

Lectures on Multivariable Mathematics: Bases for Vector Spaces

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Table of Contents

- 1 Objectives
- 2 Systems of Equations
- 3 Span
- 4 Independence
- 5 Bases
- 6 Changing Bases

Objectives

After this lesson you should be able to:

- 1 Recall using matrix reductions to solve systems of equations,

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- 6 Change calculations from one basis to another.

Table of Contents

- 1 Objectives
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This System is Too “Small”

$$\begin{aligned}2x - y + 3z &= 9 \\ y - z &= -1\end{aligned}$$

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$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 9 \\ 0 & 1 & -1 & -1 \end{array} \right) \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 0 & 2 & 8 \\ 0 & 1 & -1 & -1 \end{array} \right)$$
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$x = 4 - z$, $y = z - 1$, & z is a free variable

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$$\begin{aligned}\left(\begin{array}{cc|c}1 & 0 & 6 \\2 & 7 & 0 \\0 & 1 & 9\end{array}\right) &\rightsquigarrow \left(\begin{array}{cc|c}1 & 0 & 6 \\0 & 7 & -12 \\0 & 1 & 9\end{array}\right) \\&\rightsquigarrow \left(\begin{array}{cc|c}1 & 0 & 6 \\0 & 0 & -75 \\0 & 1 & 9\end{array}\right)\end{aligned}$$

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$$\left(\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 0 & -75 \\ 0 & 1 & 9 \end{array} \right)$$

This is an inconsistent system. The only time there will be a solution is if the values on the right are of the form

$$\langle a_1, 2a_1 + 7a_2, a_2 \rangle.$$

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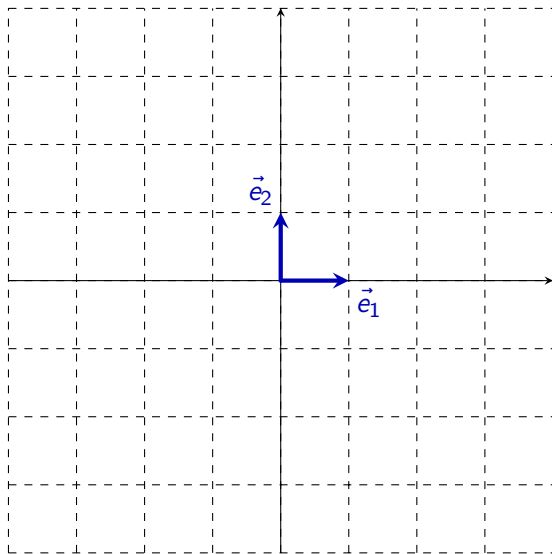
$$\begin{aligned} \left(\begin{array}{cc|c} 2 & -1 & 13 \\ 1 & -1 & -4 \end{array} \right) &\rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & -4 \\ 2 & -1 & 13 \end{array} \right) \\ &\rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & -4 \\ 0 & 1 & 21 \end{array} \right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 17 \\ 0 & 1 & 21 \end{array} \right) \end{aligned}$$

