Trees

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STATE UNIVERSITY
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& SCIENCES

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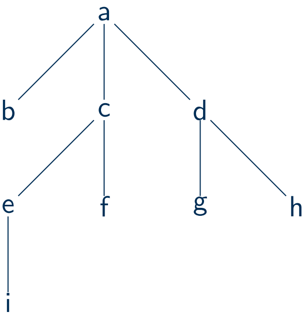
Tree Terminology



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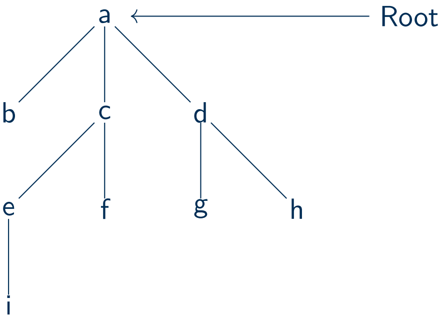

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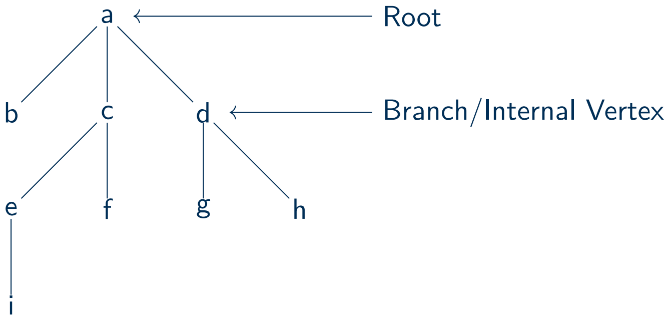

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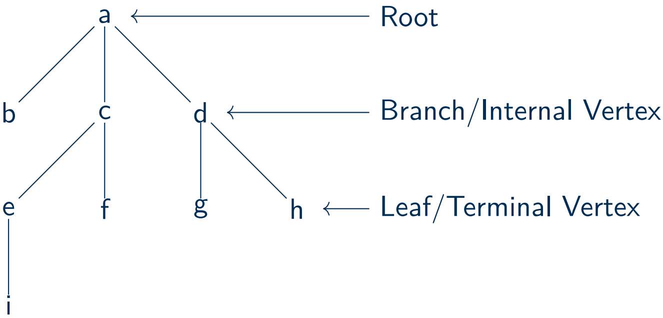

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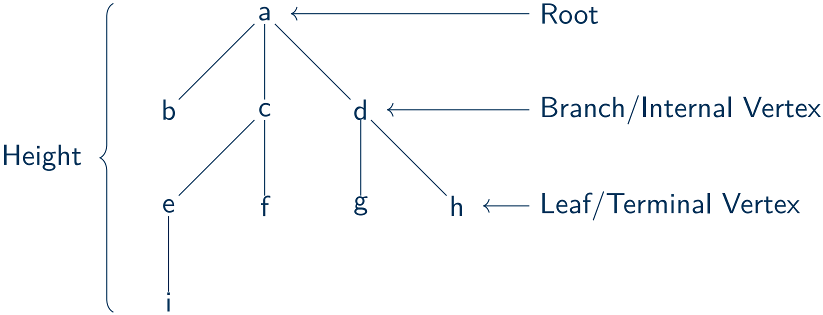

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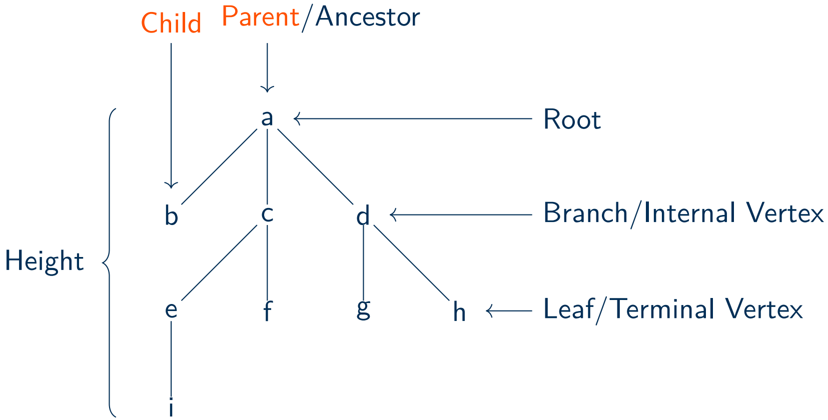

