Limits of Turing Machines

Dr. Chuck Rocca
roccac@wcsu.edu
http://sites.wcsu.edu/roccac

WESTERN CONNECTICUT STATE UNIVERSITY
MACRICOSTAS SCHOOL OF ARTS & SCIENCES

Table of Contents

Table of Contents

Algorithm

Definition

An *algorithm* is a computational process that is describable in terms of a Turing machine.

Connected Graphs Example

M="On input $\langle G \rangle$, the string encoding of a graph G:

0. Check that $\langle G \rangle$ is in the correct format
1. Select the first node in G and mark it.
2. Repeat the following until no new nodes are marked:
3. For each node in G, mark it if is is attached by an edge to a node that is already marked.
4. Scan all the nodes of G, if they are all marked, *accept*; otherwise *reject*."

Decidable Unions

Given two *decidable* languages L_1 and L_2 and corresponding Turing machines M_1 and M_2 the union of the languages can be decided by:
M=“On input w:

0. Check that w is in the correct format.
1. Run M_1 on w. If it accepts, *accept*.
2. Run M_2 on w. If it accepts, *accept*.
3. Otherwise *reject*”

Recognizable Unions

Given two *Turing recognizable* languages L_1 and L_2 and corresponding Turing machines M_1 and M_2 the union of the languages can be recognized by:
M=“On input w:

0. Check that w is in the correct format.
1. Run M_1 and M_2 alternately on w step by step. If either accepts, *accept*.
2. If both halt or reject, *reject*.”

Decidable vs. Recognizable

Definition

A Language is *Turing-decidable* or simply *decidable* if some Turing machine decides it; the machine always reaches an accept or reject state. Given any word there is a TM that can tell if the word is or is not in the language.

Decidable vs. Recognizable

Definition

A Language is *Turing-decidable* or simply *decidable* if some Turing machine decides it; the machine always reaches an accept or reject state. Given any word there is a TM that can tell if the word is or is not in the language.

Definition

A Language is *Turing-recognizable* if some Turing machine recognizes it; in this case the machine reaches an accept state, reject state, or it may loop (fail to accept). There is a TM that accepts words in the language, but may fail to reach a verdict if a word is not in the language.

Decidable Intersections

Given two *decidable* languages L_1 and L_2 and corresponding Turing machines M_1 and M_2 the intersections of the languages can be decided by:
M="On input w:

0. Check that w is in the correct format.
1. Run M_1 and M_2 on w. If they both accept, *accept*.
2. Otherwise *reject*"

Decidable Intersections

Given two *decidable* languages L_1 and L_2 and corresponding Turing machines M_1 and M_2 the intersections of the languages can be decided by: M="On input w:

0. Check that w is in the correct format.
1. Run M_1 and M_2 on w. If they both accept, *accept*.
2. Otherwise *reject*"

Why didn't we say "Run M_1 and M_2 alternately on w step by step?"

Decidable Complements

Given a *decidable* language L_1 and corresponding Turing machine M_1 the complement of the languages can be decided by:
M="On input w:

0. Check that w is in the correct format.
1. Run M_1 on w. If it accepts, *reject*.
2. Otherwise *accept*"

Table of Contents

Deterministic Finite Automaton

Theorem

The set

$$A_{DFA} = \{\langle B, w\rangle \,|\, B \text{ is a DFA that accepts string } w\}$$

is a decidable language.

Deterministic Finite Automaton

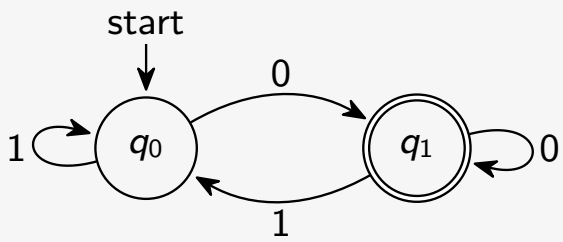
M="On input $\langle B, w \rangle$, where B is a DFA and w is a string:

0. Check the format of the input.
1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*; otherwise, *reject*."

Deterministic Finite Automaton

M="On input $\langle B, w \rangle$, where B is a DFA and w is a string:

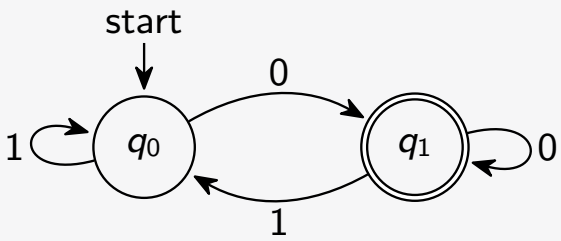
0. Check the format of the input.
1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*; otherwise, *reject*."

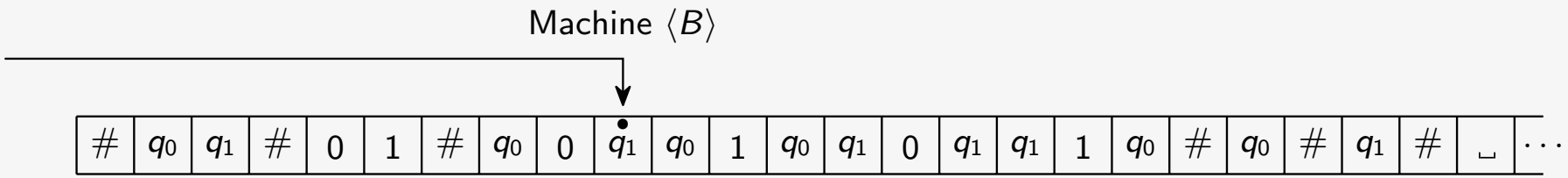


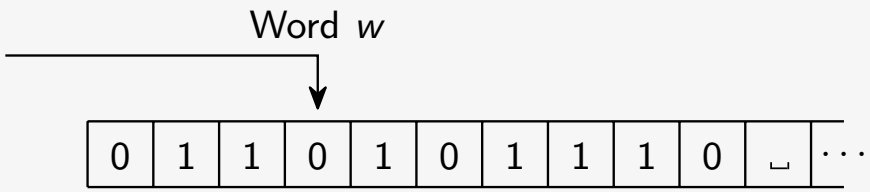
Deterministic Finite Automaton

M="On input $\langle B, w \rangle$, where B is a DFA and w is a string:

0. Check the format of the input.
1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*; otherwise, *reject*."







Nondeterministic Finite Automaton

Theorem

The set

$$A_{NFA} = \{\langle B, w\rangle \,|\, B \text{ is an NFA that accepts string } w\}$$

is a decidable language.

N="On input $\langle B, w\rangle$, where B is a NFA and w is a string:

1. Convert B to a DFA C.
2. Run M from the previous theorem on input $\langle C, w\rangle$.
3. If the simulation ends in an accept state, *accept*; otherwise, *reject*."

Regular Expressions

Theorem

The set

$$A_{REX} = \{\langle B, w\rangle \,|\, B \text{ is a regex that accepts string } w\}$$

is a decidable language.

P="On input $\langle B, w\rangle$, where B is a RegEx and w is a string:

1. Convert B to a NFA C.
2. Run N from the previous theorem on input $\langle C, w\rangle$.
3. If the simulation ends in an accept state, *accept*; otherwise, *reject*."

