Scaffolded Induction Extra Credit

Below is a scaffolded proof by induction exercise, with sample solutions given. Complete these exercises using the given work.

1. Read over all the given work carefully.
2. Write out explanations for all the lines with detail markers (the little red numbers "**(1)**"). Remember, for each number you should answer ***two*** questions:
   (a) Why the statement is true?
   (b) Why is the statement there?
3. Using the given material, write out a proof by induction for this theorem in grammatically correct sentences and paragraphs, and using proper mathematical terminology, notation, and formatting.
4. Prove Theorem 2, the extension of Theorem 1.

Theorem 1. *If* $n \in \mathbb{N}$, *then* 7 *divides* $8^n - 1$.

- Show that Theorem 1 is true for $n = 2$. Then, show it is true for 3 using the fact that it is true for 2.

$$
\begin{aligned}
n = 2 \text{ implies } 8^n - 1 &= 8^2 - 1 = 63 = 7 \cdot 9, \text{ so } 7|8^n - 1 \\
n = 3 \text{ implies } 8^n - 1 &= 8^3 - 1 = (8^2 \cdot 8) - 1^{\mathbf{(1)}} \\
&= ((7 \cdot 9 + 1)(7 + 1)) - 1^{\mathbf{(2)}} = (49 \cdot 9 + 7 \cdot 9 + 7 + 1) - 1 \\
&= 7(7 \cdot 9 + 9 + 1), \text{ so } 7|8^n - 1^{\mathbf{(3)}}
\end{aligned}
$$

- Write out the mathematical steps of the *base case* for a *proof by induction* of Theorem 1 (i.e. don't worry about sentences for this question).

Base Case: $n = 1$ implies $8^n - 1 = 8 - 1 = 7$, so $7|8^n - 1$

- Write out the mathematical steps of the *induction step* for a *proof by induction* of Theorem 1 (i.e. don't worry about sentences for this question).

Assumption: For $n = k$, assume $7|8^k - 1$, i.e. $8^k - 1 = 7q^{\mathbf{(4)}}$
Induction Step: Let $n = k + 1$, then

$$
\begin{aligned}
8^n - 1 = 8^{k+1} - 1 &= (8^k \cdot 8) - 1^{\mathbf{(5)}} \\
&= ((7 \cdot q + 1)(7 + 1)) - 1^{\mathbf{(6)}} = (49 \cdot q + 7 \cdot q + 7 + 1) - 1 \\
&= 7(7 \cdot q + q + 1), \text{ so } 7|8^n - 1^{\mathbf{(7)}}
\end{aligned}
$$

Theorem 2 (Extension)**.** *Given* $m \in \mathbb{N}$, *if* $n \in \mathbb{N}$, *then* m *divides* $(m + 1)^n - 1$.