Grammars and Pumping Lemmas

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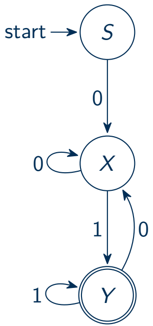
Finite Automaton



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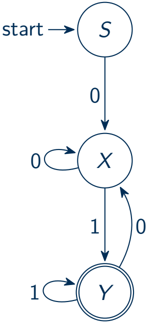
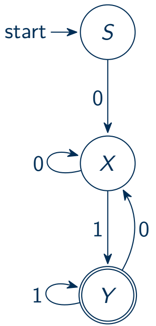


$$S \rightarrow 0X$$

Finite Automaton



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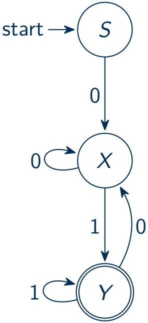

$$S \rightarrow 0X$$
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Regular Grammar

Definition (Regular Grammar)

A ***regular grammar*** is a 4-tuple (V, Σ, R, S), where

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4. $S \in V$ is a ***start variable***.

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Regular Grammar to Automaton

$$S \to aX|bY$$
$$X \to aX|bX|b$$
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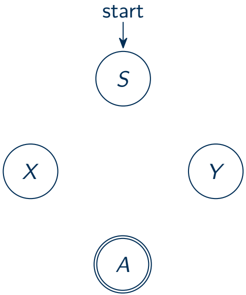

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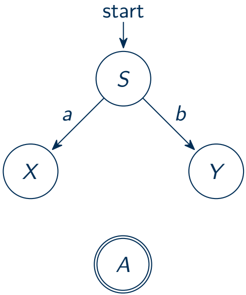

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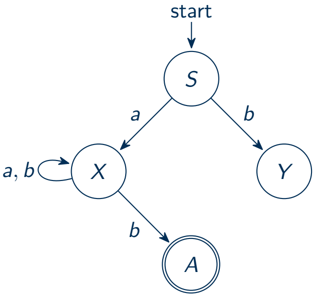

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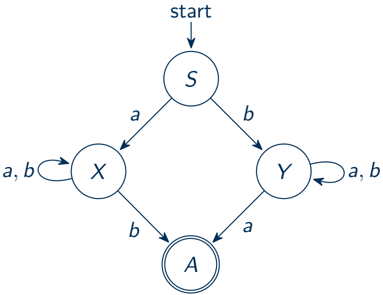

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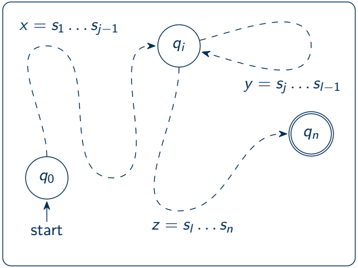
Pumping Lemma for Regular Languages

Theorem (Pumping Lemma)

If A is a regular language, then there exists $p \in \mathbb{N}$ such that, any string $s \in A$ of length at least p may be written as $s = xyz$ such that:

1. *for each $i \geq 0$, $xy^iz \in A$,*
2. *$|y| > 0$, and*
3. *$|xy| \leq p$.*



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Context-Free-Grammar to NPDA