Empty Languages

Theorem

The set

$$E_{DFA} = \{\langle A \rangle \,|\, A \text{ is a DFA and } L(A) = \emptyset\}$$

is a decidable language.

T="On input $\langle A \rangle$, the string encoding of DFA A:

1. Select the start state in A and mark it.
2. Repeat the following until no new states are marked:
3. For each state in A, mark it if there is a transition from a marked state.
4. Scan the accept states of A, if any are marked, *reject*; otherwise *accept*."

Equal Languages

Theorem

The set

$$EQ_{DFA} = \{\langle A, B\rangle \,|\, A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$

is a decidable language.

- Note, DFAs are closed under *unions*, *intersections*, and *compliments*.
- Construct the *symmetric difference* of A and B, $C \equiv A \; XOR \; B$ or

$$L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right).$$

- Check if C is empty using T from the previous theorem.

CFLs

Theorem

The set

$$A_{CFG} = \{\langle G, w\rangle \,|\, G \text{ is a CFG that generates the string } w\}$$

is a decidable language.

$S =$"On input $\langle G, w\rangle$, where G is a CFG and w is a string:

1. Convert G to Chomsky Normal Form.
2. List all derivations with $2n-1$ steps, $n = |w|$; except for n=0, then list derivations with one step. (Why $2n-1$?)
3. If w is generated, *accept*; otherwise *reject*."

CFLs

Theorem

The set

$$E_{CFG} = \{\langle G \rangle \,|\, G \text{ is a CFG and } L(G) = \emptyset\}$$

is a decidable language.

R="On input $\langle G \rangle$, the string encoding of CFG G:

1. Mark the terminals in G.
2. Repeat the following until no new variables get marked:
3. Mark any variable A where $A \to U_1 U_2 \cdots U_k$ is in G and the U_i are all marked already.
4. If the start variable of G is not marked, *accept*; otherwise *reject*."

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle \,|\, G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

is not a decidable language.

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle \,|\, G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

is not a decidable language. (Proof held until after Chapter 5.)

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle \,|\, G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

is not a decidable language. (Proof held until after Chapter 5.)

- Recall, CFLs are not necessarily closed under intersections and complementation.

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle \,|\, G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

is not a decidable language. (Proof held until after Chapter 5.)

- Recall, CFLs are not necessarily closed under intersections and complementation.
- $L(G) = \{a^m b^n c^n | m, n \geq 0\}$

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle \,|\, G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

is not a decidable language. (Proof held until after Chapter 5.)

- Recall, CFLs are not necessarily closed under intersections and complementation.
- $L(G) = \{a^m b^n c^n | m, n \geq 0\}$
- $L(H) = \{a^n b^n c^m | m, n \geq 0\}$

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle \,|\, G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

is not a decidable language. (Proof held until after Chapter 5.)

- Recall, CFLs are not necessarily closed under intersections and complementation.
- $L(G) = \{a^m b^n c^n | m, n \geq 0\}$
- $L(H) = \{a^n b^n c^m | m, n \geq 0\}$
- $L(G) \cap L(H) = \{a^n b^n c^n | n \geq 0\}$ is not context-free by the pumping lemma

Hierarchy of Languages

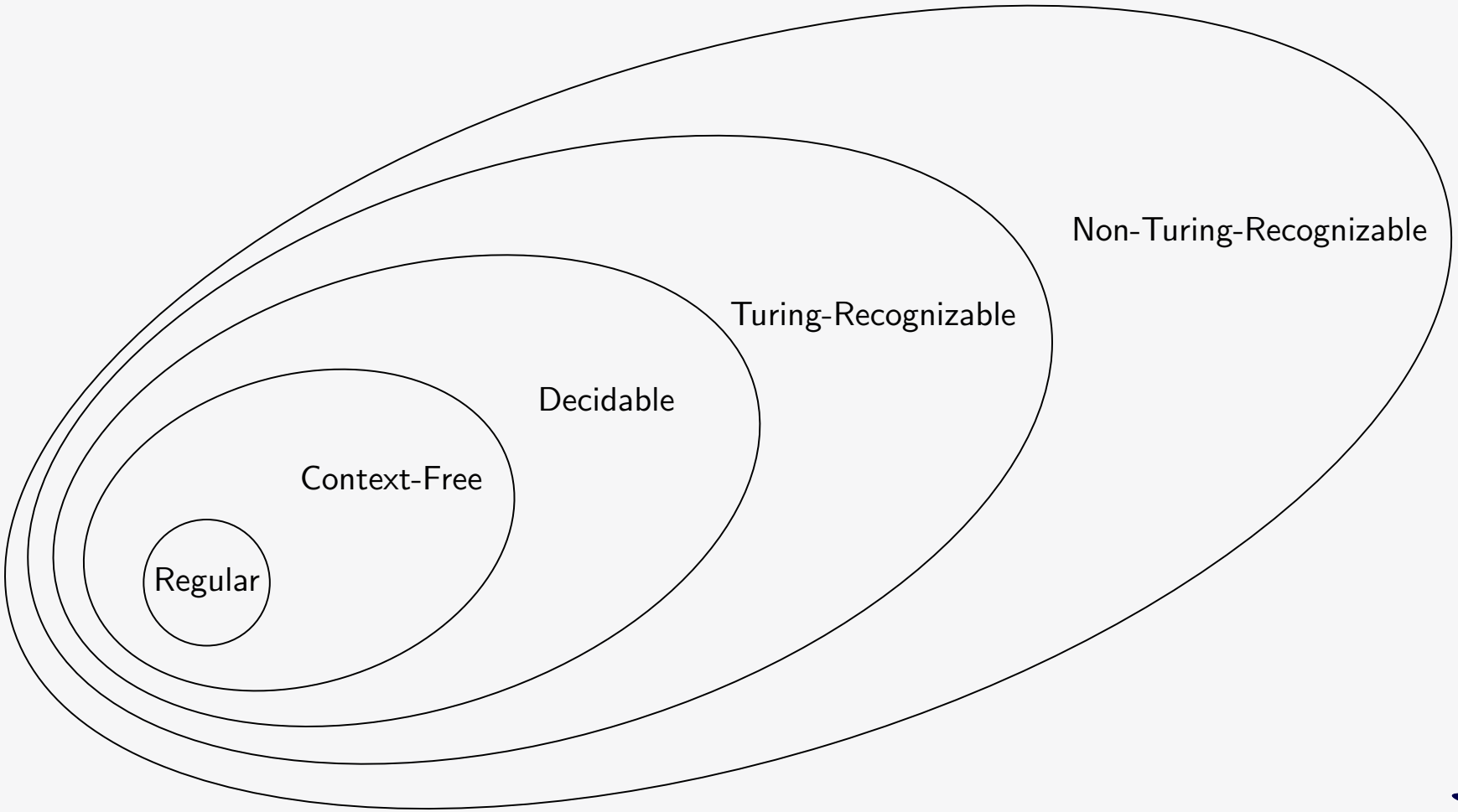


Table of Contents

Cardinality: Naive Definition

Definition

The *cardinality* of a set is the number of elements in the set and two sets have the same cardinality if they have the same number of elements.

