Discrete Math Review

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MACRICOSTAS SCHOOL OF ARTS & SCIENCES

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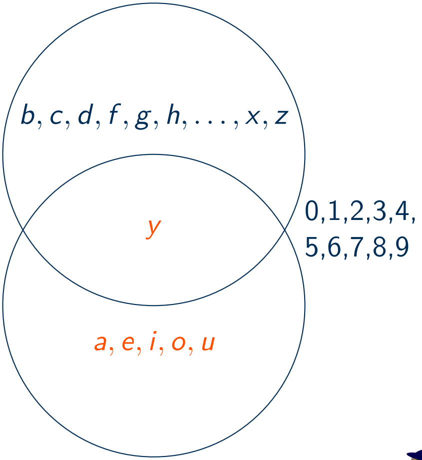
Sets

- Sets:
  $A = \{a, e, i, o, u, y\}$ and
  $B = \{b, c, d, f, g, h, \ldots, z\}$



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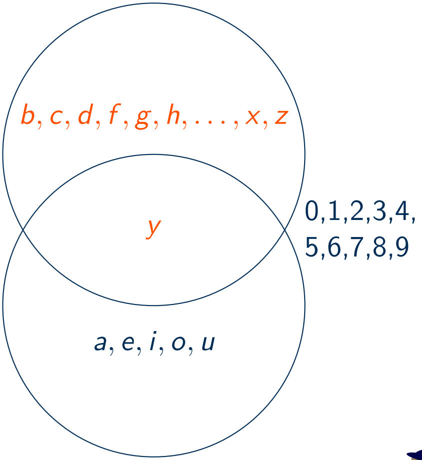

Sets

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  $A = \{a, e, i, o, u, y\}$ and
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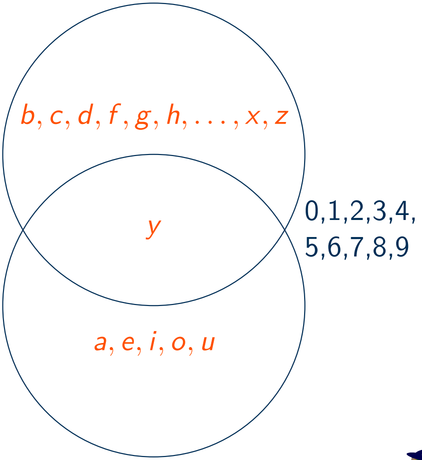
- Union:
  $A \cup B = \{a, b, c, d, e, \ldots, z\}$



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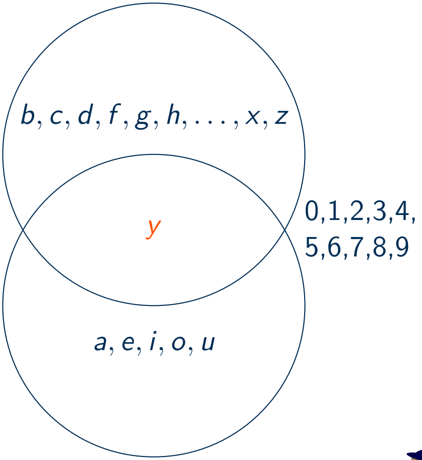

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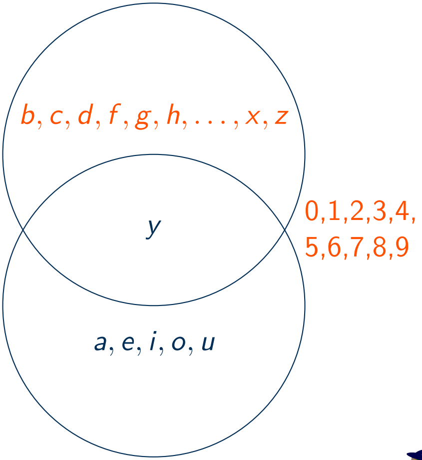
- Complement:
  $A^c = (B \setminus \{y\}) \cup \{0, 1, \ldots, 9\}$



