Exploring the Infinite

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Abstract

In this assignment we will be trying to understand the concept of infinity, that there are multiple different infinities, and that they are not all the same size. [1]

Contents

Directions

There are a lot of questions in the following sections. As you work through this packet you need to answer all the questions since the ideas build on one another. However, when you write up your observations for this packet, summarize your conclusions from each group of questions rather than answering each question individually.

[1]Much of the material here is taken from or inspired by questions in *Discovering the Art of Mathematics: The Infinite* by Fleron (with Ecke, Hotchkiss, and von Renesse). The idea for the game of Dodge Ball is taken from the text *The Heart of Mathematics* by Burger and Starbird.

1 Set Arithmetic

A set is, roughly speaking, a collection of objects such as:

- $D = \{\spadesuit, \heartsuit, \clubsuit, \diamondsuit\}$ the set of suits in a deck of cards,
- $S = \{*, \bullet, \star, \bowtie, \nabla\}$ is a set of random shapes,
- $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of positive whole numbers, called the *Natural Numbers*, or
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of all integers (the positive and negative whole numbers together with 0).

Sometimes sets have infinitely many elements and sometimes finitely many, the number of elements is called the ***cardinality*** of the set. In this section you will explore how we can do arithmetic with sets. First write down a first answer to this big question (there is no right or wrong, this is baseline for an idea).

Big Question 1. How large is the set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ compared to the set $\mathbb{N} = \{1, 2, 3, \ldots\}$?

1.1 Finite Sets

Answer the following by reading a + as "together with" and a - as "take away." For example

$$\{*, \bullet, \star, \bowtie, \nabla\} + \{\ll\} = \{*, \bullet, \star, \bowtie, \nabla, \ll\}$$

because we put the first set together with the second. Or,

$$\{*, \bullet, \star, \bowtie, \nabla\} - \{\bowtie\} = \{*, \bullet, \star, \nabla\}$$

because we took away the second set from the first.[2] Use these ideas to answer the following.

1.1. What is $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\} - \{\heartsuit, \spadesuit\}$?

1.2. What is $\{*, \bullet, \star, \bowtie, \nabla\} - \{\nabla, \star, \bullet\}$?

1.3. What is $\{\star, \nabla, \vdash\} + \{\dashv, \sharp, \mho\}$?

1.4. What is $\{\Box, \Diamond, \bigodot, \bigsqcup, \bigcirc\} - \{\textit{shapes with four sides}\}$?

1.5. What is $\{\textit{the alphabet}\}$ - $\{a, e, i, o, u\}$?

For each of the exercises 1.1 to 1.5, determine the ***cardinality*** (size) of each set, including the size of your answers. Using cardinality we can translate

$$\{\geq, \alpha, \eta\} - \{\geq\} = \{\alpha, \eta\}$$

into the arithmetical statement

$$3 - 1 = 2.$$

[2] Formally these are called a *set union* and *set difference*

Use this idea to answer the following:

1.6. What arithmetical statement corresponds to question 1.1

1.7. What arithmetical statement corresponds to question 1.2

1.8. What arithmetical statement corresponds to question 1.3

1.9. What arithmetical statement corresponds to question 1.4

1.10. What arithmetical statement corresponds to question 1.5

1.2 Infinity, a first look

Let's do this again with infinitely large sets. In particular can we make sense of $\infty - \infty$?

1.11. A set of three dots "..." is called an ellipses and they mean "and so on." We use them here to say a pattern goes on forever. To get a handle on this, list the next four elements in this set $\mathbb{N} = \{1, 2, 3, \ldots\}$.

1.12. What are the next four elements in this set $E = \{2, 4, 6, 8, \ldots\}$?

1.13. What are the next four elements in this set $F = \{4, 5, 6, 7, \ldots\}$?

1.14. What are the next four elements in this set $S = \{6, 7, 8, 9 \ldots\}$?

1.15. What is left if we take the difference of the sets in 1.11 and 1.13, $\mathbb{N} - F$?

1.16. What is left if we take the difference of the sets in 1.11 and 1.14, $\mathbb{N} - S$?

1.17. What is left if we take the difference of the sets in 1.11 and 1.12, $\mathbb{N} - E$?

1.18. If you try to repeat what you did in 1.6 to 1.10 with the previous three questions what do you get?

1.19. What does this tell you about $\infty - \infty$?

1.20. What does your previous answer tell you about trying to do arithmetic with infinity?

2 Defining Infinity

In this section we develop ways to think about infinity.