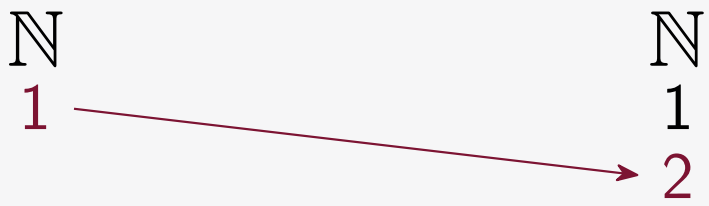
$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{a, b, c, d, e\}$$
$$C = \{!, @, \#, \$, \%, \wedge\}$$
$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$$
$$\mathbb{Q} = \{a/b | a, b \in \mathbb{Z} \text{ and } b \neq 0\}$$

Cardinality: Improved Definition

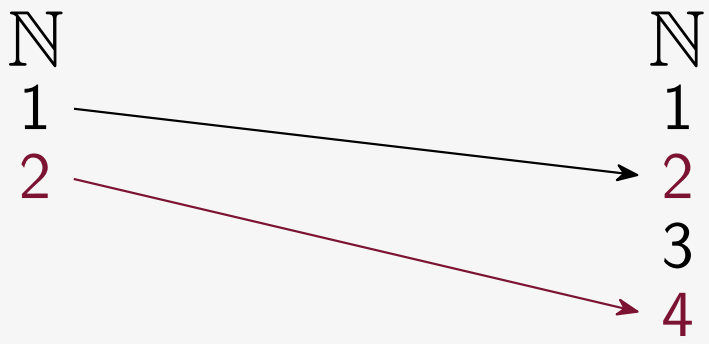
Definition

The *cardinality* of a finite set is the number of elements in the set. A set is *infinite* if there is a one-to-one correspondence between the set and a proper subset of the set. And, two sets have the same cardinality if there exists a one-to-one correspondence between their elements.

