# Reducibility and Information

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# Oracles

**Definition**

An **oracle** for a language $B$ is an external device that is capable of always deciding membership in $B$. An **oracle Turing machine** is a modification of a Turing machine $M$ denoted $M^B$ that has the added ability to query an oracle for $B$.

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# $E_{TM}$ and $A_{TM}$

$T^{A_{TM}}$="*On input* $\langle M \rangle$*, where* $M$ *is a TM:*

1. *Construct the TM* $N$.
   $N$ ="*On any input:*
   1. *Run* $M$ *in parallel on all strings in* $\Sigma^*$.
   2. *If* $M$ *accepts any of these strings,* **accept**."
2. *Query the oracle for* $A_{TM}$ *to determine if* $\langle N, 0 \rangle \in A_{TM}$.
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Can an oracle for $A_{TM}$ actually exist?

# Turing Reducible

## Definition

A language $A$ is **Turing decidable** relative to $B$ if there exists an oracle machine $M^B$ which decides $A$. If this is the case we say that $A$ is Turing reducible to $B$ and write $A \leq_T B$.

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- $E_{TM} \leq_T A_{TM}$

# Turing Reduction vs. Mapping Reduction

Recall, $A$ is mapping reducible to $B$, $A \leq_m B$, if there is a computable function $f$ such that

$$w \in A \Leftrightarrow f(w) \in B.$$

That is, if we can change questions about $A$ into questions about $B$ in some computable way. It is "clear" that $A \leq_m B \Rightarrow A \leq_T B$.

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**Theorem**

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- $A = \overline{A_{TM}}$
- $B = A_{TM}$
- $\overline{A_{TM}} \leq_T A_{TM}$
- If $\overline{A_{TM}} \leq_m A_{TM}$, then $\overline{A_{TM}}$ would be recognizable, a contradiction.

# Problem 6.3 Transitivity

**Theorem**

*If $A \leq_T B$ and $B \leq_T C$, then $A \leq_T C$*

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- Suppose $M_1^B$ decides if $w \in A$ using an oracle for $B$.
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**Theorem**

*If $A \leq_T B$ and $B \leq_T C$, then $A \leq_T C$*

- Suppose $M_1^B$ decides if $w \in A$ using an oracle for $B$.
- Suppose $M_2^C$ decides if $w \in B$ using an oracle for $C$.
- Define $M_3^C$ by replacing the oracle for $B$ in $M_1^B$ with $M_2^C$.

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$M_1^B =$ *"On input $w$: Query the oracle for $B$ to determine if $w \in B$ and so also in $A$."*

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$M_2^C =$ *"On input $w$: Query the oracle for $C$ to determine if $w \in C$ and so also in $B$."*

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Given an oracle TM $M^B$ which decides $A$, replace the oracle for $B$ with the decider for $B$.

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- $\langle M, \text{"01"}, 14\rangle$

$$|\langle M, \text{"10"}, 14\rangle| = \underbrace{1011\,0110}_{|\langle M\rangle|}\ \underbrace{11101010001\cdots 11101010}_{\langle M\rangle}\ \underbrace{10}_{\text{"10"}}\ \underbrace{1110}_{14}$$

# Information and Minimal Descriptions

## Definition

The **minimal description** of a binary string $x$, written $d(x)$, is the shortest string $\langle M, w \rangle$ where the TM $M$ prints $x$ on input $w$. The **descriptive complexity** of $x$, $K(x)$, is $K(x) = |d(x)|$.

# Upper Bound for $x$ and $xx$

## Theorem

*For any binary string $x$*

$$\exists c_0 \forall x : K(x) \leq |x| + c_0$$

*and*

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- $c_0$ is the length of a Turing machine that outputs its input $x$ (the identity function).
- $c_1$ is the length of a machine that runs a Turing machine $N$ on input $w$ and prints the result twice; we run this on $d(x)$, the minimal description of $x$.