$$x = 17 \text{ \& } y = 21$$

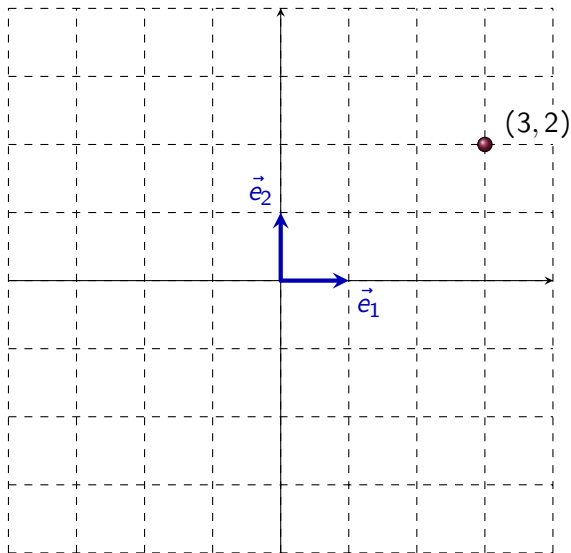
Table of Contents

- 1 Objectives
- 2 Systems of Equations
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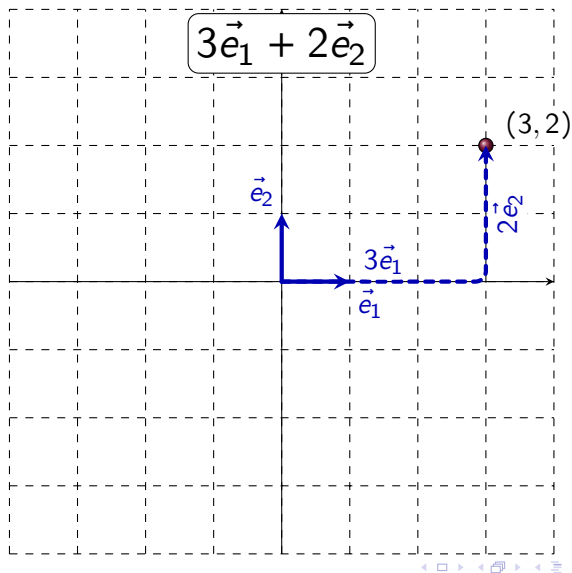
Navigating in two Dimensions