Sets

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- Intersection:
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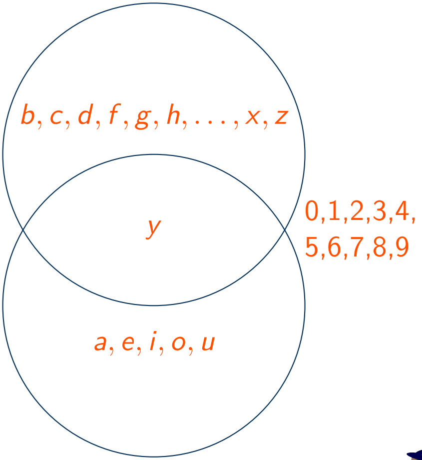
  $A^c = (B \setminus \{y\}) \cup \{0, 1, \ldots, 9\}$
- Universal Set:
  $U \ = A \cup B \cup \{0, 1, \ldots, 9\}$



New Sets from Old

- $A = \{a, b, c\}$ and $B = \{0, 1, 2\}$

New Sets from Old

- $A = \{a, b, c\}$ and $B = \{0, 1, 2\}$
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$$A \times B = \{(a,0),(a,1),(a,2),(b,0),(b,1),(b,2),(c,0),(c,1),(c,2)\}$$

New Sets from Old

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- Cartesian Product:

$$A \times B = \{(a,0),(a,1),(a,2),(b,0),(b,1),(b,2),(c,0),(c,1),(c,2)\}$$

- Power Set:

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$
$$|P(A)| = 2^{|A|}$$

Cardinality of a Power Set: $|P(S)| = 2^{|S|}$

- $A = \{0, 1\}$ and $B = \{0, 1, 2\}$

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- $A = \{0, 1\}$ and $B = \{0, 1, 2\}$
- $P(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
- $P(B) = ?$

$$
\begin{aligned}
P(B) &= P(A) \cup \left(\bigcup_{s2\ P(A)} \{s \cup \{2\}\} \right) \\
&= \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} \cup \{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\} \\
&= \{\emptyset, \{0\}, \{1\}, \{0, 1\}, \{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}
\end{aligned}
$$

Cardinality of a Power Set: $|P(S)| = 2^{|S|}$

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- $|P(B)| =?$

$$\begin{aligned} |P(B)| &= |P(A)| + \left| \bigcup_{s \in P(A)} \{s \cup \{2\}\} \right| \\ &= |P(A)| + \sum_{s \in P(A)} |\{s \cup \{2\}\}| \\ &= |P(A)| + |P(A)| \\ &= 2 \cdot |P(A)| \end{aligned}$$

Cardinality of a Power Set: $|P(S)| = 2^{|S|}$

- $A = \{0, 1\}$ and $B = \{0, 1, 2\}$
- $P(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
- $P(B) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
- $|P(B)| = 2 \cdot |P(A)| = 2 \cdot 2^{|A|} = 2^{|A|+1} = 2^{|B|}$

Relations

Definition (Relation)

A **relation** between two sets is a subset of their Cartesian product.

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A **relation** between two sets is a subset of their Cartesian product.

Given:

$$A \times B = \{(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2), (c, 0), (c, 1), (c, 2)\}$$

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A **relation** between two sets is a subset of their Cartesian product.

Given:

$$A \times B = \{(a,0),(a,1),(a,2),(b,0),(b,1),(b,2),(c,0),(c,1),(c,2)\}$$

A sample relation might be:

$$\mathcal{R} = \{(a,0),(a,1),(a,2),(b,1),(b,2),(c,2)\}$$

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A **relation** between two sets is a subset of their Cartesian product.

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Definition (Relation)

A **relation** between two sets is a subset of their Cartesian product.

Given:

$$A \times A = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

A sample relation might be:

$$\mathcal{O} = \{(a, b), (a, c), (b, c)\}$$

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

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A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

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Equivalence Relation

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A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{O} = \{(a, b), (a, c), (b, c)\}$$

Since a does not relate to its self ($a \not\sim a$) this is not **reflexive**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{O} = \{(a, b), (a, c), (b, c)\}$$

Since a relates to b ($a \sim b$) but b does not relate to a ($b \not\sim a$) this is not **symmetric**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{O} = \{(a, b), (a, c), (b, c)\}$$