$S \rightarrow aSc|bSc|\varepsilon$

Context-Free-Grammar to NPDA

$$S \rightarrow aSc|bSc|\varepsilon$$

$$S_0 \rightarrow S$$
$$S \rightarrow aU|bU|\varepsilon$$
$$U \rightarrow Sc$$

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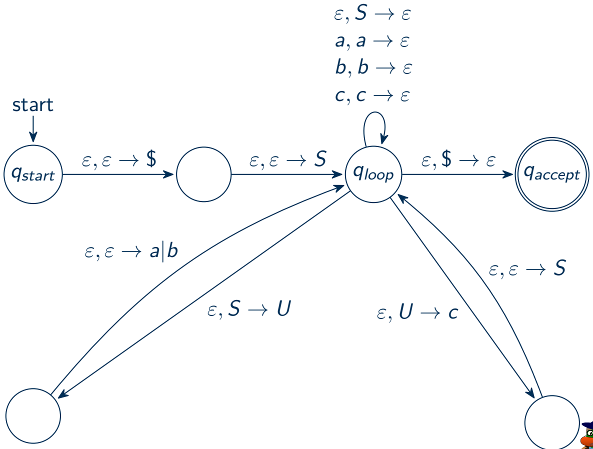
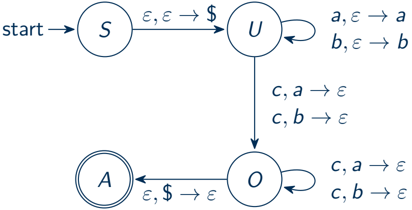


NPDA to Context-Free-Grammar



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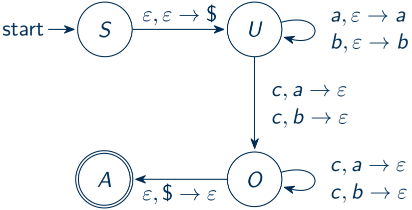

$$\delta(U, a, \varepsilon) = \{(U, a)\}$$
$$\delta(U, b, \varepsilon) = \{(U, b)\}$$
$$\delta(U, c, a) = \{(O, \varepsilon)\}$$
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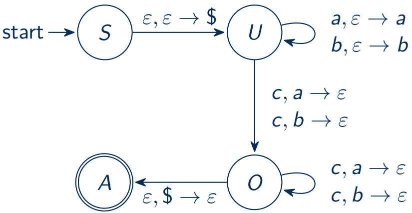
NPDA to Context-Free-Grammar



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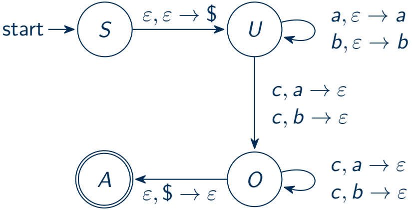
1. $S \rightarrow A_{start,accept}$
2. Add $A_{pq} \rightarrow aA_{rs}b.$, if $(r, u) \in \delta(p, a, \varepsilon)$ and $(q, \varepsilon) \in \delta(s, b, u)$
3. Add $A_{pq} \rightarrow A_{pr}A_{rq}$.
4. Add $A_{pp} \rightarrow \varepsilon$.

NPDA to Context-Free-Grammar



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$$S \to R_{SA}$$
$$R_{SA} \to R_{UO}$$
$$R_{UO} \to aR_{UO}c | bR_{UO}c$$
$$R_{UO} \to aR_{UU}c | bR_{UU}c$$
$$R_{UU} \to \varepsilon$$

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Pumping Example

$$S \rightarrow A|B|C|SS|\varepsilon$$
$$A \rightarrow aBc|aCc|aSc$$
$$B \rightarrow bAc|bCc|bSc$$
$$C \rightarrow cAa|cBb|cSa|cSb$$

Consider the parse tree for

$$abcabccbcc$$

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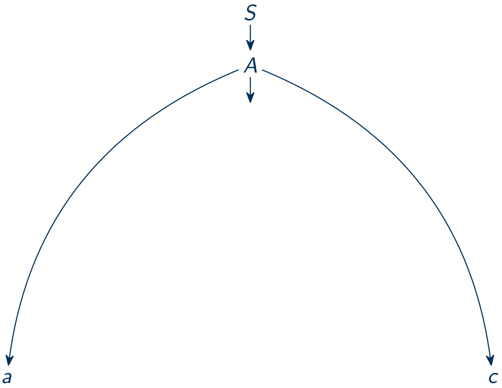

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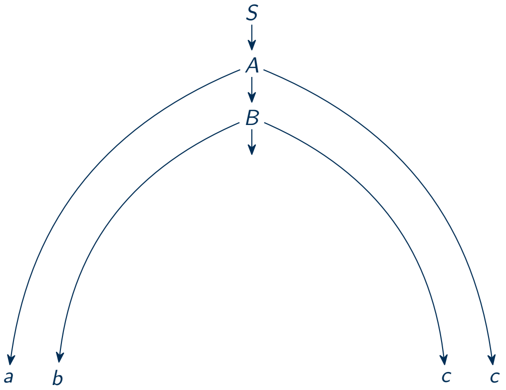