2.1 Hilbert's Hotel

To begin watch the following short video about

Hilbert's Hotel (https://youtu.be/faQBrAQ87l4)

2.1. In the video how did Hilbert squeeze in one extra guest?

2.2. How would Hilbert be able to fit in two extra guests?

2.3. How would Hilbert be able to fit in ten extra guests?

2.4. Would the strategy you described work for any finite number of guests?

2.5. Why didn't Hilbert use the same strategy when a bus arrived with an infinite number of new guests? (It might help to think in terms of the work you did in section 1)

2.6. How does figure 1 describe Hilbert's strategy for fitting in infinitely many new guests?

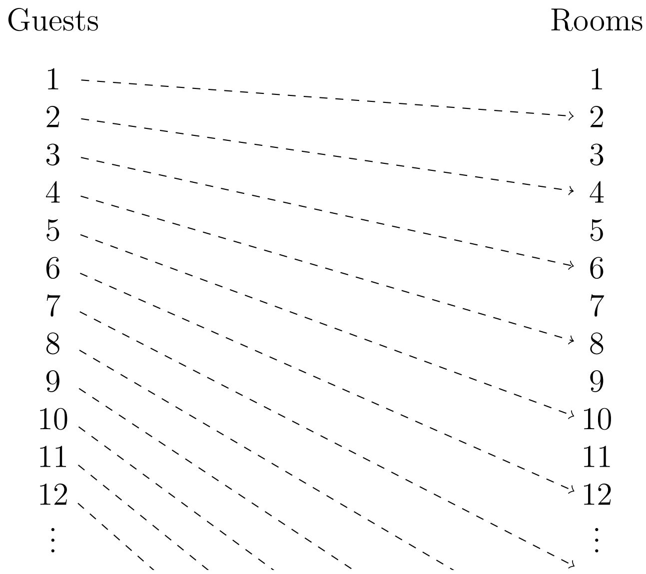

Figure 1: Infinite Guests

2.7. Could Hilbert repeat this strategy more than once?

Figure 2 gives a way to place guests into Hilbert's Hotel if the rooms are numbered $\{1, 2, 3, \ldots\}$ but the guests are numbered $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots\}$. Use this to answer the next set of questions.

Guests		Rooms
⋮		⋮
-5	→	11
-4	→	9
-3	→	7
-2	→	5
-1	→	3
0	→	1
1	→	2
2	→	4
3	→	6
4	→	8
5	→	10
⋮		⋮

Figure 2: Integer and Counting Numbers

2.8. Looking at the positively numbered guests, describe the rooms they end up in.

2.9. Looking at the negatively numbered guests, describe the rooms they end up in.

2.10. Where would you place guest number -6?

2.11. Where would you place guest number 6?

2.12. Where would you place guest number -7?

2.13. Where would you place guest number 7?

The previous examples show how we can *map* one infinite set into another. In figure 1 we took all the positive whole numbers and mapped them to just the even numbers. In figure 2 we took all the integers (positive and negative whole numbers and 0) and mapped them to just the positive whole numbers. We can even do this for fractions, even though there are infinitely many fractions between any two whole numbers. Looking at figure 3 if you follow the squiggling red path and number the steps you get a map of the positive whole numbers to the fractions.

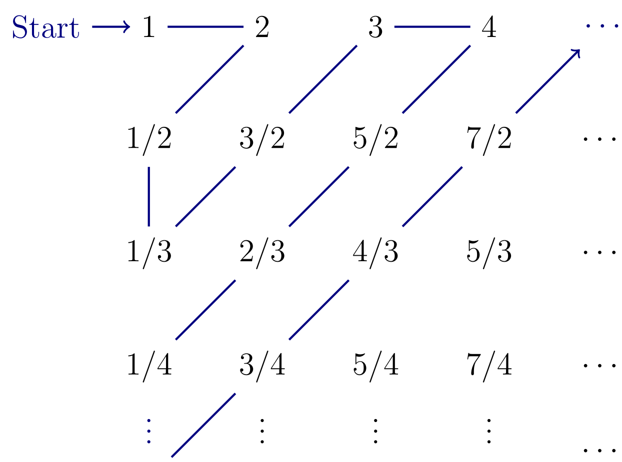

Figure 3: Positive Fractions and Whole Numbers

This idea, that we can map one infinite set onto or into another infinite set, leads us to a definition of infinity.

Definition 1 (Infinity)**.** A set is called *infinite* if there is a map between the set and a proper subset of its self. For example, how we mapped the all integers to the positive whole numbers in figure 2 or the positive whole numbers to the even numbers in figure 1.

2.14. What was your original answer to the Big Question 1 on page 2?

2.15. Looking at your answers in this section, at figure 2, and at your answer to question 1.19, how would you answer Big Question 1?

Now we are ready to consider another big question.

