Grammars and Pushdown Automata

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A Simple Example

Grammar:

$$A \;!\; 0A1$$
$$A \;!\; B$$
$$B \;!\; \#$$

A

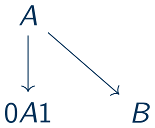
A Simple Example

Grammar:

A ! $0A1$

A ! B

B ! $\#$



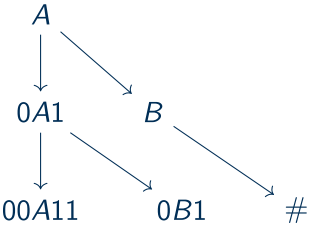
A Simple Example

Grammar:

$A \;\; ! \;\; 0A1$

$A \;\; ! \;\; B$

$B \;\; ! \;\; \#$



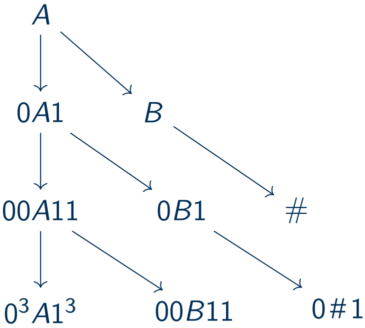
A Simple Example

Grammar:

A ! $0A1$

A ! B

B ! $\#$



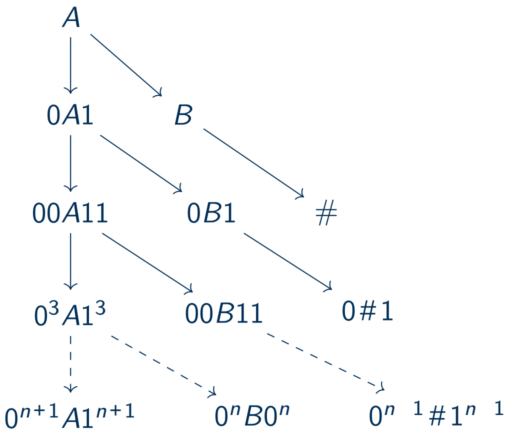
A Simple Example

Grammar:

A ! $0A1$

A ! B

B ! #



A Simple Example: Derivation and Parse Tree

Grammar:

$$A \; ! \; 0A1$$
$$A \; ! \; B$$
$$B \; ! \; \#$$

Derivation of 000#111:

$$A \;)$$

Parse Tree for 000#111:

$$A$$

A Simple Example: Derivation and Parse Tree

Grammar:

$A\ !\ 0A1$

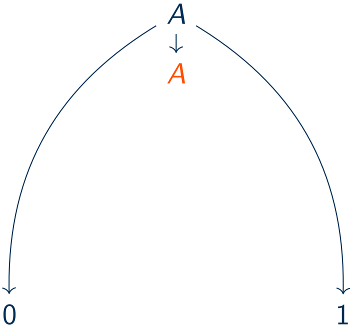
$A\ !\ B$

$B\ !\ \#$

Derivation of 000#111:

$A\ )\ 0A1$

Parse Tree for 000#111:



A Simple Example: Derivation and Parse Tree

Grammar:

A ! 0*A*1

A ! *B*

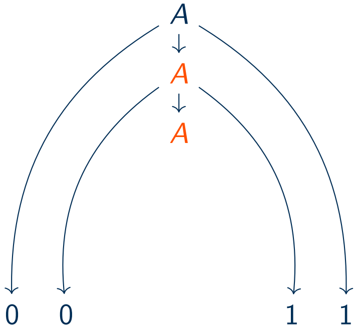
B ! #

Derivation of 000#111:

A) 0*A*1

) 00*A*11

Parse Tree for 000#111:



A Simple Example: Derivation and Parse Tree

Grammar:

$A \; ! \; 0A1$

$A \; ! \; B$

$B \; ! \; \#$

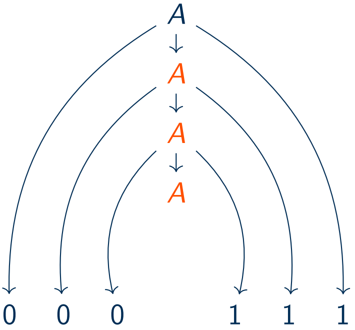
Derivation of 000#111:

$A \;)\; 0A1$

$\;)\; 0\,0A1\,1$

$\;)\; 00\,0A1\,11$

Parse Tree for 000#111:



A Simple Example: Derivation and Parse Tree

Grammar:

A ! $0A1$

A ! B

B ! #

Derivation of 000#111:

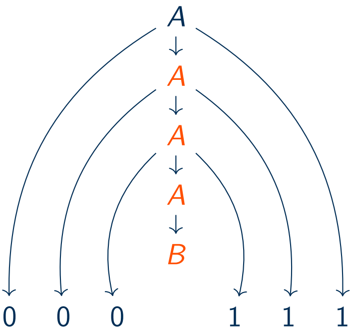
A) $0A1$

) $0\,0A1\,1$

) $00\,0A1\,11$

) $000\,B\,111$

Parse Tree for 000#111:



A Simple Example: Derivation and Parse Tree

Grammar:

$$A \;\; ! \;\; 0A1$$
$$A \;\; ! \;\; B$$
$$B \;\; ! \;\; \#$$

Derivation of 000#111:

$$A \;\;) \;\; 0A1$$
$$) \;\; 0\,0A1\,1$$
$$) \;\; 00\,0A1\,11$$
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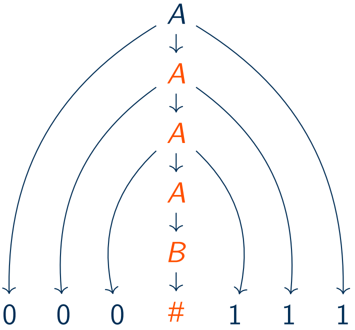
$$) \;\; 000\,\#\,111$$

Parse Tree for 000#111:



Formal Definition

Definition (Context-Free Grammar)

A ***context-free grammar*** is a 4-tuple $(V; \quad ; R; S)$, where

1. V is a finite set called the ***variables***,

Grammar:

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Variables: $V = \{A, B\}$

Formal Definition

Definition (Context-Free Grammar)

A ***context-free grammar*** is a 4-tuple $(V;\quad ;R;S)$, where

1. V is a finite set called the ***variables***,
2. is a finite set, disjoint from V, called the ***terminals***,

Grammar:

A ! 0A1

A ! B

B ! #

Terminals: = $f0;1;\#g$

Formal Definition

Definition (Context-Free Grammar)

A ***context-free grammar*** is a 4-tuple $(V; \ ; R; S)$, where

1. V is a finite set called the ***variables***,
2. is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule associating a variable to a string of variables and/or terminals, and

Grammar:

$$A \ ! \ 0A1$$
$$A \ ! \ B$$
$$B \ ! \ \#$$

Rules

Formal Definition

Definition (Context-Free Grammar)

A ***context-free grammar*** is a 4-tuple $(V; \quad ; R; S)$, where

1. V is a finite set called the ***variables***,
2. is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule associating a variable to a string of variables and/or terminals, and
4. S 2 V is a ***start variable***.

Grammar:

$$A \ ! \ 0A1$$
$$A \ ! \ B$$
$$B \ ! \ \#$$

Start Variable: $S = A$

Designing a CFG

$w \quad w^R$

Give a CFG what recognizes the language of all words in $= \{0, 1\}$ which end with the reverse of their beginnings, e.g. 10110 0101010 01101.

Arbitrary Word:

$$S_1 \to 0S_1 | 1S_1 | \varepsilon$$

Designing a CFG

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Even Length Palindrome, ww^R:

$$S_2 \rightarrow 0X0 | 1X1$$
$$X \rightarrow S_2 | \varepsilon$$

Designing a CFG

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Arbitrary Word:

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Even Length Palindrome, ww^R:

$$S_2 \rightarrow 0X0 | 1X1$$
$$X \rightarrow S_2 | \varepsilon$$

Combining Them, $w \quad w^R$:

$$S \rightarrow S_2$$
$$S_2 \rightarrow 0X0 | 1X1$$
$$X \rightarrow S_2 | S_1$$
$$S_1 \rightarrow 0S_1 | 1S_1 | \varepsilon$$

A Couple Variations

Combining Grammars

$$S_1 \;\; ! \;\; 0S_1 j 1 S_1 j "$$

$$\begin{aligned} S_2 \;\; &! \;\; 0X0j1X1 \\ X \;\; &! \;\; S_2 j " \end{aligned}$$

$$\begin{aligned} S \;\; &! \;\; S_2 \\ S_2 \;\; &! \;\; 0X0j1X1 \\ X \;\; &! \;\; S_2 j S_1 \\ S_1 \;\; &! \;\; 0S_1 j 1 S_1 j " \end{aligned}$$

A Couple Variations

Combining Grammars

$$S_1 \;!\; 0S_1 j 1 S_1 j''$$

$$S_2 \;!\; 0X0 j 1X1$$
$$X \;!\; S_2 j''$$

$$S \;!\; S_2$$
$$S_2 \;!\; 0X0 j 1X1$$
$$X \;!\; S_2 j S_1$$
$$S_1 \;!\; 0S_1 j 1 S_1 j''$$

$$S \;!\; S_1 j S_2$$
$$S_2 \;!\; 0X0 j 1X1$$
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A Couple Variations

Combining Grammars

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$$\begin{aligned} S \;\; &! \;\; S_1jS_2 \\ S_2 \;\; &! \;\; 0X0j1X1 \\ X \;\; &! \;\; S_2j" \\ S_1 \;\; &! \;\; 0S_1j1S_1j" \end{aligned}$$

$$\begin{aligned} S \;\; &! \;\; S_1S_2 \\ S_2 \;\; &! \;\; 0X0j1X1 \\ X \;\; &! \;\; S_2j" \\ S_1 \;\; &! \;\; 0S_1j1S_1j" \end{aligned}$$

$$\begin{aligned} S \;\; &! \;\; S_2S_2 \\ S_2 \;\; &! \;\; 0X0j1X1jS \\ X \;\; &! \;\; S_2j" \end{aligned}$$

Example with Ambiguity

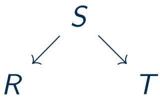
Given the grammar:

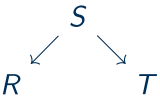
S ! RT

R ! aRjRbj"

T ! Tbj"

there are multiple ways to derive the string *aab*. This is an ***ambiguous*** grammar.





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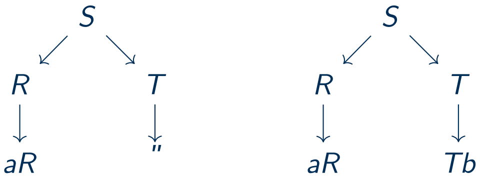

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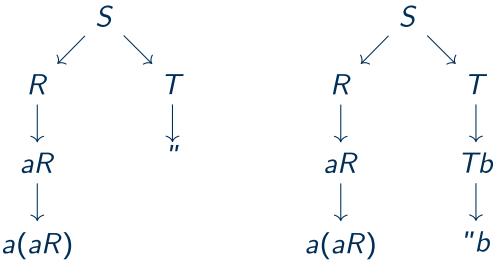

Example with Ambiguity

Given the grammar:

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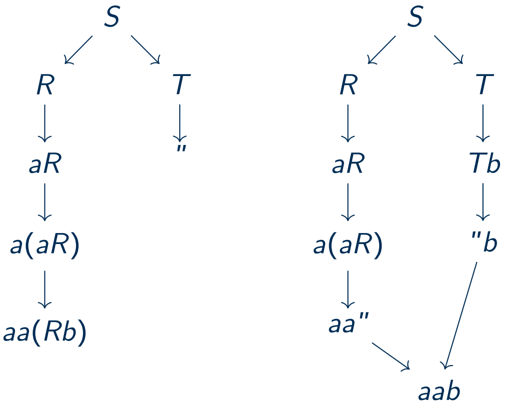