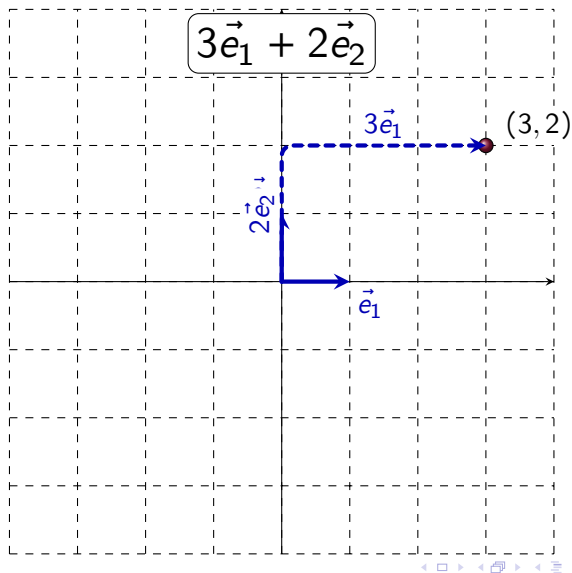
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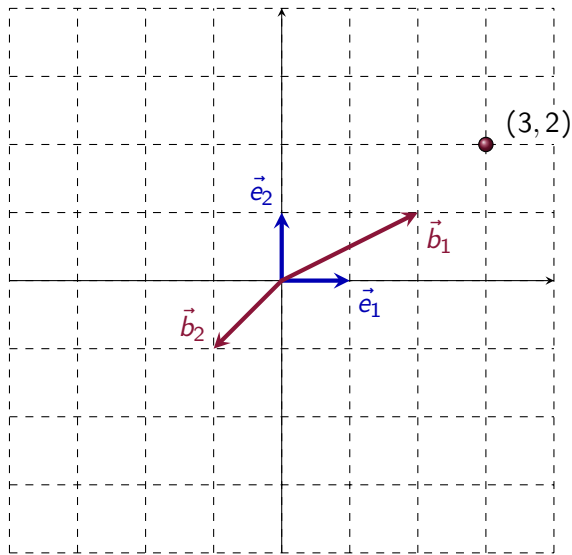
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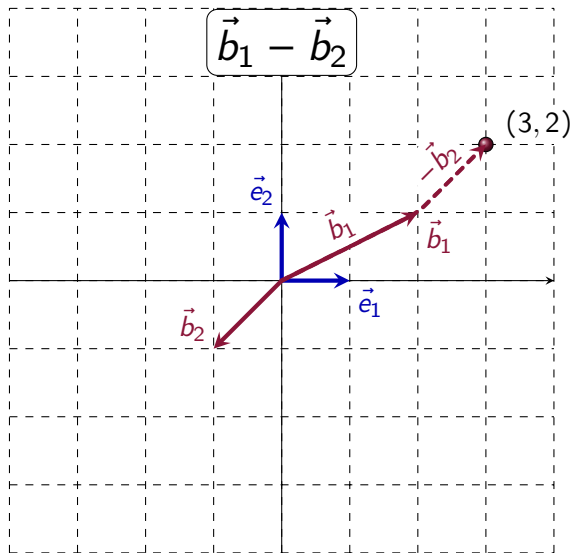
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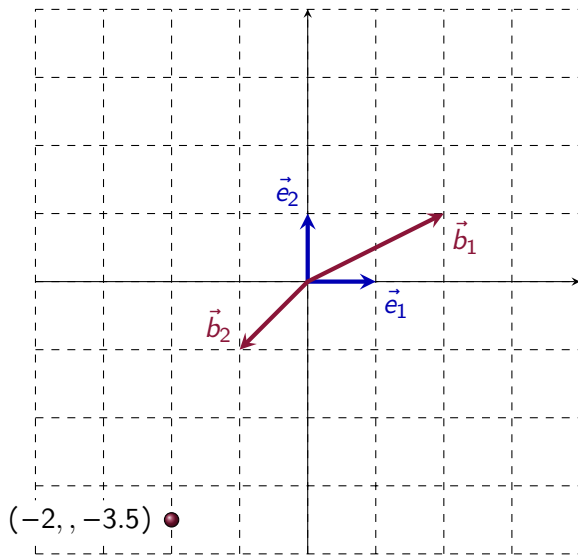
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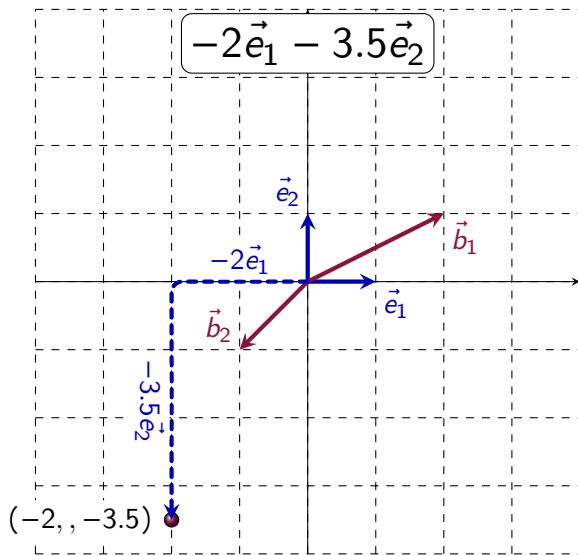
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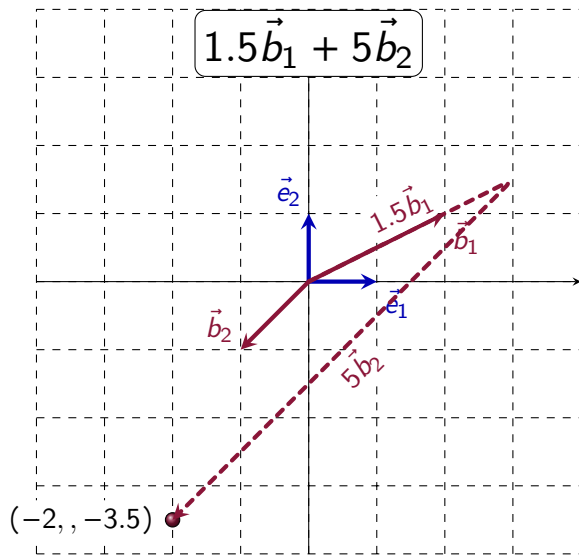
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Span of Vectors

Definition (Span)