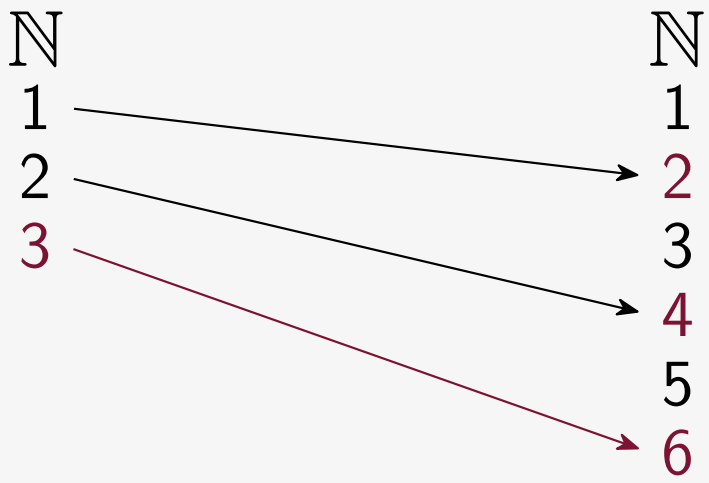
$\mathbb{N}$ to $2\mathbb{N}$



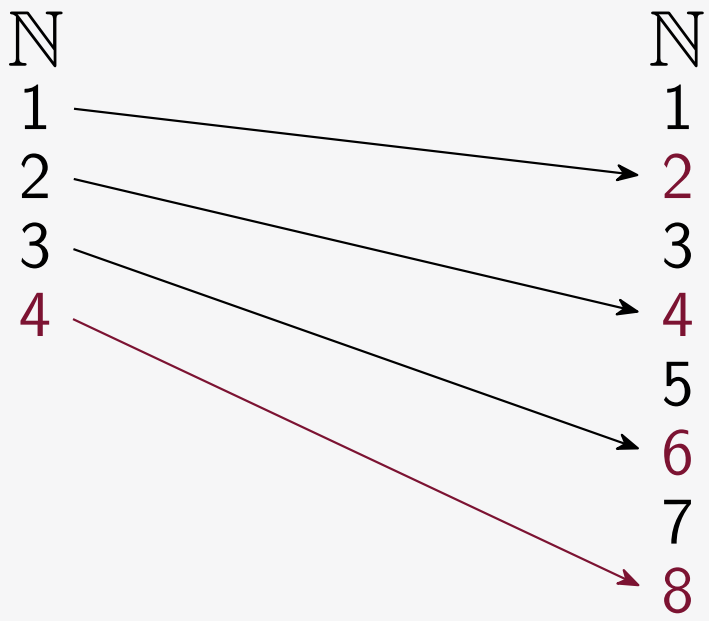
$\mathbb{N}$ to $2\mathbb{N}$



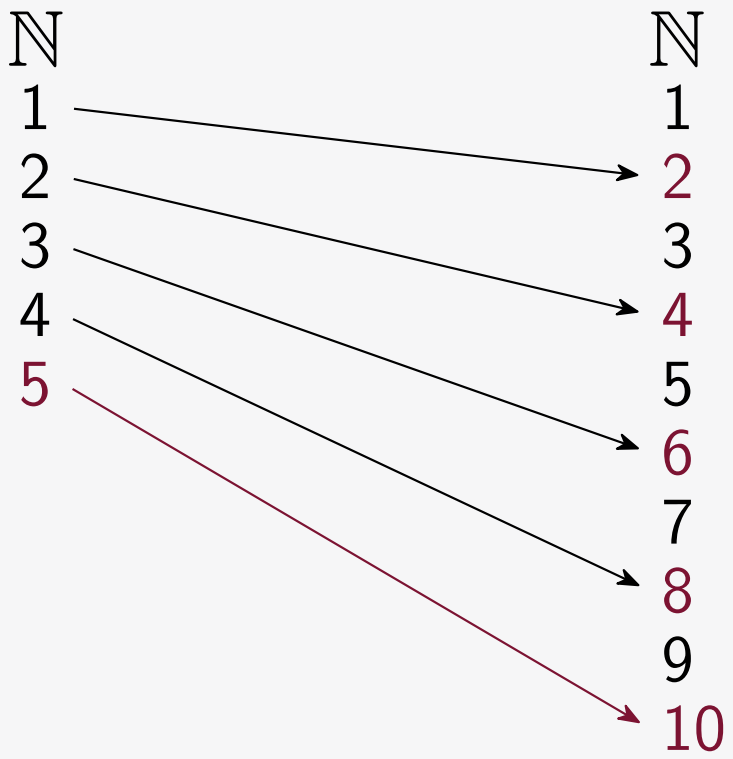
$\mathbb{N}$ to $2\mathbb{N}$



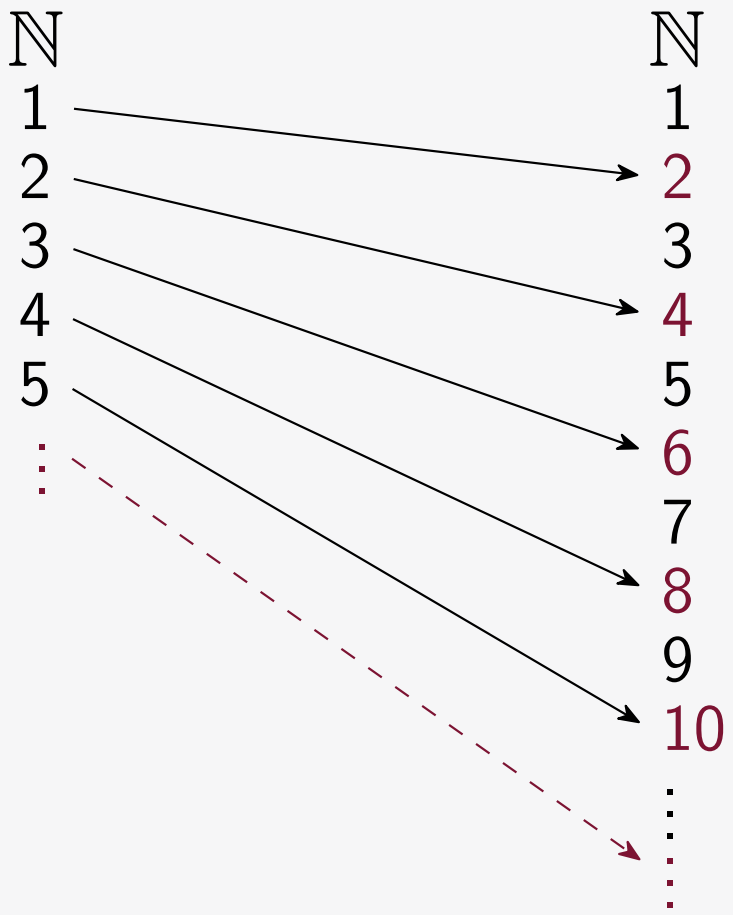
$\mathbb{N}$ to $2\mathbb{N}$



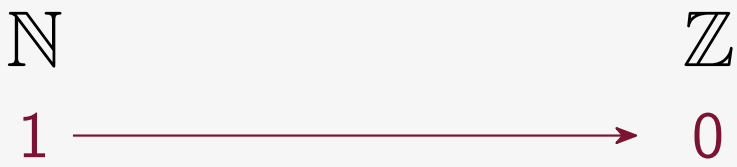
$\mathbb{N}$ to $2\mathbb{N}$



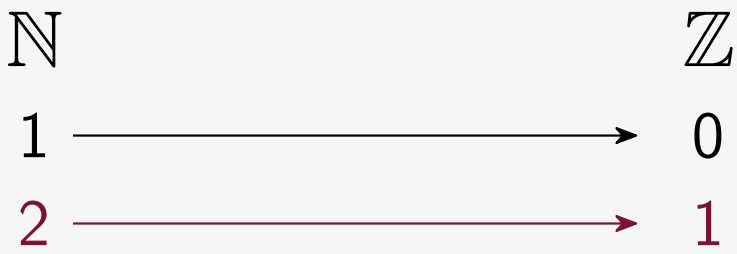
$\mathbb{N}$ to $2\mathbb{N}$



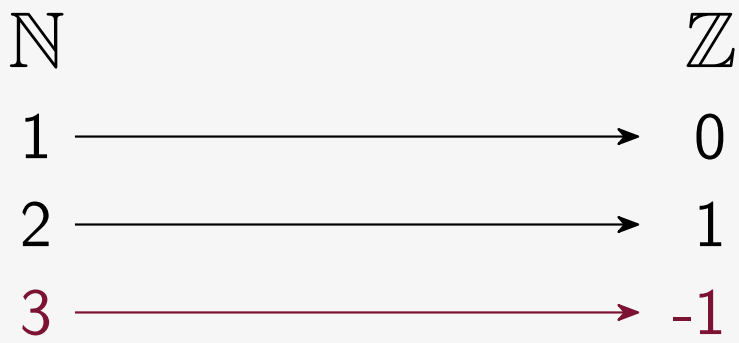
$\mathbb{N}$ to $\mathbb{Z}$



$\mathbb{N}$ to $\mathbb{Z}$



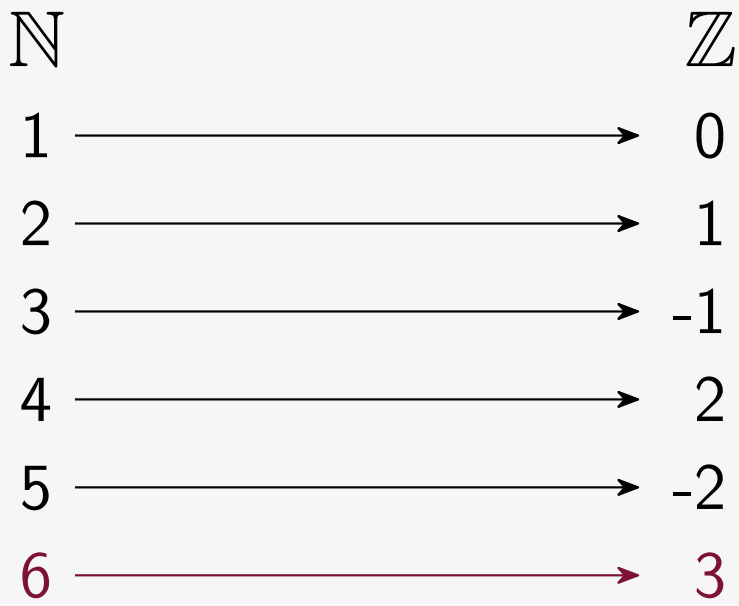
$\mathbb{N}$ to $\mathbb{Z}$



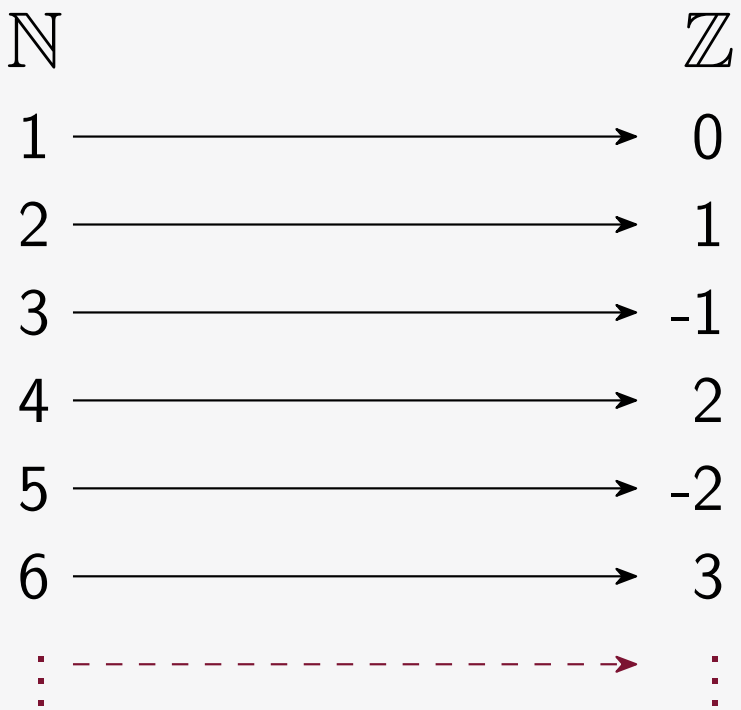
$\mathbb{N}$ to $\mathbb{Z}$

$\mathbb{N}$ to $\mathbb{Z}$

$\mathbb{N}$ to $\mathbb{Z}$



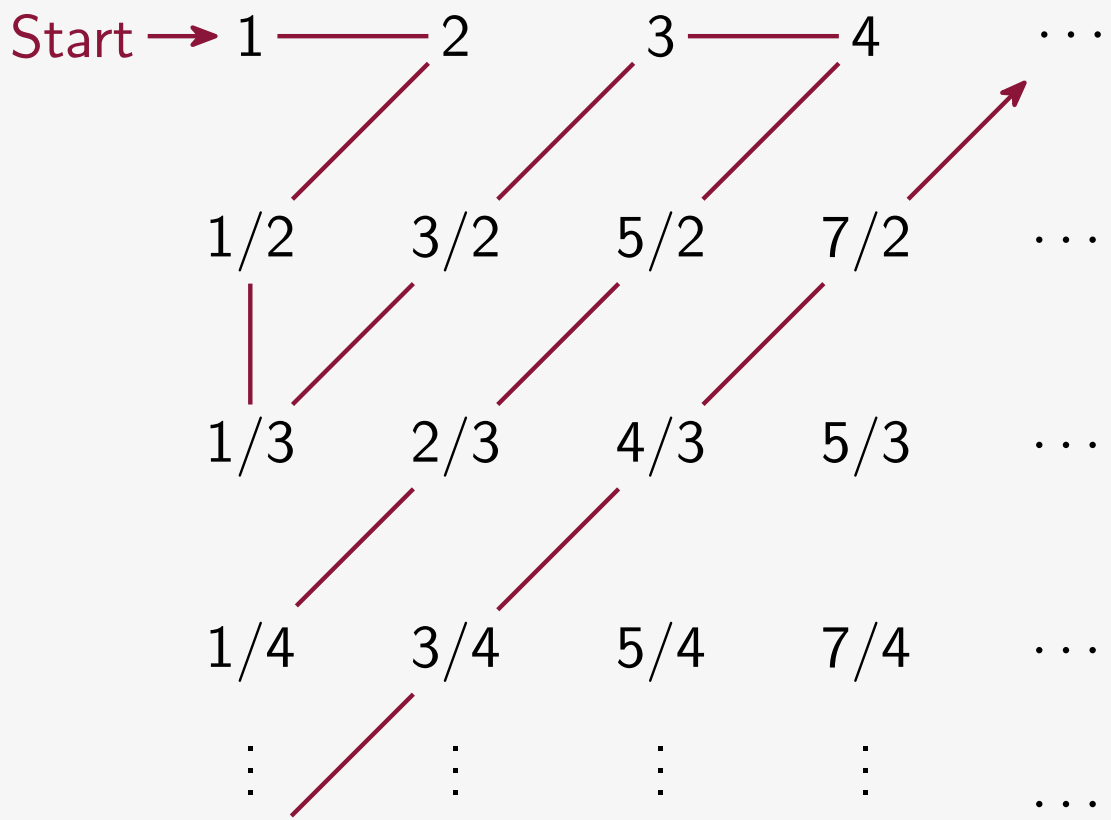
$\mathbb{N}$ to $\mathbb{Z}$



$\mathbb{N}$ to $\mathbb{Q}^+$

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & \cdots \\
1/2 & 3/2 & 5/2 & 7/2 & \cdots \\
1/3 & 2/3 & 4/3 & 5/3 & \cdots \\
1/4 & 3/4 & 5/4 & 7/4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

$\mathbb{N}$ to $\mathbb{Q}^+$



Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6577\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6577\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.7$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6657\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.74$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6577\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.741$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6577\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.7412$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6577\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.74128$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6577\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.741287$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6577\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.741287\mathbf{3}$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6577\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.74128731$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6577\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.74128731\mathbf{8}$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6577\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.7412873187\ldots$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.\mathbf{6}5395501314\cdots$