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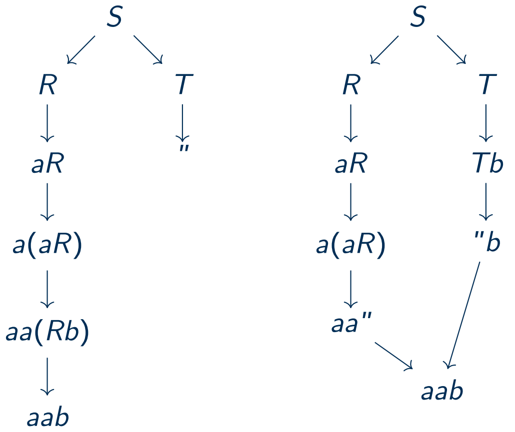

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Chomsky Normal Form

Definition

A context-free grammar is in ***Chomsky Normal Form*** if every rule is of the form

$$A \to BC \text{ or}$$
$$A \to a$$

where a is a terminal, $A, B,$ and C are variables, B and C are not start variables, and only a start variable my point to the empty string ε

Converting to Chomsky Normal Form

New Start:
Replace

$$S \;\; ! \;\; Ajb$$
$$A \;\; ! \;\; aSja$$

With

$$S_0 \;\; ! \;\; S$$
$$S \;\; ! \;\; Ajb$$
$$A \;\; ! \;\; aSja$$

Converting to Chomsky Normal Form

New Start:
Replace

$$S \ ! \ Aj b$$
$$A \ ! \ aSj a$$

With

$$S_0 \ ! \ S$$
$$S \ ! \ Aj b$$
$$A \ ! \ aSj a$$

No ":
Replace

$$A \ ! \ XBY$$
$$B \ ! \ "$$

With

$$A \ ! \ XBY$$
$$A \ ! \ XY$$

Converting to Chomsky Normal Form

New Start:
Replace

$$S \; ! \; Ajb$$
$$A \; ! \; aSja$$

With

$$S_0 \; ! \; S$$
$$S \; ! \; Ajb$$
$$A \; ! \; aSja$$

No ":
Replace

$$A \; ! \; XBY$$
$$B \; ! \; "$$

With

$$A \; ! \; XBY$$
$$A \; ! \; XY$$

No Units:
Replace

$$A \; ! \; B$$
$$B \; ! \; XYja$$

With

$$A \; ! \; XYja$$
$$B \; ! \; XYja$$

Converting to Chomsky Normal Form

New Start:
Replace

$$S \; ! \; Ajb$$
$$A \; ! \; aSja$$

With

$$S_0 \; ! \; S$$
$$S \; ! \; Ajb$$
$$A \; ! \; aSja$$

No ":
Replace

$$A \; ! \; XBY$$
$$B \; ! \; "$$

With

$$A \; ! \; XBY$$
$$A \; ! \; XY$$

No Units:
Replace

$$A \; ! \; B$$
$$B \; ! \; XYja$$

With

$$A \; ! \; XYja$$
$$B \; ! \; XYja$$

New Variables:
Replace:

$$A \; ! \; BXY$$
$$B \; ! \; bX$$

With:

$$A \; ! \; BU$$
$$U \; ! \; XY$$
$$B \; ! \; VX$$
$$V \; ! \; b$$

Chomsky Example Steps 1-3

$S_0 \; ! \; 0X0j1X1$
$X \; ! \; S_0 j S_1 j"$
$S_1 \; ! \; 0S_1 j 1S_1 j"$

$S \; ! \; S_0$
$S_0 \; ! \; 0X0j1X1$
$X \; ! \; S_0 j S_1 j \; "$
$S_1 \; ! \; 0S_1 j 1S_1 j \; "$

$S \; ! \; S_0$
$S_0 \; ! \; 0X0j1X1$
$S_0 \; ! \; 00j11$
$X \; ! \; S_0 j S_1$
$S_1 \; ! \; 0S_1 j 1S_1$
$S_1 \; ! \; 0j1$

Chomsky Example Steps 3-5

$S \ ! \ S_0$
$S_0 \ ! \ 0X0j1X1$
$S_0 \ ! \ 00j11$
$X \ ! \ S_0jS_1$
$S_1 \ ! \ 0S_1j1S_1$
$S_1 \ ! \ 0j1$

$S \ ! \ S_0$
$S_0 \ ! \ 0W_0j1W_1$
$W_0 \ ! \ X0$
$W_1 \ ! \ X1$
$S_0 \ ! \ 00j11$
$X \ ! \ S_0jS_1$
$S_1 \ ! \ U_0S_1jU_1S_1$
$U_0 \ ! \ 0$
$U_1 \ ! \ 1$
$S_1 \ ! \ 0j1$

$S \ ! \ S_0$
$S_0 \ ! \ 0W_0j1W_1$
$W_0 \ ! \ X0$
$W_1 \ ! \ X1$
$S_0 \ ! \ 00j11$
$X \ ! \ S_0jS_1$
$S_1 \ ! \ U_0S_1jU_1S_1$
$U_0 \ ! \ 0$
$U_1 \ ! \ 1$
$S_1 \ ! \ 0j1$

Chomsky Example Steps 6-7

$$
\begin{aligned}
S &\ !\ S_0 \\
S_0 &\ !\ U_0 W_0 j U_1 W_1 \\
W_0 &\ !\ X U_0 \\
W_1 &\ !\ X U_1 \\
S_0 &\ !\ U_0 U_0 j U_1 U_1 \\
X &\ !\ S_0 j S_1 \\
S_1 &\ !\ U_0 S_1 j U_1 S_1 \\
U_0 &\ !\ 0 \\
U_1 &\ !\ 1 \\
S_1 &\ !\ 0 j 1
\end{aligned}
$$

$$
\begin{aligned}
S &\ !\ U_0 W_0 j U_1 W_1 j U_0 U_0 j U_1 U_1 \\
S_0 &\ !\ U_0 W_0 j U_1 W_1 \\
W_0 &\ !\ X U_0 \\
W_1 &\ !\ X U_1 \\
S_0 &\ !\ U_0 U_0 j U_1 U_1 \\
X &\ !\ U_0 U_0 j U_1 U_1 j U_0 S_1 j U_1 S_1 \\
S_1 &\ !\ U_0 S_1 j U_1 S_1 \\
U_0 &\ !\ 0 \\
U_1 &\ !\ 1 \\
S_1 &\ !\ 0 j 1
\end{aligned}
$$