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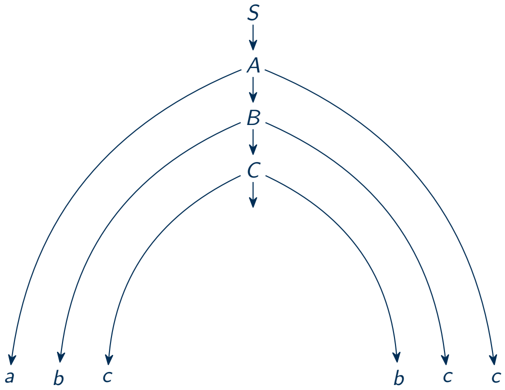

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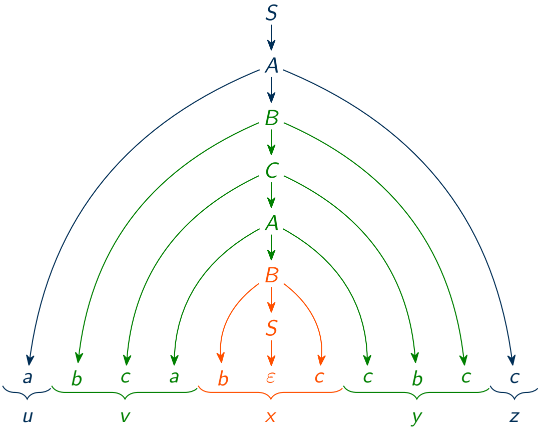
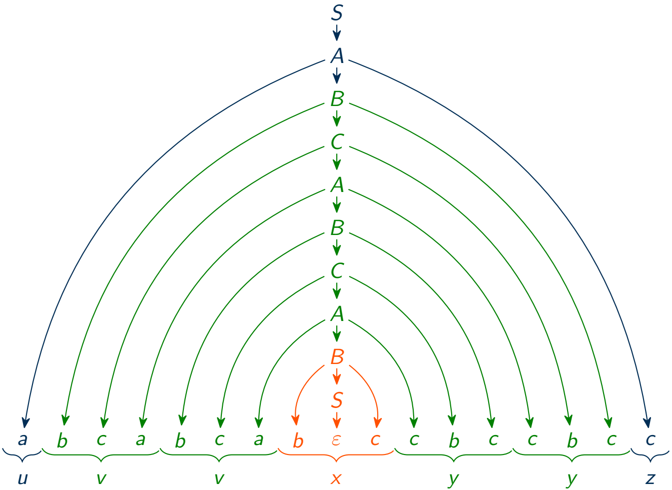
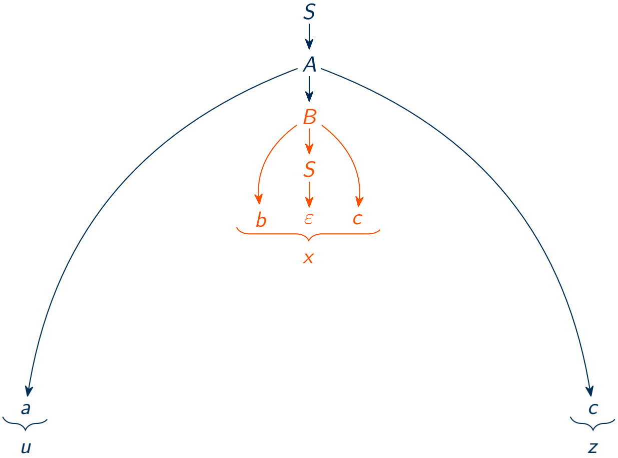


Pumping Up



Pumping Down



Pumping Lemma for CFL's

Theorem (Pumping Lemma for Context-Free Languages)

If A is a context-free language, then there is a number p (the pumping length) *such that, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ satisfying the conditions:*

1. *for each $i \geq 0$, $uv^ixy^iz \in A$,*
2. $|vy| > 0$, *and*
3. $|vxy| \leq p$.