Big Question 2. Are all infinities the same size?[3]

3 A New Infinity

3.1 Dodge Ball

Let's play a game. In this game there are two players. In each round

- Player 1 creates a row of six X's or O's like "X O X X O X"
- Player 2 then chooses exactly one new X or O like so "O," but can not change their previous choices.

After six rounds, the objective for Player 1 is to have a row that matches Player 2 (it can be any of their rows it doesn't have to be the last one), Player 2 is trying to stop this.

[3] Spoiler, the answer is no, the real question is why not.

For example in this match player 2 would win.

	Player 1						Player 2					
Round 1	X	X	X	X	X	X	O					
Round 2	O	O	O	O	O	O	O	X				
Round 3	O	X	O	X	O	X	O	X	X			
Round 4	O	X	X	O	X	X	O	X	X	X		
Round 5	O	X	X	X	X	X	O	X	X	X	O	
Round 6	O	X	X	X	O	X	O	X	X	X	O	O

Table 1: Player 2 Wins

But in this match Player 1 wins.

	Player 1						Player 2					
Round 1	X	X	X	X	X	X	O					
Round 2	O	O	O	O	O	O	O	O				
Round 3	O	O	O	X	O	X	O	O	X			
Round 4	O	O	X	O	X	X	O	O	X	X		
Round 5	O	O	X	X	O	X	O	O	X	X	O	
Round 6	O	O	X	X	O	O	O	O	X	X	O	O

Table 2: Player 1 Wins

Try playing the game with another person or against yourself. You can do this on paper or there is an online version here:

Dodge Ball (http://thewessens.net/ClassroomApps/Main/dodgeball.html)

3.1. Suppose that at the end of round 3 Player 2 has "X O X" and Player 1 plays "X O X O O O" at the start of round 4, what should Player 2 choose to stay ahead?

3.2. Suppose that at the end of round 4 Player 2 has "X O X X" and Player 1 plays "X O X X X X" at the start of round 5, what should Player 2 choose to stay ahead?

3.3. Suppose that at the end of round 5 Player 2 has "X O X X O" and Player 1 plays "X O X X O O" at the start of round 6, what should Player 2 play to win as long as they are careful?

3.4. Did you notice that Player 2 can always win? (If you didn't go back and play again.)

3.5. What strategy can Player 2 use to ensure victory?

3.2 Diagonalization

Consider this variation on the game of Dodge Ball. On the left is a list of numbers, this is like Player 1's lists. On the right are blank spaces for Player 2 to build a new number one digit at a time.

	Player 1	Player 2
Round 1	0. ***1*** 2543649294726847...	0. ***2***
Round 2	0.3 ***4*** 573924572345724...	0.2 ***5***
Round 3	0.34 ***5*** 74748758235892...	0.25 ***6***
Round 4	0.018 ***3*** 7450345723450...	0.256 ***4***
Round 5	0.3425 ***2*** 304509899100...	0.2564 ***3***
Round 6	0.10305 ***4*** 83095023832...	0.25643 ***5***
Round 7	0.034835 ***0*** 2345800234...	0.256435 ***1***
Round 8	0.9823357 ***2*** 300754030...	0.2564351 ***3***
Round 9	0.03458345 ***0*** 48305943...	0.25643513 ***1***
Round 10	0.347249092 ***0*** 2345920...	0.256435131 ***1***
⋮	⋮	⋮

Table 3: Infinite Dodge Ball

3.6. If Player 1 next played 0.9472684762 ***5*** 949273..., what digit might Player 2 pick next?

3.7. If Player 1 then played 0.85635209847 ***9*** 23624..., what digit might Player 2 pick next?

3.8. If Player 1 then played 0.274901865083 ***1*** 2384..., what digit might Player 2 pick next?

3.9. What is Player 2's strategy for winning?

3.10. For as long as they play why can player 2 always win?

This infinite game of dodge ball is a variation of what is called ***Cantor's Diagonalization Process***. What it shows is that even if Player 1 created an infinite list of decimal numbers there would be a number that is not on the list. And if they tried adding that number to the list, Player 2 could create a whole new number that wasn't on the revised list. So, the set of all possible *real* numbers (integers, fractions, square roots, things like π, etc.), all the things we can write with infinitely long decimal parts, is a fundamentally bigger set than the counting numbers; we can't list all of them. In the last section we look at one more big question:

Big Question 3. How many different infinities are there?

4 Infinite Infinities

4.1 Power Sets

Definition 2. Given a set X, the *power set of* X, denoted $\mathscr{P}(X)$, is the set of all subsets of X including the empty set.