Chomsky Example Steps 7-8

$$
\begin{aligned}
S &\;!\; U_0W_0jU_1W_1jU_0U_0jU_1U_1 \\
S_0 &\;!\; U_0W_0jU_1W_1 \\
W_0 &\;!\; XU_0 \\
W_1 &\;!\; XU_1 \\
S_0 &\;!\; U_0U_0jU_1U_1 \\
X &\;!\; U_0U_0jU_1U_1jU_0S_1jU_1S_1 \\
S_1 &\;!\; U_0S_1jU_1S_1 \\
U_0 &\;!\; 0 \\
U_1 &\;!\; 1 \\
S_1 &\;!\; 0j1
\end{aligned}
$$

$$
\begin{aligned}
S &\;!\; U_0W_0jU_1W_1jU_0U_0jU_1U_1 \\
W_0 &\;!\; XU_0 \\
W_1 &\;!\; XU_1 \\
X &\;!\; U_0U_0jU_1U_1jU_0S_1jU_1S_1 \\
S_1 &\;!\; U_0S_1jU_1S_1j0j1 \\
U_0 &\;!\; 0 \\
U_1 &\;!\; 1
\end{aligned}
$$

Converting to Chomsky Normal Form

New Start:
Replace

$$S \;!\; Ajb$$
$$A \;!\; aSja$$

With

$$S_0 \;!\; S$$
$$S \;!\; Ajb$$
$$A \;!\; aSja$$

No ":
Replace

$$A \;!\; XBY$$
$$B \;!\; "$$

With

$$A \;!\; XBY$$
$$A \;!\; XY$$

No Units:
Replace

$$A \;!\; B$$
$$B \;!\; XYja$$

With

$$A \;!\; XYja$$
$$B \;!\; XYja$$

New Variables:
Replace:

$$A \;!\; BXY$$
$$B \;!\; bX$$

With:

$$A \;!\; BU$$
$$U \;!\; XY$$
$$B \;!\; VX$$
$$V \;!\; b$$

Practice Grammar:

$$S \;!\; XYjZ$$
$$X \;!\; aXbj"$$
$$Y \;!\; Ycjcj"$$
$$Z \;!\; aZcjWj"$$
$$W \;!\; bWjbj"$$

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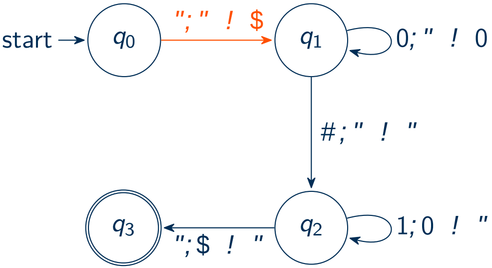
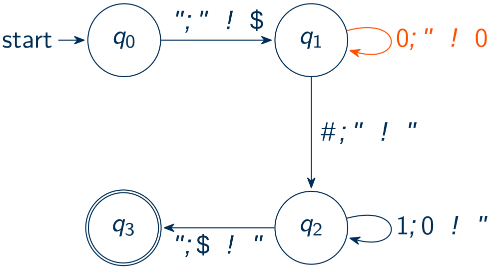
A Simple Example: $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



$ – Empty Stack Symbol
$a; " \ ! \ "$ – read
$"; " \ ! \ b$ – push
$"; b \ ! \ "$ – pop
$a; " \ ! \ b$ – read and push
$a; b \ ! \ "$ – read and pop
$a; b \ ! \ c$ – read, pop, and push

A Simple Example: $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



$ – Empty Stack Symbol
a; " ! " – read
"; " ! *b* – push
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a; " ! *b* – read and push
a; *b* ! " – read and pop
a; *b* ! *c* – read, pop, and push

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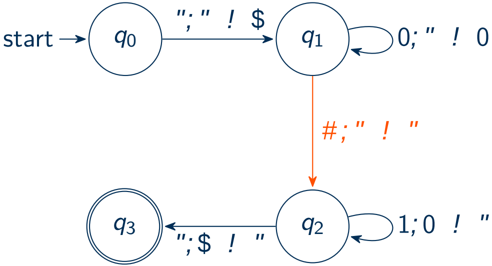

$ – Empty Stack Symbol
a; " ! " – read
"; " ! *b* – push
"; *b* ! " – pop
a; " ! *b* – read and push
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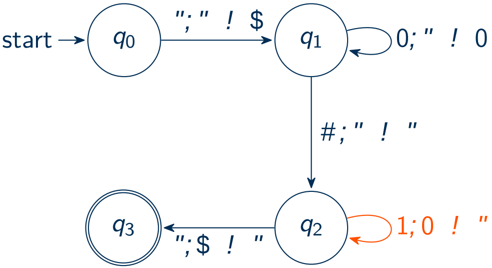

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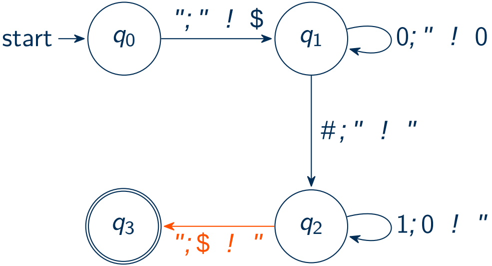

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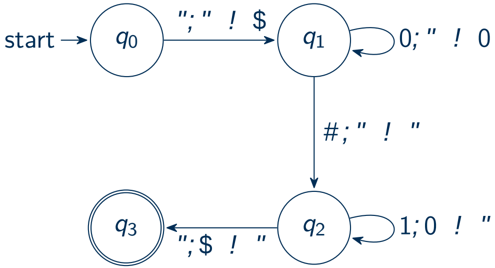
$a;b\ !\ c$ – read, pop, and push