General Idea (Finding p)

1. Let b equal the max. length on the right-hand of the rules (≥ 2).

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3. Given at least $b^h + 1$ leaves, the parse tree is at least $h + 1$ tall.
4. Let $p = b^{|V|+1}$, where $|V|$ is the number of variables.

General Idea (Finding a Repeated Variable)

5. If $s \in A$ and $|s| \geq p$, then its parse tree is at least $|V| + 1$ tall.

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5. If $s \in A$ and $|s| \geq p$, then its parse tree is at least $|V| + 1$ tall.
6. Paths of length $|V| + 1$ or more have at least $|V| + 1$ variables.
7. At least one variable is repeated, call it R.

General Idea (Finding *vxy*)

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10. Sub-trees of height no more than $|V|+1$ have at most $b^{|V|+1}$ leaves.
11. The string vxy under the first R has length at most $p = b^{|V|+1}$.
12. Finally, $|vy| > 0$ because τ is minimal.

Pumping Example

$S \to A|B|C|SS|\varepsilon$

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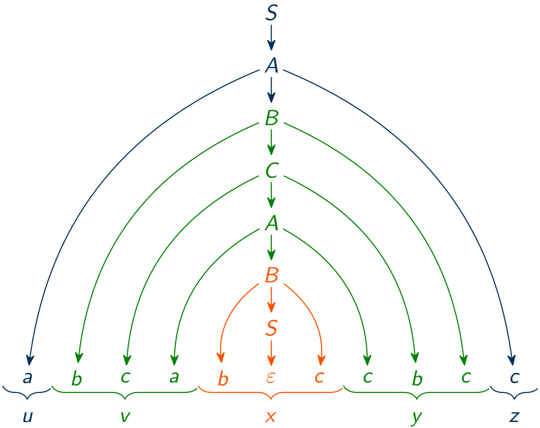

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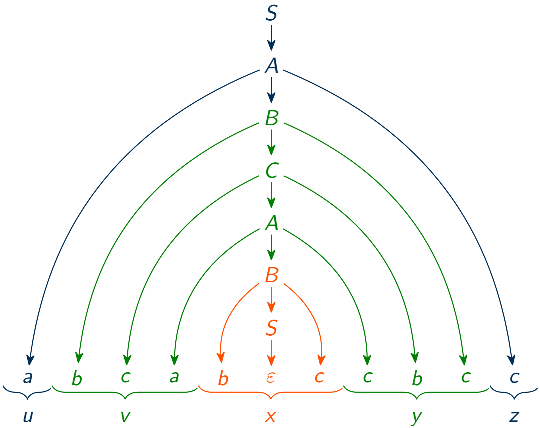
- $b = 3$



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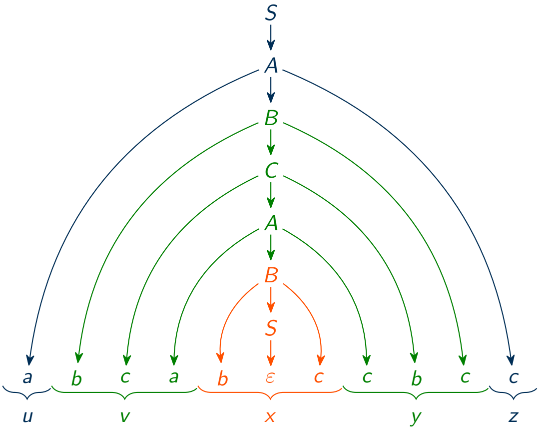
- $b = 3$
- $|V| = 4$



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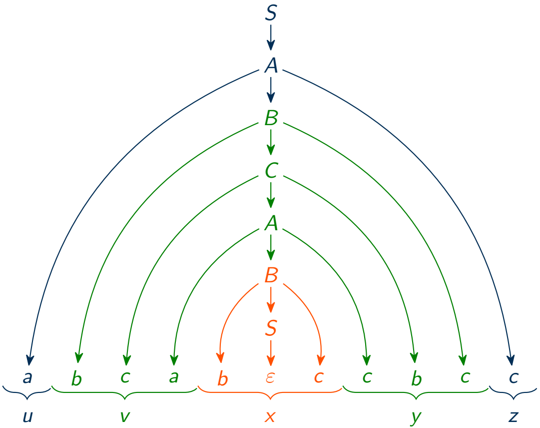
- $b = 3$
- $|V| = 4$
- $p \leq b^{|V|+1} = 81$