Given a set of vectors

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \dots, \vec{b}_k\},$$

the span of the set is the set of all possible linear combinations of the vectors

$$\vec{v} = a_1 \vec{b}_1 + a_2 \vec{b}_2 + a_3 \vec{b}_3 + \dots + a_k \vec{b}_k.$$

Example 1

Let $\mathcal{B} = \{\langle 2, 1 \rangle, \langle -1, -1 \rangle\}$, then the goal is to see what vectors we can write like so:

$$\vec{v} = a_1 \vec{b}_1 + a_2 \vec{b}_2 = a_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

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In this case we can use a matrix inverse to get

$$\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \vec{v} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

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So if $\vec{v} = \langle 7, -3 \rangle$, then

$$\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ 13 \end{pmatrix}.$$

And we have

$$\vec{v} = 10\vec{b}_1 + 13\vec{b}_2 = 10 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 13 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \end{pmatrix} + \begin{pmatrix} -13 \\ -13 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}. \quad \checkmark$$

Example 2

Now let $\mathcal{C} = \{\langle 2, 0 \rangle, \langle -1, 1 \rangle, \langle 3, -1 \rangle\}$; what vectors can be written as a linear combination of these like

$$\vec{v} = a_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}?$$

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For example

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \vec{0}.$$

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But we also have

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = \vec{0}.$$

Solutions are not unique!

Example 3

Finally if $\mathcal{D} = \{\langle 1, 2, 0 \rangle, \langle 0, 7, 1 \rangle\}$ then the linear combinations look like

$$\vec{v} = a_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

But, this will only ever give vectors of the form $\langle a_1, 2a_1 + 7a_2, a_2 \rangle$, and no combination will give, for example, the vector $\langle 1, 0, 1 \rangle$.

Table of Contents

- 1 Objectives
- 2 Systems of Equations
- 3 Span
- 4 Independence**
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Independence of Vectors

Definition (Linear Independence)

A set of vectors

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \dots, \vec{b}_k\},$$

is **linearly independent** if and only if none of the vectors $\vec{b}_i$ can be written as a linear combination of the other vectors, or equivalently the only solution to

$$a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3 + \dots + a_k\vec{b}_k = \vec{0}$$

is the trivial solution when $a_i = 0$ for all i .

Example 1 Revisited

We saw that linear combinations of the set of vectors $\mathcal{B} = \{\langle 2, 1 \rangle, \langle -1, -1 \rangle\}$ can be written as a matrix product:

$$\vec{v} = a_1 \vec{b}_1 + a_2 \vec{b}_2 = a_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

And, that the matrix is inevitable

$$B = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix},$$

so that solutions to $B\vec{a} = \vec{v}$ will be unique.

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Alternately when B is row reduced we get the identity matrix

$$B = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} \rightsquigarrow I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which also gives unique solutions.

Example 2 Revisited

Similarly linear combinations of the vectors in $\mathcal{C} = \{\langle 2, 0 \rangle, \langle -1, 1 \rangle, \langle 3, -1 \rangle\}$ can be written as a matrix product

$$\vec{v} = a_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

However when we row reduce this matrix we get:

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

with a pivot position in each row and a free variable we will always have infinitely many solutions. In particular we could write $\vec{0}$ multiple ways.

Example 3 Revisited

With $\mathcal{D} = \{\langle 1, 2, 0 \rangle, \langle 0, 7, 1 \rangle\}$ we had

$$\vec{v} = a_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Row reducing this matrix we get:

$$\begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

with a pivot position in each column, but not each row, and no free variables we won't always have a solution, but when we do it will be unique.