A Simple Example: $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



$ – Empty Stack Symbol
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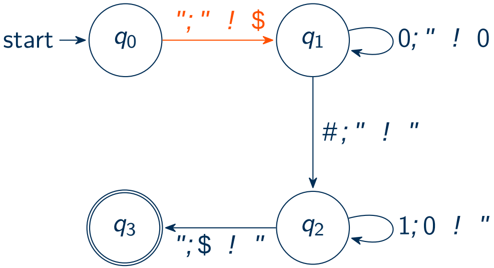
Input Tape:

000#111

Stack: ⊔

A Simple Example: $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



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Input Tape:

000#111

Stack: $

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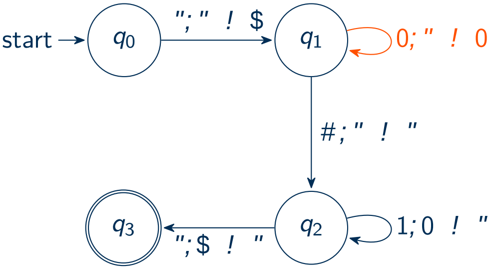

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$"; b \ ! \ "$ – pop
$a; " \ ! \ b$ – read and push
$a; b \ ! \ "$ – read and pop
$a; b \ ! \ c$ – read, pop, and push

Input Tape:

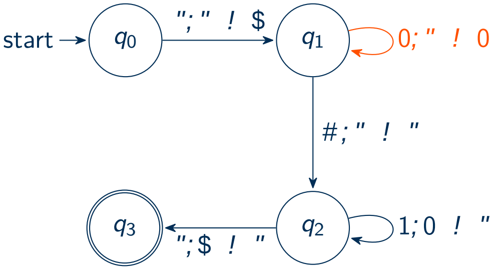
000#111

Stack:

0

\$

A Simple Example: $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



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a; b ! c – read, pop, and push

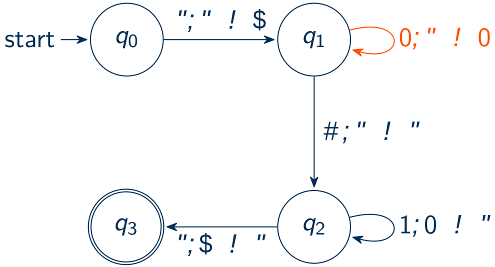
Input Tape:

000#111

Stack:

0
0
$

A Simple Example: $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



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Input Tape:

000#111

Stack:

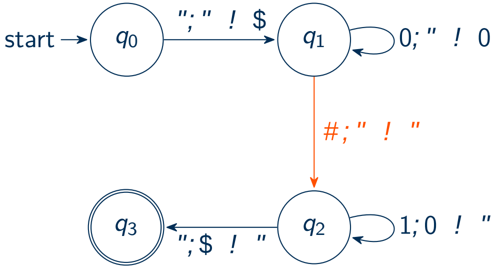
0

0

0

$

A Simple Example: $L = f0^n\#1^n jn\ 2\ \mathbb{N}g$



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$"; b \; ! \; "$ – pop
$a; " \; ! \; b$ – read and push
$a; b \; ! \; "$ – read and pop
$a; b \; ! \; c$ – read, pop, and push

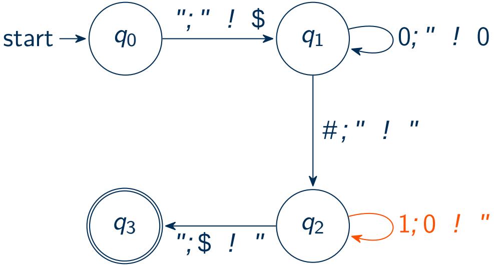
Input Tape:

000#111

Stack:

0
0
0
$

A Simple Example: $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



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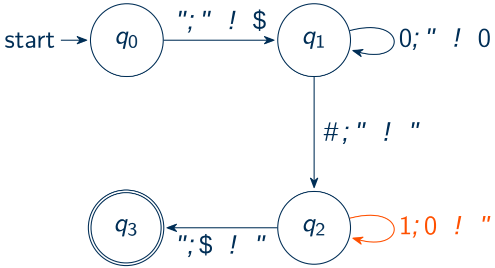
Input Tape:

000#111

Stack:

0
0
$

A Simple Example: $L = f0^n\#1^n jn \ 2 \ \mathrm{N}g$



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a; b ! " – read and pop
a; b ! c – read, pop, and push

Input Tape:

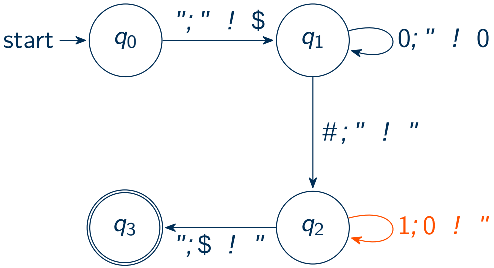
000#111

Stack:

0

$

A Simple Example: $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



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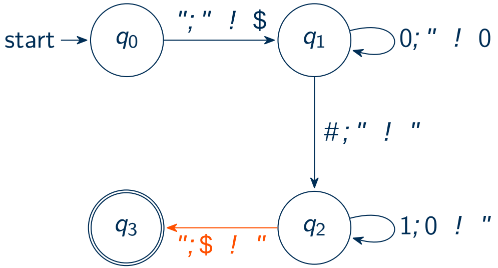
Input Tape:

000#111**1**

Stack: $

A Simple Example: $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



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Input Tape:

000#111

Stack: ␣

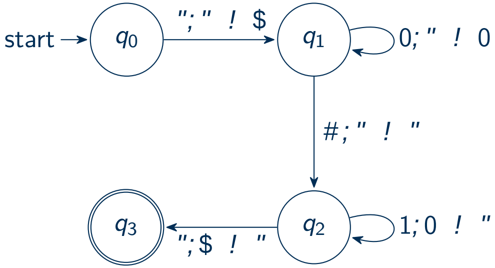
Formal Definition

Definition (Pushdown Automaton)

A **pushdown automaton** is a 6-tuple $(Q; \quad ; \quad ; \quad ; q_0; F)$, where $Q; \quad ; \quad ;$ and F are all finite sets, and

1. Q is the set of states,
2. is the input alphabet,
3. is the stack alphabet,
4. $: Q \quad\quad _\varepsilon \quad\quad _\varepsilon \; ! \; \mathscr{P}(Q \quad\quad _\varepsilon)$ is the transition function,
5. $q_0 \; 2 \; Q$ is the start state, and
6. $F \quad\quad Q$ is the set of accept states.