# Bounding $xy$

Suppose $x = 1010$ and $y = 1101$;
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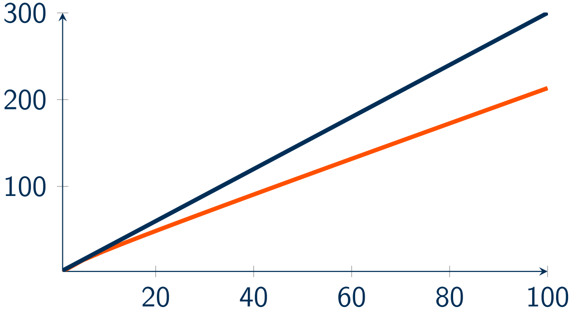

# Descriptive Languages

**Theorem**

*For any descriptive language p (programming language or language language), a fixed constant exists that depends only on p such that*

$$\forall x : K(x) \leq K_p(x) + c,$$

*where $K_p(x) = |d_p(x)|$ is the descriptive complexity of a program in language p which prints x.*

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- Let $K_p(x) = |d_p(x)|$ as above.
- Let $M$ be an interpreter for $p$ with $|\langle M \rangle| = c$.
- Then, $\langle M \rangle\, d_p(x)$ prints $x$ and

$$K(x) \leq K_p(x) + |\langle M \rangle| = K_p(x) + c$$

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# $c$-Compressible

**Definition**

Given a string $x$ and constant $c$ we say that $x$ is $c$-**compressible** if

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- $\therefore$ there exists strings incompressible by $c$
- Specifically, $2^n - (2^{n-c+1} - 1)$

# Incompressible Strings

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- $\therefore$ there exists incompressible strings

# Computable Properties and Compressibility

**Definition**

A **property** of a string $x$ is a binary function $f$, i.e. $x$ either has the property or it does not. Given $F_n = \{x | f(x) = FALSE \ and \ |x| \leq n\}$, a property holds for **almost all** strings if

$$\frac{|F_n|}{|All \ Strings \ with \ |x| \leq n|} \longrightarrow 0$$

as $n$ goes to infinity.

# Computable Properties and Compressibility

**Theorem**

*Given a computable property $f$ that holds for almost all strings, and $b > 0$, then the property $f$ is FALSE for only finitely many strings that are incompressible by $b$.*

# Computable Properties and Compressibility (proof part 1)

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- Thus $\langle M, i_x \rangle$ returns $x$ and $|d(x)| = K(x) \leq |i_x| + c$

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- Thus $\langle M, i_x \rangle$ returns $x$ and $|d(x)| = K(x) \leq |i_x| + c$
- Choose $n$ so that

$$\frac{|F_n|}{|All\ Strings\ with\ |x| \leq n|} \leq \frac{1}{2^{b+c+1}}$$

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- There are $2^{n+1} - 1$ strings of length $n$, so

$$i_x \leq \frac{2^{n+1}-1}{2^{b+c+1}} < \frac{2^{n+1}}{2^{b+c+1}} = 2^{n-b-c}$$

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- Then for all $x$ such that $|x| \geq n$ and $f(x) = FALSE$

$$|d(x)| = K(x) \leq |i_x| + c \leq (n - b - c) + c = n - b$$

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$$i_x \leq \frac{2^{n+1} - 1}{2^{b+c+1}} < \frac{2^{n+1}}{2^{b+c+1}} = 2^{n-b-c}$$

- Then for all $x$ such that $|x| \geq n$ and $f(x) = \mathit{FALSE}$

$$|d(x)| = K(x) \leq |i_x| + c \leq (n - b - c) + c = n - b$$

- $\therefore$ If $x$ is incompressible by $b$, then its length is less than $n$, so there are only finitely many such $x$.

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- But $d(x)$ is minimal, so this is a contradiction.

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# Next Class

- Have a Good Summer

# Reducibility and Information

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