Table of Contents

- 1 Objectives
- 2 Systems of Equations
- 3 Span
- 4 Independence
- 5 Bases**
- 6 Changing Bases

Basis for a Vector Space

Definition (Basis of a Vector Space)

A set of vectors

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \dots, \vec{b}_k\},$$

is **basis** for a vector space if and only if for every vector $\vec{v}$ the linear combination

$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3 + \dots + a_k\vec{b}_k$$

has a unique solution. That is $\mathcal{B}$ is a linearly independent set which spans the entire vector space. Note that the coefficients a_i are called the **$\mathcal{B}$ – coordinates** of $\vec{v}$ and we write

$$[\vec{v}]_{\mathcal{B}} = \langle a_1, a_2, \dots, a_n \rangle.$$

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- but if $\mathcal{B} = \{\langle 2, 1 \rangle, \langle -1, -1 \rangle\}$ every vector in $\mathbb{R}^2$ can be written as a unique linear combination (This set is just right)

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Theorem

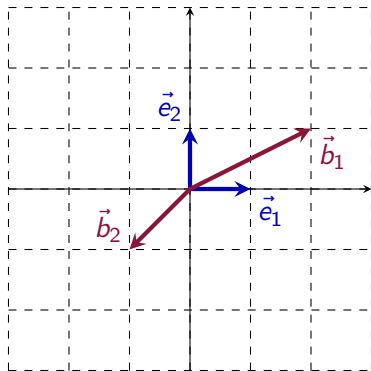
A set of vectors is a basis for a vector space if and only if a matrix whose columns are the elements of the set is invertible.

Table of Contents

- 1 Objectives
- 2 Systems of Equations
- 3 Span
- 4 Independence
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Calculations in One Basis or Another

Depending on what we want to do sometimes $\mathcal{E} = \{\langle 1, 0 \rangle, \langle 0, 1 \rangle\}$ is easier to work with and sometimes something like $\mathcal{B} = \{\langle 2, 1 \rangle, \langle -1, -1 \rangle\}$ is what we might want. So, how can we always change to a more convenient basis for the job?



Change of Basis Matrix

Theorem (Change of Basis Matrix)

Given two bases for the same vector space, $\mathcal{B}$ and $\mathcal{C}$, and matrices B and C whose columns are the vectors in each of the bases, then the *change of basis matrix* from $\mathcal{B}$ to $\mathcal{C}$ is $C^{-1}B$. That, is if

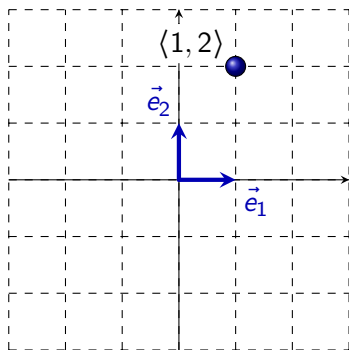
$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3 + \cdots + a_k\vec{b}_k$$

so the $\mathcal{B}$ – coordinates of $\vec{v}$ are

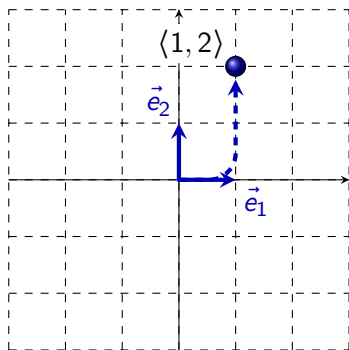
$$[\vec{v}]_{\mathcal{B}} = \langle a_1, a_2, \dots, a_n \rangle,$$

then the $\mathcal{C}$ – coordinates of $\vec{v}$ will be

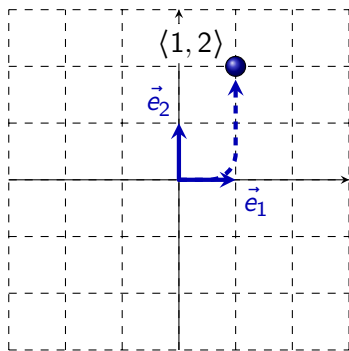
$$[\vec{v}]_{\mathcal{C}} = C^{-1}B[\vec{v}]_{\mathcal{B}} = C^{-1}B\langle a_1, a_2, \dots, a_n \rangle.$$