Second Look at $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



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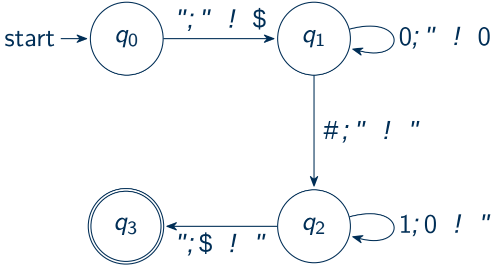
$a;b\ !\ c$ – read, pop, and push

- $(q_0;";") = f(q_1;\$)g$

Second Look at $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



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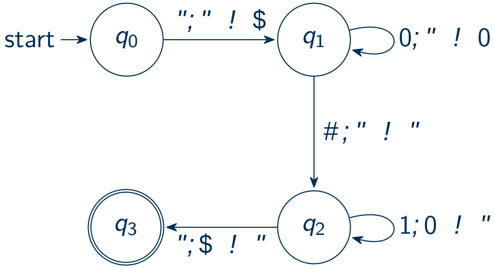
a; *b* ! " – read and pop
a; *b* ! *c* – read, pop, and push

- $(q_0; "; ") = f(q_1; \$)g$
- $(q_1; 0; ") = f(q_1; 0)g$

Second Look at $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



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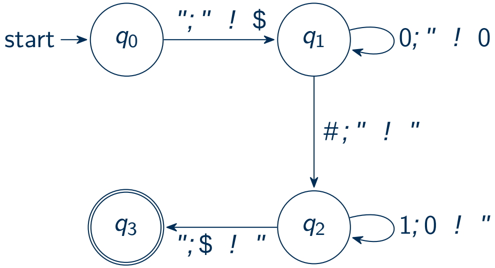
- $(q_0; "; ") = f(q_1; \$)g$
- $(q_1; 0; ") = f(q_1; 0)g$
- $(q_1; \#; ") = f(q_2; ")g$

Second Look at $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



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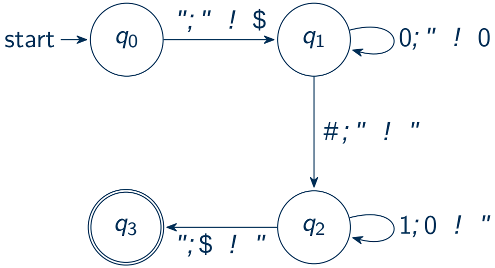
- $(q_1; \#; ") = f(q_2; ")g$
- $(q_2; 1; 0) = f(q_2; ")g$

Second Look at $L = f0^n\#1^n jn\ 2\ \mathrm{N}g$



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- $(q_2; 1; 0) = f(q_2; ")g$
- $(q_2; "; \$) = f(q_3; ")g$

A More Complicated Example:

$$\left\{ a^i b^j c^k \,\middle|\, i; j; k \quad 0 \textit{ and } i = j \textit{ or } i = k \right\}$$

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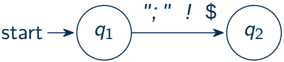

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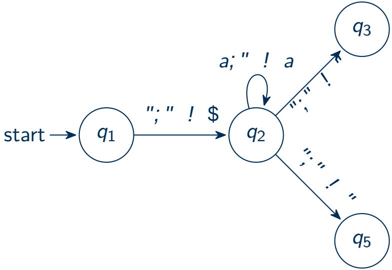

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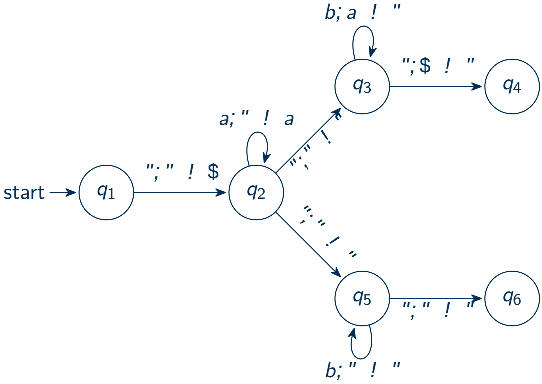

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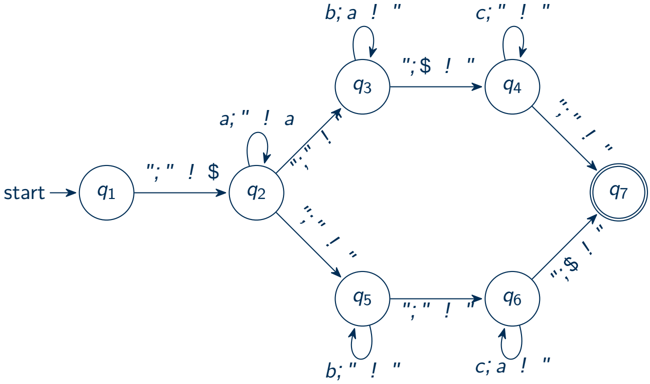

Table of Contents

CFL implies PDA

Context Free
Grammar

$$X \ ! \ XX$$
$$X \ ! \ aXbjbXa$$
$$X \ ! \ "$$

CFL implies PDA

Context Free Grammar

$X\ !\ XX$

$X\ !\ aXbjbXa$

$X\ !\ "$

OR

$X\ !\ XXjaBjbA$

$A\ !\ Xaja$

$B\ !\ Xbjb$

$X\ !\ "$

CFL implies PDA

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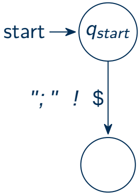