Since $a \sim b$ and $b \sim c$ and $a \sim c$ this is **transitive**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{C} = \{(a,a),(a,b),(b,a),(b,b),(c,c)\}$$

Equivalence Relation

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A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{C} = \{(a,a),(a,b),(b,a),(b,b),(c,c)\}$$

Since $a \sim a$, $b \sim b$, and $c \sim c$ this is **reflexive**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{C} = \{(a,a),(a,b),(b,a),(b,b),(c,c)\}$$

Since $a \sim b$ and $b \sim a$ this is **symmetric**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{C} = \{(a,a),(a,b),(b,a),(b,b),(c,c)\}$$

Since $a \sim b$, $b \sim a$ and $a \sim a$ (also, $b \sim a, a \sim b,$ and $b \sim b$) this is **transitive**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{C} = \{(a,a),(a,b),(b,a),(b,b),(c,c)\}$$

This relation is am **equivalence relation**.

Function

Definition (Function)

A **function** is a relation between two sets, the first called the **domain** and the second the **co-domain**, such that for all x in the domain there exists a unique y in the co-domain such that (x, y) is in the relation.

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Given:

$$A \times B = \{(a,0),(a,1),(a,2),(b,0),(b,1),(b,2),(c,0),(c,1),(c,2)\}$$

The relation:

$$\mathcal{R} = \{(a,0),(a,1),(a,2),(b,1),(b,2),(c,2)\}$$

is not a function

Function

Definition (Function)

A **function** is a relation between two sets, the first called the **domain** and the second the **co-domain**, such that for all x in the domain there exists a unique y in the co-domain such that (x, y) is in the relation.

Given:

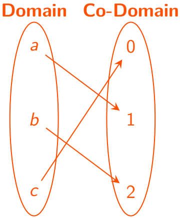
$$A \times B = \{(a,0),(a,1),(a,2),(b,0),(b,1),(b,2),(c,0),(c,1),(c,2)\}$$

But, the relation:

$$\mathcal{S} = \{(a,1),(b,2),(c,0)\}$$

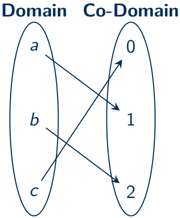
is a function

Visualizing Functions



$1-1$ & *onto*

Visualizing Functions



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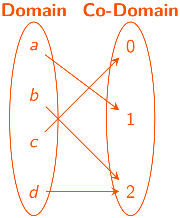

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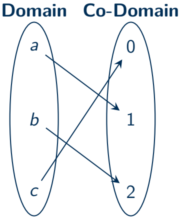

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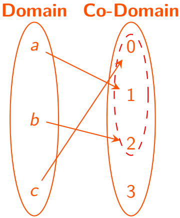

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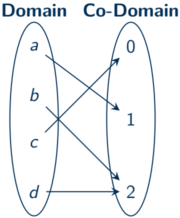

onto

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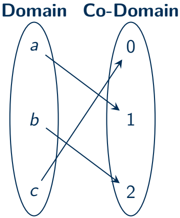

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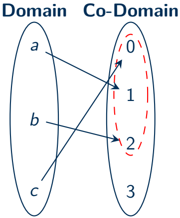

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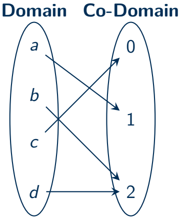

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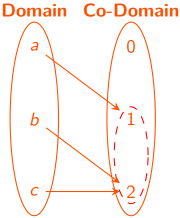

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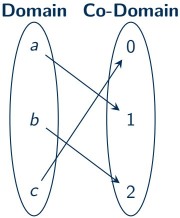

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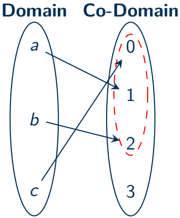

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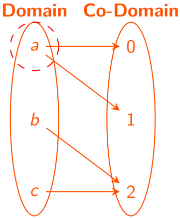
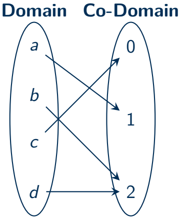


Not a function



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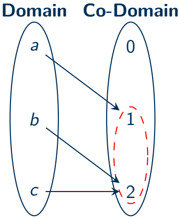