Elementary to $\mathcal{B}$ 

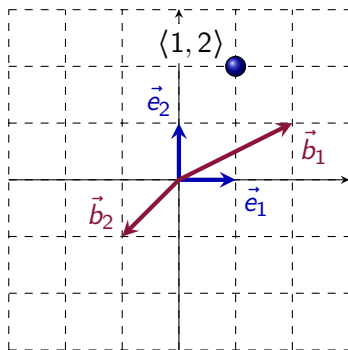
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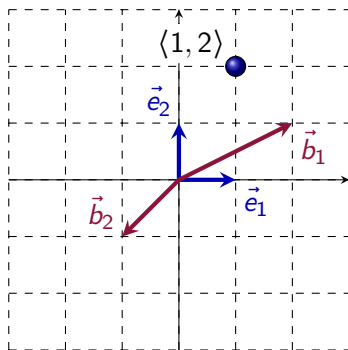
- $\mathcal{E} = \{\langle 1, 0 \rangle, \langle 0, 1 \rangle\}$
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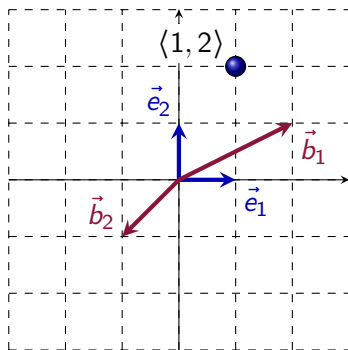
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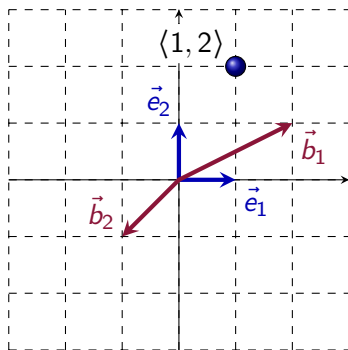
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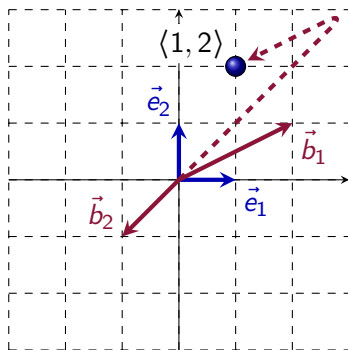
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Elementary to $\mathcal{B}$ 

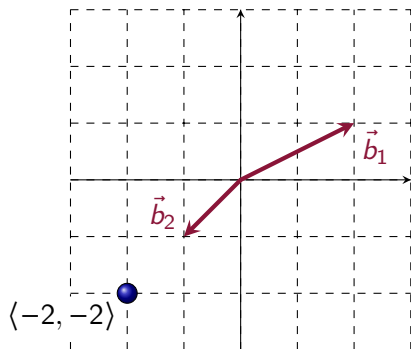
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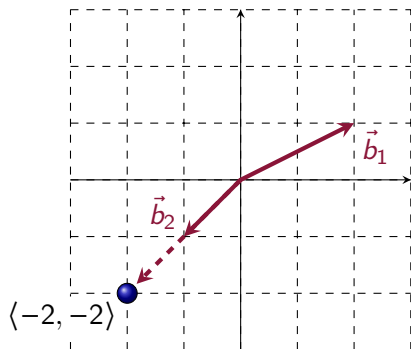
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$\mathcal{B}$ to Elementary

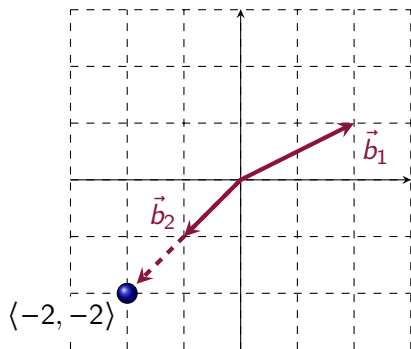
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$\mathcal{B}$ to Elementary