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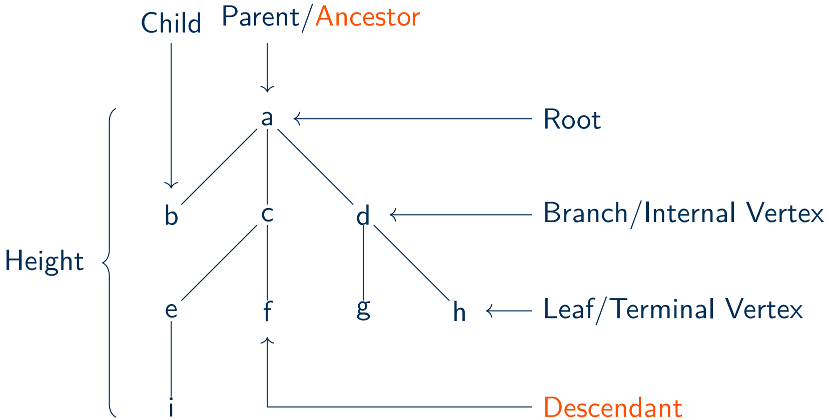

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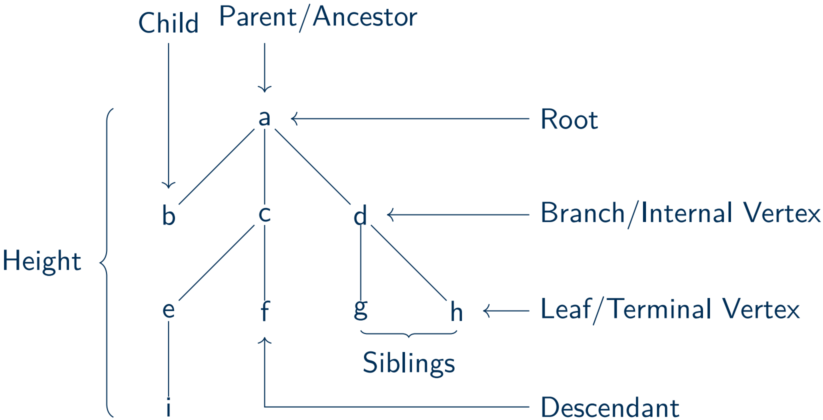

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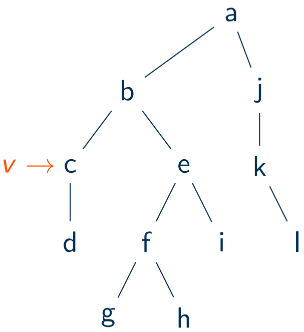
Vertices and Edges

Lemma

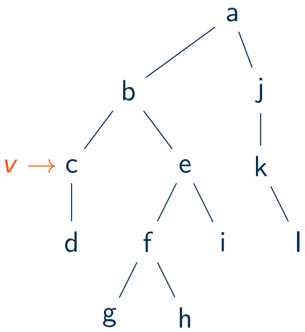
Trees with two or more vertices has a vertex of degree 1.

Degree 1 Vertex



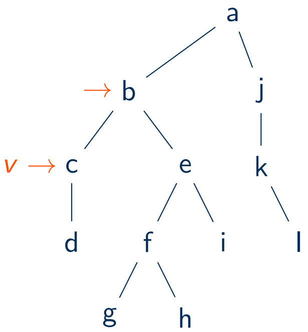
- pick a vertex v

Degree 1 Vertex



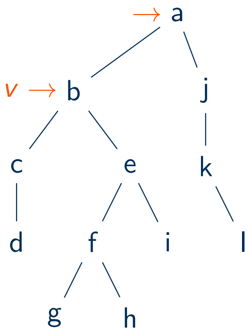
- pick a vertex v
- while $deg(v) > 1$:

Degree 1 Vertex



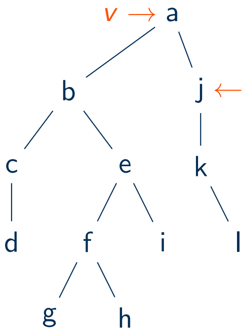
- pick a vertex v
- while $deg(v) > 1$:
    - move to an un-visited vertex