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Non-Context-Free Language (Example 1)

Suppose the language $B = \{a^n b^n c^n | n \geq 0\}$ has pumping length p and consider $w = a^p b^p c^p$:

$$\overbrace{\quad}^{u}\ \overbrace{\quad}^{v}\ \overbrace{\quad}^{x}\ \overbrace{\quad}^{y}\ \overbrace{\qquad\qquad\qquad\qquad}^{z}$$

$$\underbrace{aaaaa\cdots a}_{a^p}\ \underbrace{bbbbb\cdots b}_{b^p}\ \underbrace{ccccc\cdots c}_{c^p}$$

1. $uv^ixy^iz \in B$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

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Suppose the language $B = \{a^n b^n c^n | n \geq 0\}$ has pumping length p and consider $w = a^p b^p c^p$:

$$\underbrace{\overbrace{aaaaa \cdots a \; bb}^{u}}_{a^p}\ldots$$

$$\underbrace{aaaaa \cdots a}_{a^p} \; \underbrace{bbbbb \cdots b}_{b^p} \; \underbrace{ccccc \cdots c}_{c^p}$$

with w split above as u, v, x, y, z.

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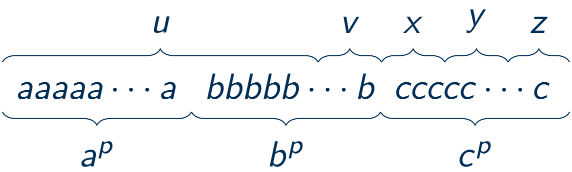

1. $uv^i xy^i z \in B$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

Non-Context-Free Language (Example 2)

Suppose the language $D = \{ww | w \in \{01\}^*\}$ has pumping length p and consider $w = 0^p1^p0^p1^p$:

$$\overbrace{00000\cdots}^{u}\overbrace{0\;\,1}^{v}\overbrace{11}^{x}\overbrace{11}^{y}\overbrace{\cdots1\;\,00000\cdots0\;\,11111\cdots1}^{z}$$

$$\underbrace{00000\cdots0}_{0^p}\;\underbrace{11111\cdots1}_{1^p}\;\underbrace{00000\cdots0}_{0^p}\;\underbrace{11111\cdots1}_{1^p}$$

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$$\overbrace{\underbrace{00000\cdots0}_{0^p}\ \underbrace{11111\cdots1}_{1^p}}^{u\ \ \ v}\ \overbrace{\underbrace{00000\cdots0}_{0^p}}^{x\ \ y}\ \overbrace{\underbrace{11111\cdots1}_{1^p}}^{z}$$

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$$\overbrace{00000\cdots0 \quad 11111\cdots1 \quad 000}^{u}\overbrace{00}^{v}\overbrace{\cdots}^{x}\overbrace{0 \quad 1}^{y}\overbrace{1111\cdots1}^{z}$$

$$\underbrace{00000\cdots0}_{0^p} \quad \underbrace{11111\cdots1}_{1^p} \quad \underbrace{00000\cdots0}_{0^p} \quad \underbrace{11111\cdots1}_{1^p}$$

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Deterministic Pushdown Automaton

Definition

A *deterministic pushdown automaton* is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are all finite sets, and

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to (Q \times \Gamma_\varepsilon) \cup \{\emptyset\}$ is the transition function,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

The Transition function δ must satisfy the following conditions, for every $q \in Q$, $a \in \Sigma$, and $x \in \Gamma$, exactly one of the values $\delta(q, a, x)$, $\delta(q, a, \varepsilon)$, $\delta(q, \varepsilon, x)$, and $\delta(q, \varepsilon, \varepsilon)$ is not $\emptyset$

Notes from the Text

- “In contrast [to regular languages], nondeterministic pushdown automata are more powerful than their deterministic counterparts.”