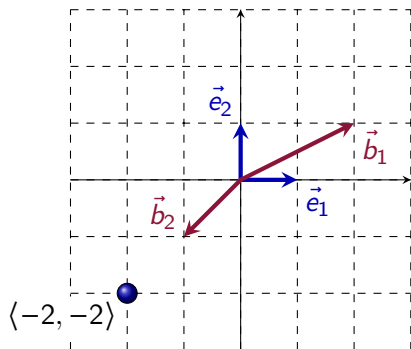
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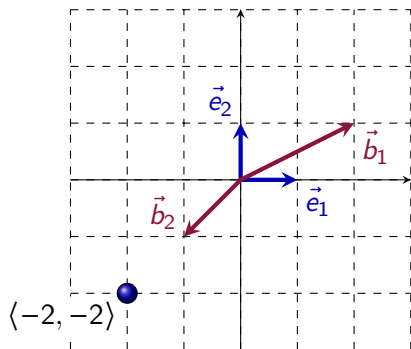
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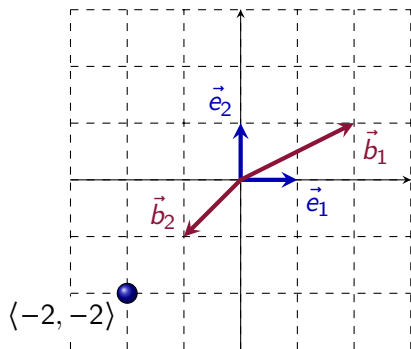
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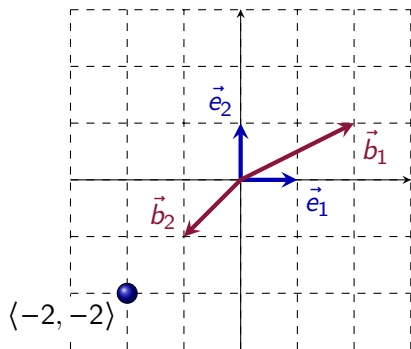
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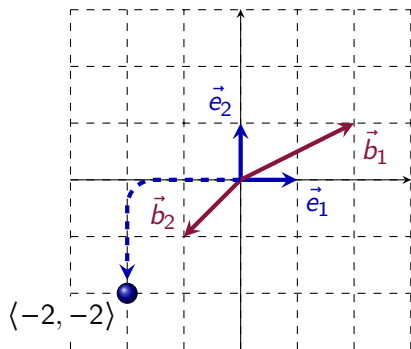
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$\mathcal{B}$ to $\mathcal{C}$ to $\mathcal{B}$ (there and back again) $\mathcal{B}$

- $\mathcal{E}, \mathcal{B}, \mathcal{C}$ bases

 $\mathcal{E}$ $\mathcal{C}$

$\mathcal{B}$ to $\mathcal{C}$ to $\mathcal{B}$ (there and back again)

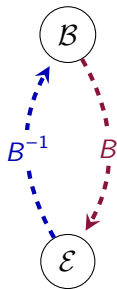


• $\mathcal{E}, \mathcal{B}, \mathcal{C}$ bases

• $\mathcal{B} : \mathcal{B} \longrightarrow \mathcal{E}$



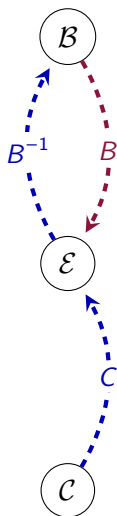
$\mathcal{B}$ to $\mathcal{C}$ to $\mathcal{B}$ (there and back again)



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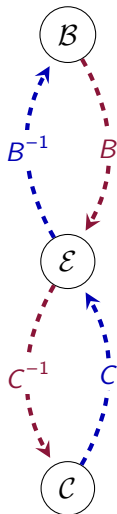