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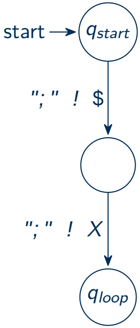

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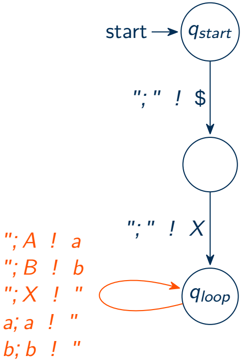

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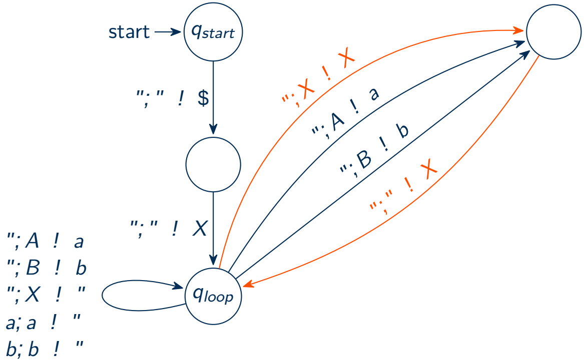

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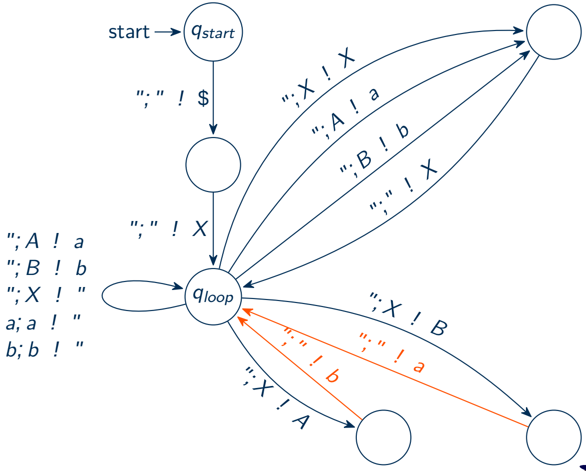

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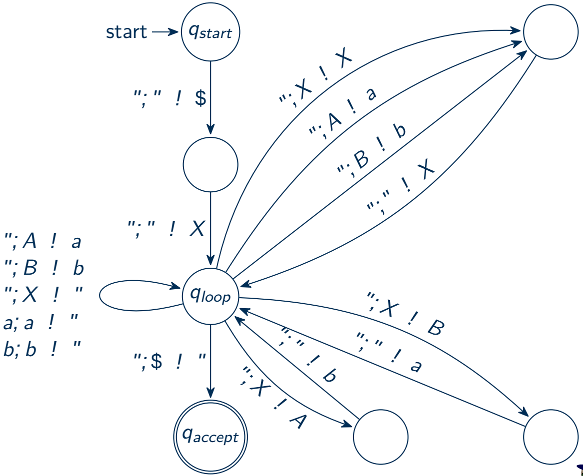

PDA implies CFL (Part 1)

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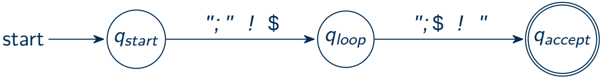

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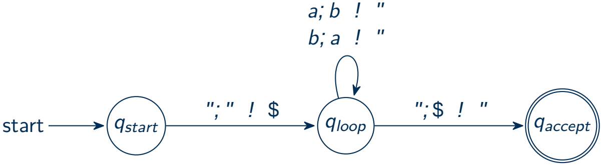

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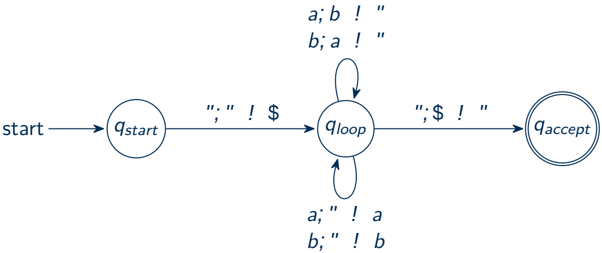
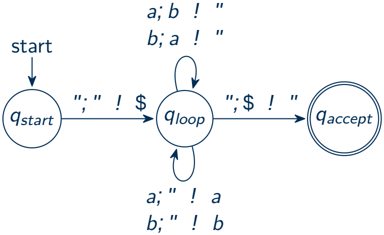
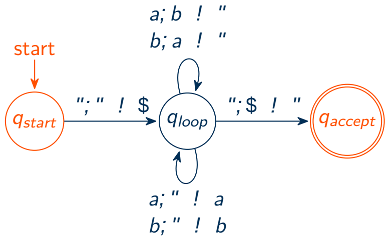


PDA implies CFL (Part 2)



1. Add S ! $A_{start;accept}$ as the start variable.
2. If $(r; u)$ 2 $(p; a; ") \wedge (q; ")$ 2 $(s; b; u)$ then add A_{pq} ! $aA_{rs}b$:
3. For each $p; q; r$ add A_{pq} ! $A_{pr}A_{rq}$:
4. For each p add A_{pp} ! ":

PDA implies CFL (Part 2)



New Rules:

- $S \; ! \; R_{SA}$
- $R_{SS} \; ! \; "$
- $R_{LL} \; ! \; "$
- $R_{AA} \; ! \; "$

1. Add $S \; ! \; A_{start;accept}$ as the start variable.
2. If $(r; u) \; 2 \;\; (p; a; ") \wedge (q; ") \; 2 \;\; (s; b; u)$ then add $A_{pq} \; ! \; aA_{rs}b$:
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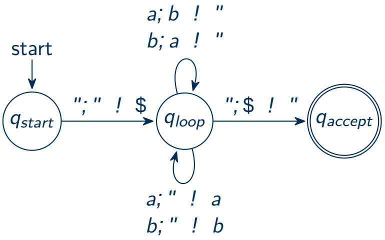