2	$0.7\mathbf{3}800613014\cdots$
3	$0.05\mathbf{0}50813247\cdots$
4	$0.108\mathbf{1}0350448\cdots$
5	$0.0458\mathbf{7}954758\cdots$
6	$0.66716\mathbf{6}6577\cdots$
7	$0.732436\mathbf{2}7345\cdots$
8	$0.2731193\mathbf{0}829\cdots$
9	$0.171772119\mathbf{0}3\cdots$
10	$0.4551827778\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.7412873187\ldots$$

Avoiding 0's and 9's when replacing digits since

$$0.19999\ldots = 0.20000\ldots$$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$
$$\mathscr{P}(\mathbb{N}) = \{\emptyset, \{1\}, \{2, 3\}, \{4\}, \{8\}, \{7, 19, 83\}, \{101, 23, 7\}, \ldots\}$$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$
$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$
$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

$$\begin{array}{ccc} 1 & \longrightarrow & s_1 \\ 2 & \longrightarrow & s_2 \\ 3 & \longrightarrow & s_3 \\ 4 & \longrightarrow & s_4 \\ 5 & \longrightarrow & s_5 \\ 6 & \longrightarrow & s_6 \\ 7 & \longrightarrow & s_7 \\ \vdots & \dashrightarrow & \vdots \end{array}$$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

$$\begin{array}{ccc} 1 & \longrightarrow & s_1 \\ 2 & \longrightarrow & s_2 \\ 3 & \longrightarrow & s_3 \\ 4 & \longrightarrow & s_4 \\ 5 & \longrightarrow & s_5 \\ 6 & \longrightarrow & s_6 \\ 7 & \longrightarrow & s_7 \\ \vdots & \dashrightarrow & \vdots \end{array}$$

Define a new set X as follows

$$X = \{n | n \notin s_n\}$$

Thus, if

- $1 \in s_1$, then $1 \notin X$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

$$\begin{array}{ccc} 1 & \longrightarrow & s_1 \\ 2 & \longrightarrow & s_2 \\ 3 & \longrightarrow & s_3 \\ 4 & \longrightarrow & s_4 \\ 5 & \longrightarrow & s_5 \\ 6 & \longrightarrow & s_6 \\ 7 & \longrightarrow & s_7 \\ \vdots & \dashrightarrow & \vdots \end{array}$$

Define a new set X as follows

$$X = \{n | n \notin s_n\}$$

Thus, if

- $1 \in s_1$, then $1 \notin X$
- $1 \notin s_1$, then $1 \in X$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

$$\begin{array}{ccc} 1 & \longrightarrow & s_1 \\ 2 & \longrightarrow & s_2 \\ 3 & \longrightarrow & s_3 \\ 4 & \longrightarrow & s_4 \\ 5 & \longrightarrow & s_5 \\ 6 & \longrightarrow & s_6 \\ 7 & \longrightarrow & s_7 \\ \vdots & \dashrightarrow & \vdots \end{array}$$

