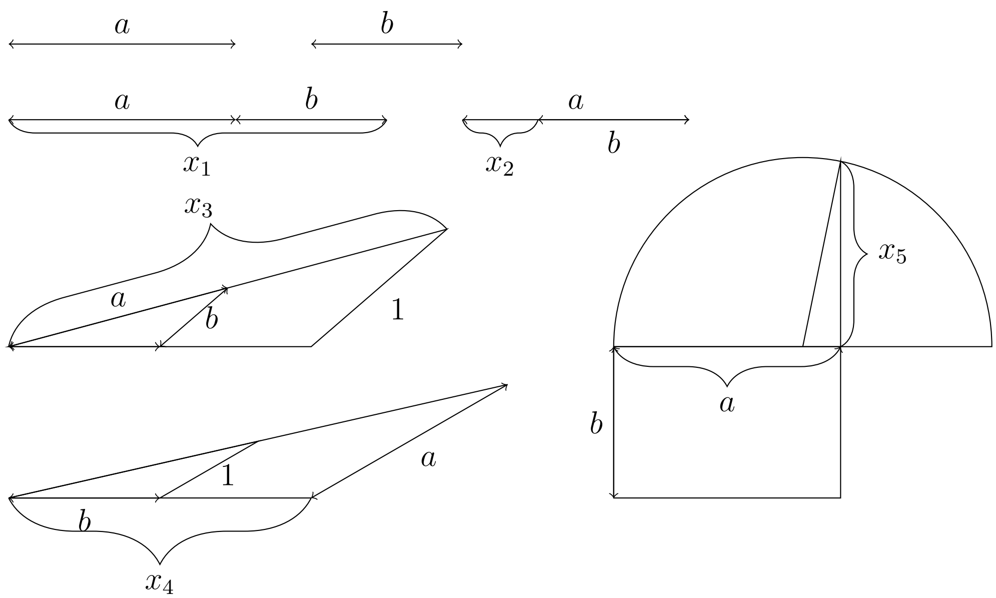
1 Constructible Numbers:

Definition 1 (Constructible Number)**.** Very roughly speaking, a *Constructible Number* is a magnitude for which we can create a straight line whose length equals that magnitude using only a straight edge and compass.

To understand what sorts of magnitudes we can construct it is helpful to understand that we can use a straightedge and compass to add, subtract, multiply, divide, and find square roots. Study the five diagrams below and try to identify which x_i represents addition, subtraction, multiplication, division, and square roots. Be sure to justify your answer.



With a straight edge we create straight lines which we can express with linear equations like

$$ax + by = c$$

and with a compass we can make circles which we can represent with quadratic equations like

$$(x - k)^2 + (y - h)^2 = r^2.$$

Given that a straight edge and compass are our only tools, so all we can do is make lines and circles, loosely why does it make since that we can't construct the cube root of most numbers?

2 Algebraic Numbers:

Definition 2. Again, very loosely, a number is *Algebraic over the Rationals* if it is a root of a polynomial with rational coefficients.

For example, $a = \sqrt[3]{3}$ is not constructible but it is *algebraic* because it is a root of $f(x) = x^3 - 3$. Using the same $f(x)$, so $f\left(\sqrt[3]{3}\right) = 0$, try and find a root of each of the following functions:

1. $g(x) = f\left(x + 3/2\right)$
2. $h(x) = f\left(7x\right)$
3. $k(x) = f\left(x^7\right)$
4. $l(x) = x^3 \cdot f\left(1/x\right)$

Try to use the same function, $f(x)$, together with the ideas you just looked at, to create a polynomial $n(x)$ with rational coefficients and a root of

$$b = \frac{2}{1 + \sqrt[3]{3}} \, .$$

Finally, if we have a polynomial $f(x)$ with rational coefficients and a root α (i.e. α is algebraic), show that $(\alpha + r)$, $(r \cdot \alpha)$, $(\sqrt[k]{\alpha})$, and $(1/\alpha)$ are also algebraic where $r \in \mathbb{Q} \setminus \{0\}$ and $k \in \mathbb{Z}$.

3 Every Thing Else

As a final note, numbers such as π and e are not constructible or algebraic, they are what is called *Transcendental*. These numbers actually make up the “majority” of numbers.