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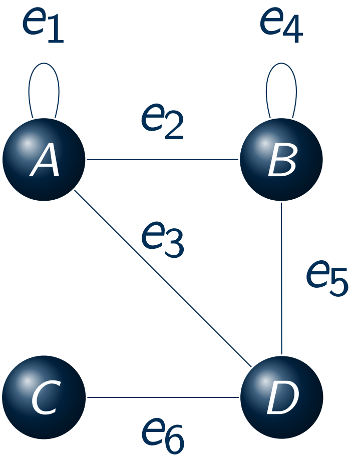
Graphs



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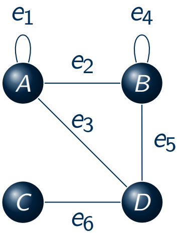
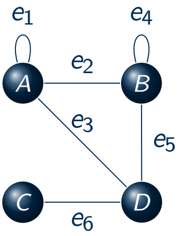


- Vertex Set:

$$V = fA, B, C, Dg$$

Graphs



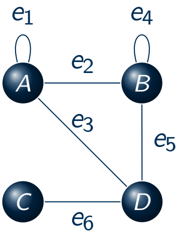
- Vertex Set:

$$V = \{A, B, C, D\}$$

- Edge Set:

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

Graphs



- Vertex Set:

$$V = fA, B, C, Dg$$

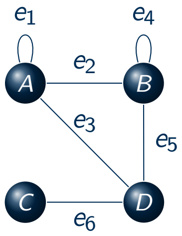
- Edge Set:

$$E = fe_1, e_2, e_3, e_4, e_5, e_6g$$

- Edge Set:

$$E = f(A, A), (A, B), (A, D), (B, B), (B, D), (C, D)g$$

Graphs



- Vertex Set:

$$V = \{A, B, C, D\}$$

- Edge Set:

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

- Edge Set:

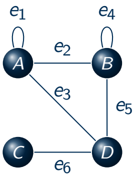
$$E = \{(A, A), (A, B), (A, D), (B, B), (B, D), (C, D)\}$$

- Graph:

$$\begin{aligned} G &= (V, E) \\ &= (\{A, B, C, D\}, \\ &\qquad \{(A, A), (A, B), (A, D), (B, B), (B, D), (C, D)\}) \end{aligned}$$

Types of Graphs

Graph



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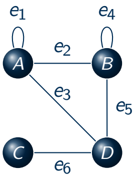
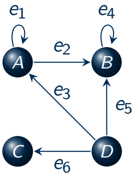


Directed Graph



Types of Graphs

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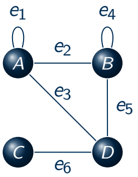
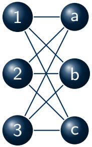


Bipartite Graph



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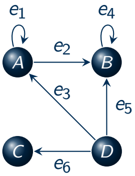

Types of Graphs

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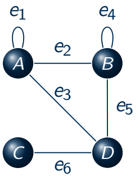

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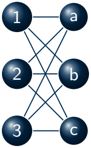

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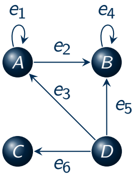
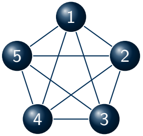


Complete Graph



Types of Graphs

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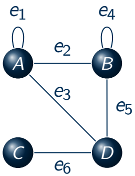

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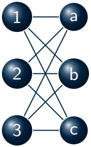
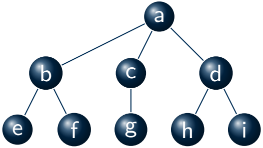
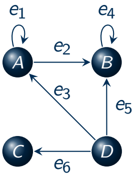