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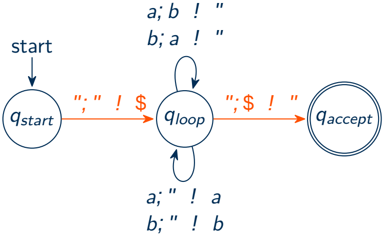
New Rules:

- $S\ !\ R_{SA}$
- $R_{SS}\ !\ "jR_{SS}R_{SS}$
- $R_{LL}\ !\ "jR_{LL}R_{LL}$
- $R_{AA}\ !\ "jR_{AA}R_{AA}$
- $R_{SA}\ !\ R_{SL}R_{LA}$
- $R_{SL}\ !\ R_{SS}R_{SL}jR_{SL}R_{LL}$
- $R_{LA}\ !\ R_{LL}R_{LA}jR_{LA}R_{AA}$

PDA implies CFL (Part 2)

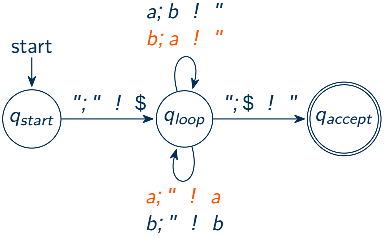


1. Add $S \to A_{start,accept}$ as the start variable.
2. If $(r, u) \in \delta(p, a, \varepsilon) \wedge (q, \varepsilon) \in \delta(s, b, u)$ then add $A_{pq} \to aA_{rs}b.$
3. For each p, q, r add $A_{pq} \to A_{pr}A_{rq}.$
4. For each p add $A_{pp} \to \varepsilon.$

New Rules:

- $S \to R_{SA}$
- $R_{SS} \to \varepsilon | R_{SS}R_{SS}$
- $R_{LL} \to \varepsilon | R_{LL}R_{LL}$
- $R_{AA} \to \varepsilon | R_{AA}R_{AA}$
- $R_{SA} \to R_{SL}R_{LA} | \varepsilon R_{LL} \varepsilon$
- $R_{SL} \to R_{SS}R_{SL} | R_{SL}R_{LL}$
- $R_{LA} \to R_{LL}R_{LA} | R_{LA}R_{AA}$

PDA implies CFL (Part 2)



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- $S \; ! \; R_{SA}$
- $R_{SS} \; ! \; "jR_{SS}R_{SS}$
- $R_{LL} \; ! \; "jR_{LL}R_{LL}jaR_{LL}b$
- $R_{AA} \; ! \; "jR_{AA}R_{AA}$
- $R_{SA} \; ! \; R_{SL}R_{LA}j"R_{LL}"$
- $R_{SL} \; ! \; R_{SS}R_{SL}jR_{SL}R_{LL}$
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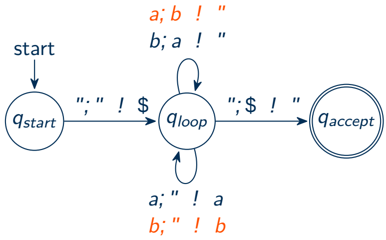
PDA implies CFL (Part 2)



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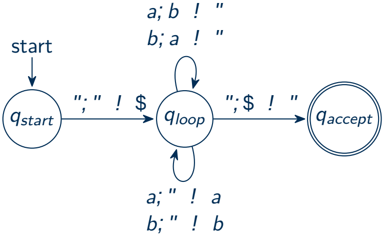
- $R_{LL} \; ! \; "jR_{LL}R_{LL}jaR_{LL}bjbR_{LL}a$
- $R_{AA} \; ! \; "jR_{AA}R_{AA}$
- $R_{SA} \; ! \; R_{SL}R_{LA}j"R_{LL}"$
- $R_{SL} \; ! \; R_{SS}R_{SL}jR_{SL}R_{LL}$
- $R_{LA} \; ! \; R_{LL}R_{LA}jR_{LA}R_{AA}$

PDA implies CFL (Part 2)



1. Add $S \; ! \; A_{start;accept}$ as the start variable.
2. If $(r; u) \; 2 \; (p; a; ") \wedge (q; ") \; 2 \; (s; b; u)$ then add $A_{pq} \; ! \; aA_{rs}b$:
3. For each $p; q; r$ add $A_{pq} \; ! \; A_{pr}A_{rq}$:
4. For each p add $A_{pp} \; ! \; "$:

New Rules:

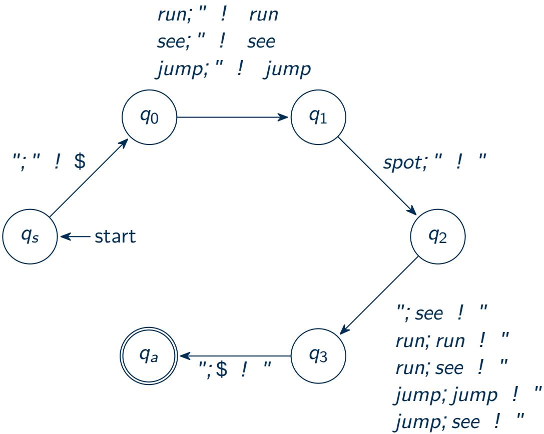
- $S \; ! \; R_{SA}$
- $R_{SS} \; ! \; "jR_{SS}R_{SS}$
- $R_{LL} \; ! \; "jR_{LL}R_{LL}jaR_{LL}bjbR_{LL}a$
- $R_{AA} \; ! \; "jR_{AA}R_{AA}$
- $R_{SA} \; ! \; R_{SL}R_{LA}j"R_{LL}"$
- $R_{SL} \; ! \; R_{SS}R_{SL}jR_{SL}R_{LL}$
- $R_{LA} \; ! \; R_{LL}R_{LA}jR_{LA}R_{AA}$

Compared to:

$$X \; ! \; XXjaXbjbXaj":$$

Practice

1. S ! $A_{start,accept}$
2. Add A_{pq} ! $aA_{rs}b$:, if $(r; u)$ 2 $(p; a; ")$ and $(q; ")$ 2 $(s; b; u)$
3. Add A_{pq} ! $A_{pr}A_{rq}$:
4. Add A_{pp} ! ":



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Grammars and Pushdown Automata

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