Define a new set X as follows

$$X = \{n | n \notin s_n\}$$

Thus, if

- $1 \in s_1$, then $1 \notin X$
- $1 \notin s_1$, then $1 \in X$
- $2 \in s_2$, then $2 \notin X$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

$$\begin{array}{ccc} 1 & \longrightarrow & s_1 \\ 2 & \longrightarrow & s_2 \\ 3 & \longrightarrow & s_3 \\ 4 & \longrightarrow & s_4 \\ 5 & \longrightarrow & s_5 \\ 6 & \longrightarrow & s_6 \\ 7 & \longrightarrow & s_7 \\ \vdots & \dashrightarrow & \vdots \end{array}$$

Define a new set X as follows

$$X = \{n | n \notin s_n\}$$

Thus, if

- $1 \in s_1$, then $1 \notin X$
- $1 \notin s_1$, then $1 \in X$
- $2 \in s_2$, then $2 \notin X$
- $2 \notin s_2$, then $2 \in X$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

$$\begin{array}{ccc} 1 & \longrightarrow & s_1 \\ 2 & \longrightarrow & s_2 \\ 3 & \longrightarrow & s_3 \\ 4 & \longrightarrow & s_4 \\ 5 & \longrightarrow & s_5 \\ 6 & \longrightarrow & s_6 \\ 7 & \longrightarrow & s_7 \\ \vdots & \dashrightarrow & \vdots \end{array}$$

Define a new set X as follows

$$X = \{n | n \notin s_n\}$$

Thus, if

- $1 \in s_1$, then $1 \notin X$
- $1 \notin s_1$, then $1 \in X$
- $2 \in s_2$, then $2 \notin X$
- $2 \notin s_2$, then $2 \in X$
- $3 \in s_3$, then $3 \notin X$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

$$\begin{array}{ccc} 1 & \longrightarrow & s_1 \\ 2 & \longrightarrow & s_2 \\ 3 & \longrightarrow & s_3 \\ 4 & \longrightarrow & s_4 \\ 5 & \longrightarrow & s_5 \\ 6 & \longrightarrow & s_6 \\ 7 & \longrightarrow & s_7 \\ \vdots & \dashrightarrow & \vdots \end{array}$$

Define a new set X as follows

$$X = \{n | n \notin s_n\}$$

Thus, if

- $1 \in s_1$, then $1 \notin X$
- $1 \notin s_1$, then $1 \in X$
- $2 \in s_2$, then $2 \notin X$
- $2 \notin s_2$, then $2 \in X$
- $3 \in s_3$, then $3 \notin X$
- $3 \notin s_3$, then $3 \in X$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

$$\begin{array}{ccc} 1 & \longrightarrow & s_1 \\ 2 & \longrightarrow & s_2 \\ 3 & \longrightarrow & s_3 \\ 4 & \longrightarrow & s_4 \\ 5 & \longrightarrow & s_5 \\ 6 & \longrightarrow & s_6 \\ 7 & \longrightarrow & s_7 \\ \vdots & \dashrightarrow & \vdots \end{array}$$

Define a new set X as follows

$$X = \{n | n \notin s_n\}$$

Thus, if

- $1 \in s_1$, then $1 \notin X$
- $1 \notin s_1$, then $1 \in X$
- $2 \in s_2$, then $2 \notin X$
- $2 \notin s_2$, then $2 \in X$
- $3 \in s_3$, then $3 \notin X$
- $3 \notin s_3$, then $3 \in X$
- etc.

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$
$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

$$\begin{array}{ccc} 1 & \longrightarrow & s_1 \\ 2 & \longrightarrow & s_2 \\ 3 & \longrightarrow & s_3 \\ 4 & \longrightarrow & s_4 \\ 5 & \longrightarrow & s_5 \\ 6 & \longrightarrow & s_6 \\ 7 & \longrightarrow & s_7 \\ \vdots & \dashrightarrow & \vdots \end{array}$$

Define a new set X as follows

$$X = \{n | n \notin s_n\}$$

Thus, if

- $1 \in s_1$, then $1 \notin X$
- $1 \notin s_1$, then $1 \in X$
- $2 \in s_2$, then $2 \notin X$
- $2 \notin s_2$, then $2 \in X$
- $3 \in s_3$, then $3 \notin X$
- $3 \notin s_3$, then $3 \in X$
- etc.

And therefore $\forall n : X \neq s_n$.

A Theorem on Power Sets

Theorem

Given a set S, the cardinality of $\mathscr{P}(S)$ is always greater than the cardinality of S; there are infinitely many infinities.

Table of Contents

A_{TM} is Undecidable

Theorem

Given the set

$$A_{TM} = \{\langle M, w\rangle \,|\, M \text{ is a Turing Machine and } M \text{ accepts } w\},$$

A_{TM} *is undecidable.*

A_{TM} is Undecidable

Theorem

Given the set

$$A_{TM} = \{\langle M, w\rangle \,|\, M \text{ is a Turing Machine and } M \text{ accepts } w\},$$

A_{TM} *is undecidable.*

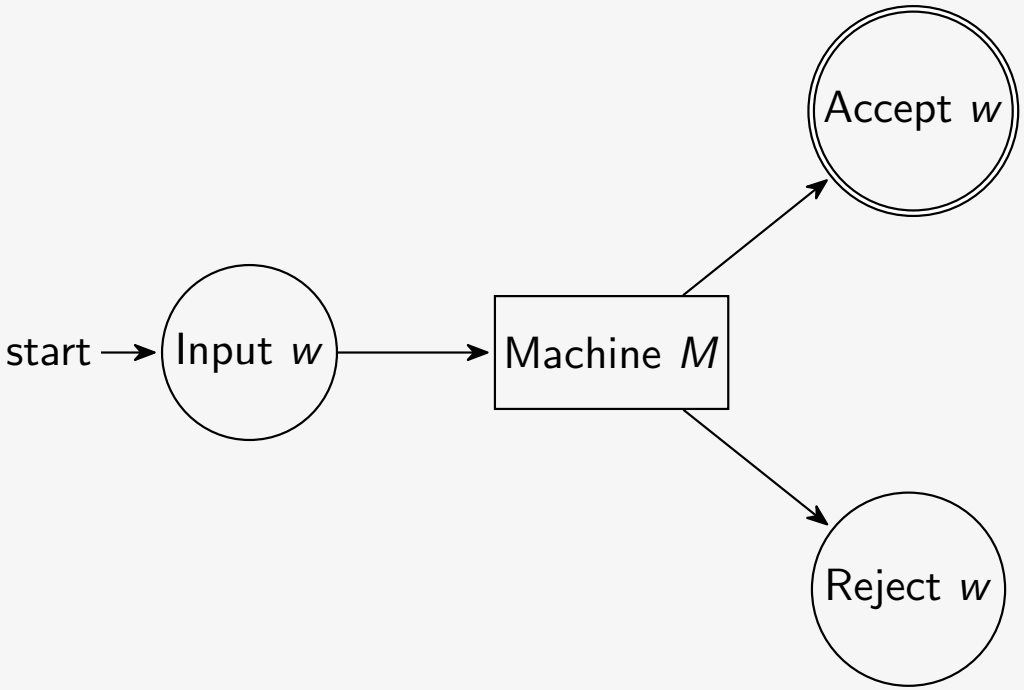
$U =$ " On input $\langle M, w\rangle$, where M is a TM and w is a string:

1. Simulate M on w.
2. If M ever enters its accept state, *accept*; If M ever enters its reject state, *reject*."

This is a *universal Turing machine* and shows that A_{TM} is recognizable.