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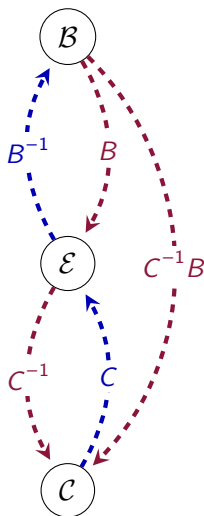
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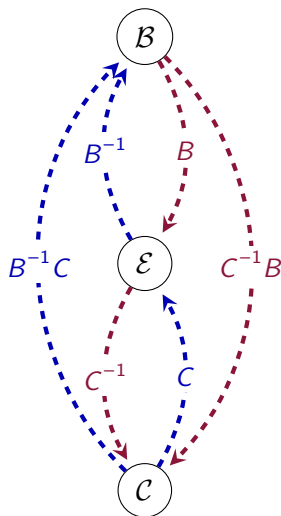
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- $C^{-1}B : \mathcal{B} \rightarrow \mathcal{C}$

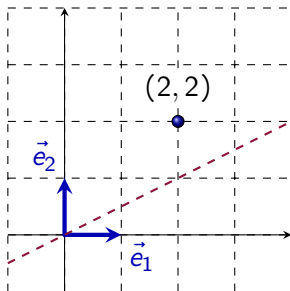
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- $B^{-1}C : \mathcal{C} \longrightarrow \mathcal{B}$

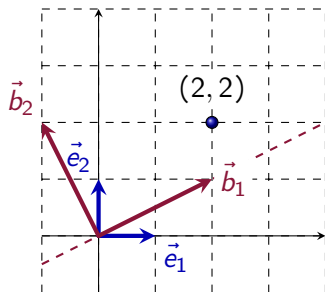
Making Calculations Easier: Reflection Problem

Reflect the point $P = (2, 3)$ in the line $y = \frac{1}{2}x$.



Making Calculations Easier: Reflection Problem

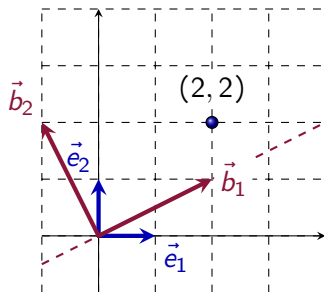
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- Create an **orthogonal basis** with a vector parallel to the line: $\mathcal{B} = \{\langle 2, 1 \rangle, \langle -1, 2 \rangle\}$

Making Calculations Easier: Reflection Problem

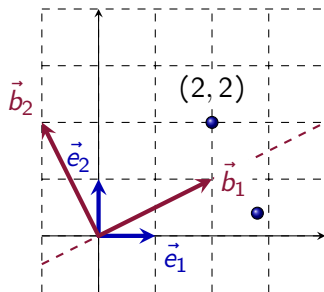
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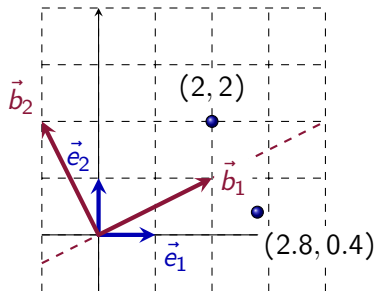


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$$Rf_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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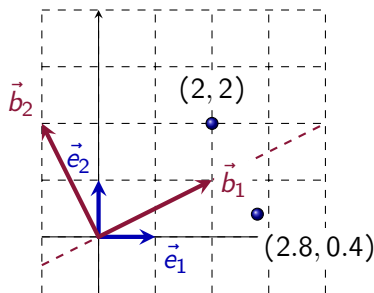
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- Change from $\mathcal{B}$ back to $\mathcal{E}$ with B
- **The complete calculation:**

$$BRf_2B^{-1} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2.8 \\ 0.4 \end{pmatrix}$$

Conjugation

Definition (Conjugation)

Given a transformation P with inverse P^{-1} and another transformation A , we say that

$$B = P \circ A \circ P^{-1}$$

is a **conjugate** of A . If A , B , and P are matrices then we say A and B are **similar matrices**.

Lectures on Multivariable Mathematics: Bases for Vector Spaces

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