Notes from the Text

- “In contrast [to regular languages], nondeterministic pushdown automata are more powerful than their deterministic counterparts.”
- “This subclass of context-free languages is relevant to practical applications, such as the design of parsers in compilers for programming languages, because the parsing problem is generally easier for DCFLs than for CFLs.”

Notes from the Text

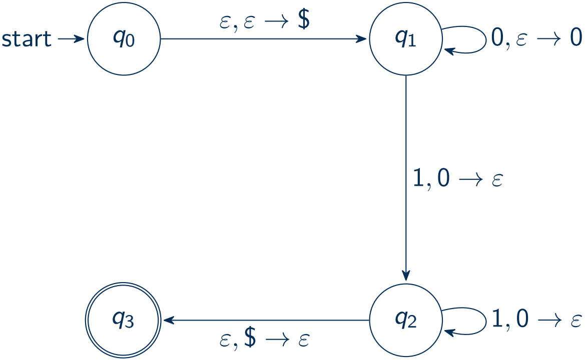
- "In contrast [to regular languages], nondeterministic pushdown automata are more powerful than their deterministic counterparts."
- "This subclass of context-free languages is relevant to practical applications, such as the design of parsers in compilers for programming languages, because the parsing problem is generally easier for DCFLs than for CFLs."
- "Arguments involving DPDAs tend to be somewhat technical in nature, ... Later material in the book doesn't depend on this section, so it may be skipped if desired."

$L = \{0^n 1^n | n \in \mathbb{N}\}$

"The Transition function δ must satisfy the following conditions, for every $q \in Q$, $a \in \Sigma$, and $x \in \Gamma$, exactly one of the values

$$\delta(q, a, x),\ (\delta(q, a, \varepsilon),\ \delta(q, \varepsilon, x),\ \textit{and}\ \delta(q, \varepsilon, \varepsilon)$$

is not $\emptyset$"



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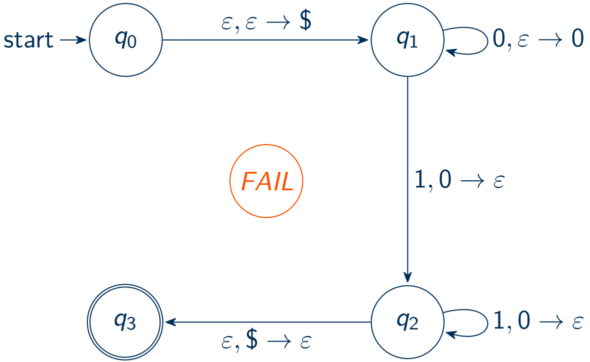

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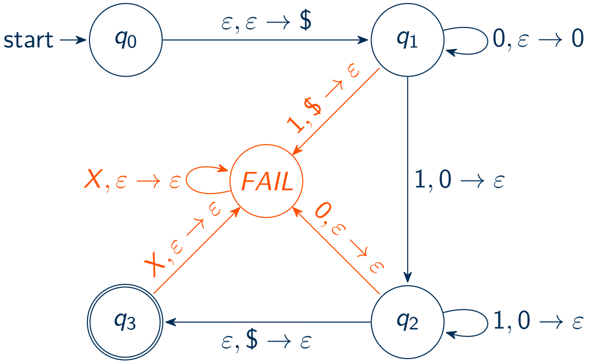

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Input:	0			1			ε		
Stack:	0	\$	ε	0	\$	ε	0	\$	ε
q_0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$(q_1, \$)$
q_1	$\emptyset$	$\emptyset$	$(q_1, 0)$	(q_2, ε)	*fail*	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
q_2	$\emptyset$	$\emptyset$	*fail*	(q_2, ε)	$\emptyset$	$\emptyset$	$\emptyset$	(q_3, ε)	$\emptyset$
q_3	$\emptyset$	$\emptyset$	*fail*	$\emptyset$	$\emptyset$	*fail*	$\emptyset$	$\emptyset$	$\emptyset$

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Grammars and Pumping Lemmas

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