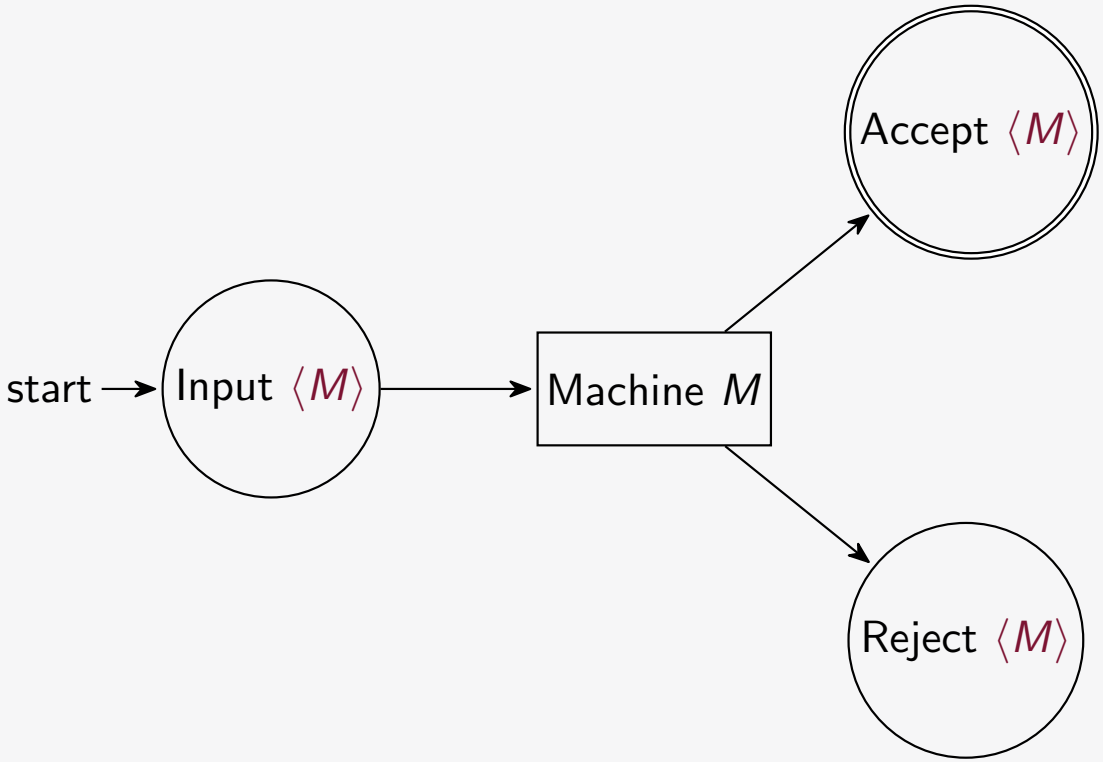
An Undecidable Language

- $A_{TM} = \{\langle M, w\rangle \,|\, M \text{ is a Turing Machine and } M \text{ accepts } w\}$



An Undecidable Language

- $A_{TM} = \{\langle M, w\rangle \,|\, M \text{ is a Turing Machine and } M \text{ accepts } w\}$



(Think C++ compiler written in C++.)

An Undecidable Language

- $A_{TM} = \{\langle M, w\rangle \,|\, M \text{ is a Turing Machine and } M \text{ accepts } w\}$
- Suppose there's a *decider* H for A_{TM}

$$H(\langle M, w\rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

An Undecidable Language

- $A_{TM} = \{\langle M, w\rangle \,|\, M \text{ is a Turing Machine and } M \text{ accepts } w\}$
- Suppose there's a *decider* H for A_{TM}
- Define $D(\langle M\rangle) = \neg H(\langle M, \langle M\rangle\rangle)$

$$D(\langle M\rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ does not accept } \langle M\rangle \\ \textit{reject} & \text{if } M \text{ accepts } \langle M\rangle \end{cases}$$

An Undecidable Language

- $A_{TM} = \{\langle M, w\rangle \,|\, M \text{ is a Turing Machine and } M \text{ accepts } w\}$
- Suppose there's a *decider* H for A_{TM}
- Define $D(\langle M\rangle) = \neg H(\langle M, \langle M\rangle\rangle)$
- D can't decide $\langle D\rangle$

$$D(\langle D\rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

An Undecidable Language

- $A_{TM} = \{\langle M, w\rangle \,|\, M \text{ is a Turing Machine and } M \text{ accepts } w\}$
- Suppose there's a *decider* H for A_{TM}
- Define $D(\langle M\rangle) = \neg H(\langle M, \langle M\rangle\rangle)$
- D can't decide $\langle D\rangle$

$$D(\langle D\rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

- $\therefore$ Neither D nor H can exist and so A_{TM} is undecidable

Machine D

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$
M_1	*accept*	*reject*	*accept*	*accept*	$\cdots$	*accept*
M_2	*reject*	*accept*	*accept*	*reject*	$\cdots$	*reject*
M_3	*accept*	*accept*	*reject*	*reject*	$\cdots$	*reject*
M_4	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$
D	*reject*	*reject*	*accept*	*reject*	$\cdots$	?

Machine D

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$
M_1	*accept*	*reject*	*accept*	*accept*	$\cdots$	*accept*
M_2	*reject*	*accept*	*accept*	*reject*	$\cdots$	*reject*
M_3	*accept*	*accept*	*reject*	*reject*	$\cdots$	*reject*
M_4	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$
D	*reject*	*reject*	*accept*	*reject*	$\cdots$	?

Machine D

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$
M_1	*accept*	*reject*	*accept*	*accept*	$\cdots$	*accept*
M_2	*reject*	*accept*	*accept*	*reject*	$\cdots$	*reject*
M_3	*accept*	*accept*	*reject*	*reject*	$\cdots$	*reject*
M_4	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$
D	*reject*	*reject*	*accept*	*reject*	$\cdots$	?

Machine D

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$
M_1	*accept*	*reject*	*accept*	*accept*	$\cdots$	*accept*
M_2	*reject*	*accept*	*accept*	*reject*	$\cdots$	*reject*
M_3	*accept*	*accept*	*reject*	*reject*	$\cdots$	*reject*
M_4	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$
D	*reject*	*reject*	*accept*	*reject*	$\cdots$	?

Machine D

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$
M_1	*accept*	*reject*	*accept*	*accept*	$\cdots$	*accept*
M_2	*reject*	*accept*	*accept*	*reject*	$\cdots$	*reject*
M_3	*accept*	*accept*	*reject*	*reject*	$\cdots$	*reject*
M_4	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ldots$
D	*reject*	*reject*	*accept*	*reject*	$\cdots$	?