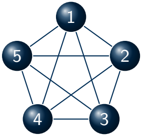
Tree



Directed Graph

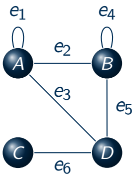


Complete Graph

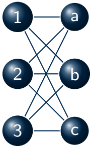


Types of Graphs

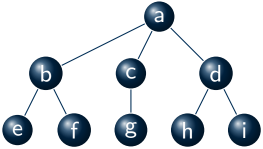
Graph



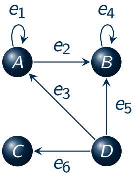
Bipartite Graph



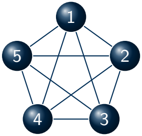
Tree



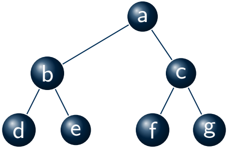
Directed Graph



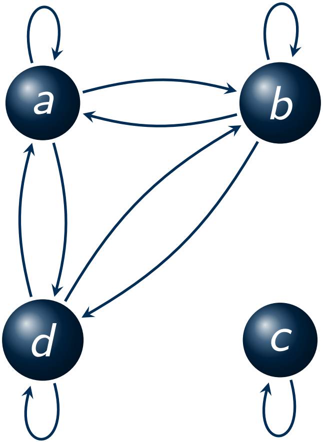
Complete Graph



Binary Tree



Relations as Graphs



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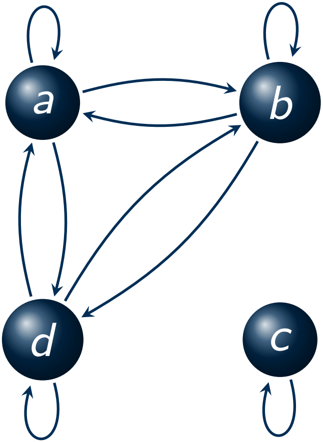
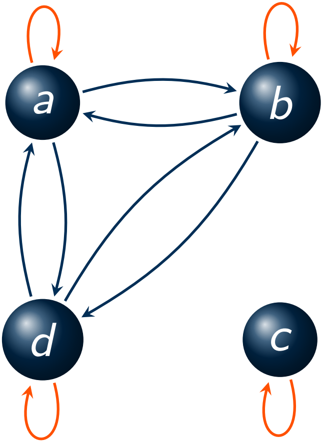


Equivalence Relation?

Relations as Graphs



Equivalence Relation?

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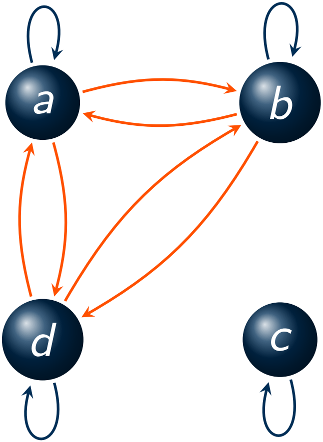
Relations as Graphs



Equivalence Relation?

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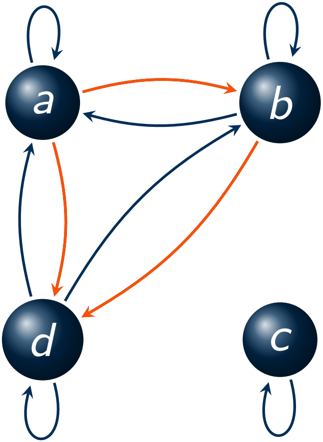
Relations as Graphs



Equivalence Relation?

- Reflexive X
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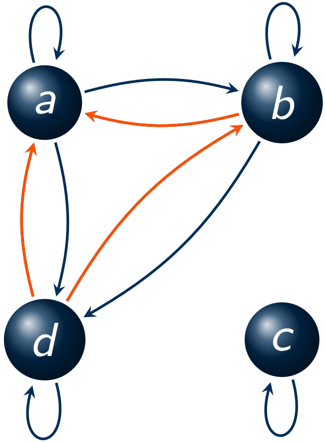
Relations as Graphs



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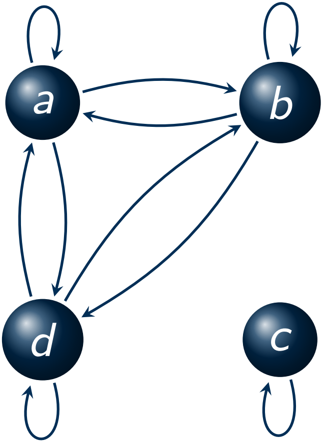
Relations as Graphs



Equivalence Relation X

- Reflexive X
- Symmetric X
- Transitive X
- Equivalence Classes

$$A = \{a, b, d\} \ \& \ C = \{c\}$$

Table of Contents

Direct Proof

Theorem (De Morgan's Law)

Given two sets A and B, the complement of their union is equal to the intersection of their complements:

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Now suppose $x \in A^c \cap B^c$ or equivalently $x \in A^c$ and $x \in B^c$. This tells us that $x \notin A$ and $x \notin B$ and thus $x \notin A \cup B$, i.e. $x \in (A \cup B)^c$.

