, 43
  squares, 43
summation, 35
summation notation, 35
symmetric, 64

tautology, 9
total degree, 104
trail, 106
transitive, 64
transitivity, 13
tree, 115
  binary, 115
  full binary, 115
truth set, 22
truth table, 11

union, 58
universal quantifier, 24
universal set, 59

vacuously true, 12
valid argument, 15
Venn diagram, 59
vertex set, 103

walk, 106