Machine D

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$
M_1	*accept*	*reject*	*accept*	*accept*	$\cdots$	*accept*
M_2	*reject*	*accept*	*accept*	*reject*	$\cdots$	*reject*
M_3	*accept*	*accept*	*reject*	*reject*	$\cdots$	*reject*
M_4	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$
D	*reject*	*reject*	*accept*	*reject*	$\cdots$	?

A_{TM} is Undecidable

Theorem

Given the set

$$A_{TM} = \{\langle M, w\rangle \,|\, M \text{ is a Turing Machine and } M \text{ accepts } w\},$$

A_{TM} *is undecidable.*

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

- $|\Sigma| < \infty$ implies $|\Sigma^*|$ is countable

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

- $|\Sigma| < \infty$ implies $|\Sigma^*|$ is countable
- Every TM can be represented by a string, $\langle TM \rangle$, in Σ^*

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

- $|\Sigma| < \infty$ implies $|\Sigma^*|$ is countable
- Every TM can be represented by a string, $\langle TM \rangle$, in Σ^*
- There are countably many TMs

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

- $|\Sigma| < \infty$ implies $|\Sigma^*|$ is countable
- Every TM can be represented by a string, $\langle TM \rangle$, in Σ^*
- There are countably many TMs
- A language, A, is a subset of Σ^*

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

- $|\Sigma| < \infty$ implies $|\Sigma^*|$ is countable
- Every TM can be represented by a string, $\langle TM \rangle$, in Σ^*
- There are countably many TMs
- A language, A, is a subset of Σ^*
- $\mathscr{L} = \{A | A \text{ is a language}\} = \mathscr{P}(\Sigma^*)$

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

- $|\Sigma| < \infty$ implies $|\Sigma^*|$ is countable
- Every TM can be represented by a string, $\langle TM \rangle$, in Σ^*
- There are countably many TMs
- A language, A, is a subset of Σ^*
- $\mathscr{L} = \{A | A \text{ is a language}\} = \mathscr{P}(\Sigma^*)$
- $\mathscr{L}$ is uncountable

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

- $|\Sigma| < \infty$ implies $|\Sigma^*|$ is countable
- Every TM can be represented by a string, $\langle TM \rangle$, in Σ^*
- There are countably many TMs
- A language, A, is a subset of Σ^*
- $\mathscr{L} = \{A | A \text{ is a language}\} = \mathscr{P}(\Sigma^*)$
- $\mathscr{L}$ is uncountable
- $\therefore$ Some languages are not Turing-recognizable

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

Alternately:

- $\Sigma^* = \{s_1, s_2, s_3, s_4, \ldots\}$

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

Alternately:

- $\Sigma^* = \{s_1, s_2, s_3, s_4, \ldots\}$
- $\mathscr{B} = \{\text{infinite binary sequences}\}$

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

Alternately:

- $\Sigma^* = \{s_1, s_2, s_3, s_4, \ldots\}$
- $\mathscr{B} = \{\text{infinite binary sequences}\}$
- $\mathscr{L} = \{\text{all languages}\} = \mathscr{P}(\Sigma^*)$

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

Alternately:

- $\Sigma^* = \{s_1, s_2, s_3, s_4, \ldots\}$
- $\mathscr{B} = \{\text{infinite binary sequences}\}$
- $\mathscr{L} = \{\text{all languages}\} = \mathscr{P}(\Sigma^*)$
- $f : \mathscr{L} \longrightarrow \mathscr{B}$

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

Alternately:

- $\Sigma^* = \{s_1, s_2, s_3, s_4, \ldots\}$
- $\mathscr{B} = \{\text{infinite binary sequences}\}$
- $\mathscr{L} = \{\text{all languages}\} = \mathscr{P}(\Sigma^*)$
- $f : \mathscr{L} \longrightarrow \mathscr{B}$
- $\forall A \in \mathscr{L} : f(A) = b_1 b_2 b_3 b_4 \cdots \in \mathscr{B}$

$$b_i = \begin{cases} 0 & s_i \notin A \\ 1 & s_i \in A \end{cases}$$

Non-Turing Recognizable Languages

Theorem

Some languages are not Turing-Recognizable.

Alternately:

- $\Sigma^* = \{s_1, s_2, s_3, s_4, \ldots\}$
- $\mathscr{B} = \{\text{infinite binary sequences}\}$
- $\mathscr{L} = \{\text{all languages}\} = \mathscr{P}(\Sigma^*)$
- $f : \mathscr{L} \longrightarrow \mathscr{B}$
- $\forall A \in \mathscr{L} : f(A) = b_1 b_2 b_3 b_4 \cdots \in \mathscr{B}$

$$b_i = \begin{cases} 0 & s_i \notin A \\ 1 & s_i \in A \end{cases}$$

- $\chi_A = f(A)$ is called the *characteristic sequence* of A

Decidability vs. Recognizably

Definition

A language, $\overline{A}$, is *co-Turing-recognizable*, if it is the complement of a Turing-recognizable language A.

Decidability vs. Recognizably

Definition

A language, $\overline{A}$, is *co-Turing-recognizable*, if it is the complement of a Turing-recognizable language A.

Theorem

A language is decidable if and only if it is Turing-recognizable and co-Turing-recognizable. (i.e. A and $\overline{A}$ are both recognizable)

Decidability vs. Recognizably

Definition

A language, $\overline{A}$, is *co-Turing-recognizable*, if it is the complement of a Turing-recognizable language A.

Theorem

A language is decidable if and only if it is Turing-recognizable and co-Turing-recognizable. (i.e. A and $\overline{A}$ are both recognizable)

Theorem (contrapositive)

A language is non-decidable if and only if it is not Turing-recognizable or not co-Turing-recognizable.

Decidability vs. Recognizably

Definition

A language, $\overline{A}$, is *co-Turing-recognizable*, if it is the complement of a Turing-recognizable language A.

Theorem (contrapositive)

A language is non-decidable if and only if it is not Turing-recognizable or not co-Turing-recognizable.

Corollary

The language $\overline{A_{TM}}$ is non-Turing-recognizable.

Proof of Corollary

- We showed that A_{TM} is Turing Recognizable and is not Turing Decidable.

Proof of Corollary

- We showed that A_{TM} is Turing Recognizable and is not Turing Decidable.
- By the theorem above, since A_{TM} is not Turing Decidable, either A_{TM} or $\overline{A_{TM}}$ is not Turing Recognizable.

Proof of Corollary

- We showed that A_{TM} is Turing Recognizable and is not Turing Decidable.
- By the theorem above, since A_{TM} is not Turing Decidable, either A_{TM} or $\overline{A_{TM}}$ is not Turing Recognizable.
- Finally, since A_{TM} is Turing Recognizable, we conclude that $\overline{A_{TM}}$ is not.

Table of Contents

Next Class

- Some Undecidable Problems

Next Class

- Some Undecidable Problems
- Specific Undecidable Problem

Next Class

- Some Undecidable Problems
- Specific Undecidable Problem
- Mapping Reducibility

Limits of Turing Machines

Dr. Chuck Rocca
roccac@wcsu.edu
http://sites.wcsu.edu/roccac

WESTERN CONNECTICUT STATE UNIVERSITY
MACRICOSTAS SCHOOL OF ARTS & SCIENCES