For example if $A = \{a\}$, then
$$\mathscr{P}(A) = \{\emptyset, \{a\}\}.$$
And, if $B = \{a, b\}$, then
$$\mathscr{P}(B) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

4.1. Looking at the examples above, if $C = \{a, b, c\}$, what is $\mathscr{P}(C)$? That is, what are all the subsets of the set of C?

4.2. What elements are in both $\mathscr{P}(A)$ and $\mathscr{P}(B)$[4], $\mathscr{P}(A) \cap \mathscr{P}(B)$?

4.3. What elements are in $\mathscr{P}(B) - \mathscr{P}(A)$?

4.4. How might we relate the sets from the previous two questions? How might we find the answer to the second question using the answer to the first?

4.5. What elements are in $\mathscr{P}(B) \cap \mathscr{P}(C)$?

4.6. What elements are in $\mathscr{P}(C) - \mathscr{P}(B)$?

4.7. How might we relate the sets from the previous two questions? How might we find the answer to the second question using the answer to the first?

4.8. How can we use $\mathscr{P}(C)$ and you observations in questions 4.4 and 4.7 to find the power set of $D = \{a, b, c, d\}$?

4.9. If you look at your answers to questions 4.4, 4.7, and 4.8, why does it make sense that
$$|\mathscr{P}(D)| = 2|\mathscr{P}(C)| = 2^2|\mathscr{P}(B) = 2^3|\mathscr{P}(A)| = 2^4?$$

4.2 More Maps and Sets

Using the set $D = \{a, b, c, d\}$, let's associate each element in D with some element in $\mathscr{P}(D)$:

$S_a = \{a, c\}$ $\qquad$ $S_c = \emptyset$

$S_b = \{a, b, d\}$ $\qquad$ $S_d = \{c\}$

Use D, the associations above, and
$$X = \{x | x \in D \text{ and } x \notin S_x\},$$
to answer the following questions.

4.10. How many elements are there in $\mathscr{P}(D)$?

4.11. How does the size of D compare to the size of $\mathscr{P}(D)$?

4.12. What elements of D are in X?

[4] This is called their intersection and denoted with a $\cap$.

4.13. Does the set X equal any of the sets S_x listed above?

4.14. Can we give a different association between the elements of D and the elements of $\mathscr{P}(D)$ which would make it possible to get a different answer to the previous question? Why?

Now suppose we repeat what we did above with $\mathbb{R}$, the set of real numbers, instead of D. Associate each $r \in \mathbb{R}$ with a set $S_r \in \mathscr{P}(\mathbb{R})$ and let

$$X = \{r | r \in \mathbb{R} \text{ and } r \notin S_r\}$$

similar to before.

4.15. If $S_5 = \{3, 6, 9, 12 \ldots\}$ is $5 \in S_5$? Is $5 \in X$?

4.16. If $S_\pi = (2.5, 3.7)$[5] is $\pi \in S_\pi$? Is $\pi \in X$?

4.17. If $S_{7.32} = (1.95, 4.7)$ is $7.32 \in S_{7.32}$? Is $7.32 \in X$?

4.18. If $S_{9.8} = \{0, 4.9, 9.8, 14.9, \ldots\}$ is $9.8 \in S_{9.8}$? Is $9.8 \in X$?

4.19. For any S_r, if $r \in S_r$, is $r \in X$?

4.20. For any S_r, if $r \notin S_r$, is $r \in X$?

4.21. Does there exists r such that $S_r = X$?

4.22. Is X in $\mathscr{P}(\mathbb{R})$? In other words, is X a subset of $\mathbb{R}$?

4.23. Since there is a different S_r for each $r \in \mathbb{R}$, what does this tell us about the number of real numbers r compared to the number of sets in $\mathscr{P}(r)$?

4.24. What does this tell us about the answer to Big Question 3?

[5] Here (a, b) is the interval from a to b not an ordered pair.

5 Glossary

Integers The set of all positive and negative whole numbers including with 0,

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Intersection The set of all objects in both sets.

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

Natural Numbers The set of all positive whole numbers.

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}.$$

Power Set The set of all subsets of a given set which includes the empty set and the entire set. Given $B = \{a, b\}$

$$\mathscr{P}(B) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Real Numbers The set of all numbers that may be written as a sum of an integer and a possibly infinite decimal fraction, i.e. integers, fractions, square roots, things like π, etc.. This set is denoted $\mathbb{R}$.

Set A collection of distinct objects

Set Minus The set of all objects in the first set which are not in the second.

$$A - B = \{x | x \in A \text{ and } x \notin B\}$$

Subset A set that contains some, none, or all elements of another set, but no elements not in the other set:

$$A \subseteq B \text{ if and only if } \; x \in A \Rightarrow x \in B.$$

Union The set of all objects in either set.

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$