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Now suppose $x \in A^c \cap B^c$ or equivalently $x \in A^c$ and $x \in B^c$. This tells us that $x \notin A$ and $x \notin B$ and thus $x \notin A \cup B$, i.e. $x \in (A \cup B)^c$. Therefore, $A^c \cap B^c \subseteq (A \cup B)^c$ and

$$(A \cup B)^c = A^c \cap B^c$$

as desired.

By Cases

Theorem

Given any integer n, either n^2 or $n^2 - 1$ is divisible by four.

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Proof: (Case 1) Let n be an even integer so that we may write $n = 2k$ for some unique k. Then

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Proof: (Case 1) Let n be an even integer so that we may write $n = 2k$ for some unique k. Then

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and n^2 is divisible by four.
(Case2) Now, if n is an odd integer then we write $n = 2k + 1$ for some unique k. Thus,

$$n^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k)$$

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Therefore, for any integer n we have shown that either n^2 or $n^2 - 1$ is divisible by four.

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$$n^2 = 4k^2 = 4k + 1 = 2(2k^2 + 2k) + 1$$

which is odd. Therefore, if n is odd, then n^2 is odd and so if n^2 is even, then n is even.

Contradiction

Theorem

No integer is both even and odd.

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Proof: Suppose that n is both even and odd so that $n = 2k$ and $n = 2l + 1$ for some unique k and l. Then we can write $2k = 2l + 1$ and $1 = 2(k - l)$. If $k - l = 0$, then $1 = 0$ and if $k - l \neq 0$, then 2 divides 1.

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Induction

Theorem

Any tree with n vertices has $n-1$ edges.

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(Base Case) When there is only one vertex there are no edges since trees do not contain loops and there is not a second vertex to connect to.

Induction

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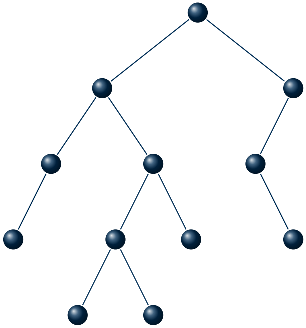

(Induction Step) Assume that the theorem is true for some $k \geq 2$ and consider a tree with $k+1 \geq 3$ vertices.

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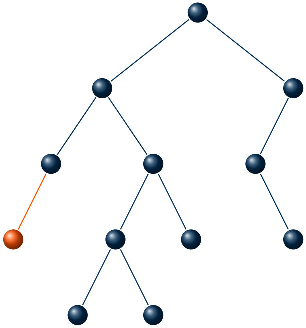

(Induction Step) Assume that the theorem is true for some $k \geq 2$ and consider a tree with $k+1 \geq 3$ vertices. Since there are at least two vertices the tree must contain at least one leaf.

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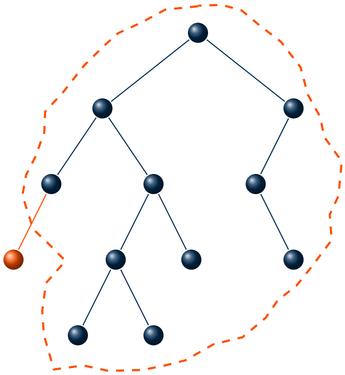

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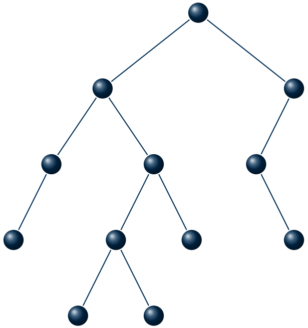

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Table of Contents

Next Class

- Deterministic and Non-Deterministic Finite State Automata

Discrete Math Review

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