Degree 1 Vertex



- pick a vertex v
- while $deg(v) > 1$:
  - move to an un-visited vertex
  - call it v

Degree 1 Vertex



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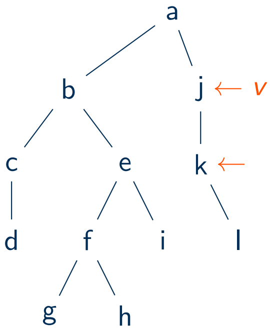

- pick a vertex *v*
- while $deg(v) > 1$:
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  - call it *v*

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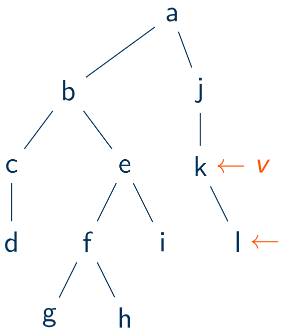

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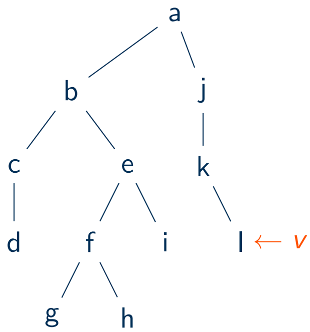

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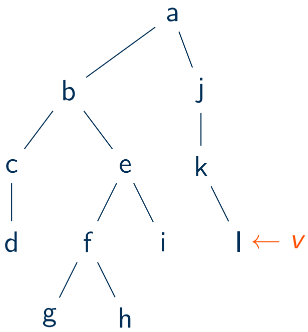


- pick a vertex v
- while $deg(v) > 1$:
    - move to an un-visited vertex
    - call it v
- Terminates as long as there are no circuits and the graph is finite.

Vertices and Edges

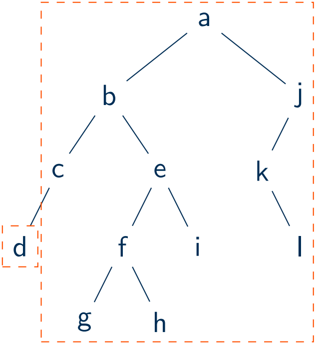
Lemma

Trees with two or more vertices has a vertex of degree 1.

Theorem

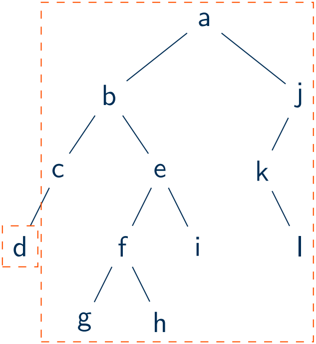
A tree with n vertices has exactly $n - 1$ edges.

$n - 1$ Edges in a Tree



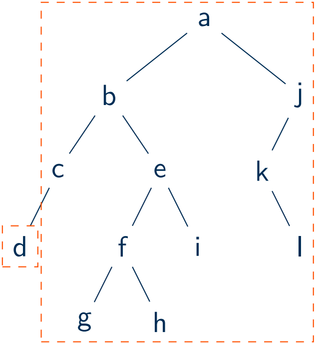
- "Obvious" if there are just two vertices.

$n - 1$ Edges in a Tree



- “Obvious” if there are just two vertices.
- By previous lemma there is at least 1 leaf

$n-1$ Edges in a Tree



- “Obvious” if there are just two vertices.
- By previous lemma there is at least 1 leaf
- Use induction with the subgraph not containing the leaf

Vertices and Edges

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A tree with n vertices has exactly $n - 1$ edges.

Theorem

A connected graph with n vertices and $n - 1$ edges is a tree.

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A connected graph with n vertices and $n - 1$ edges is a tree.

Theorem

A graph with at least as many edges as vertices has a circuit.

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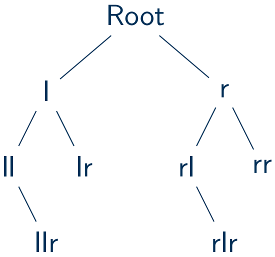
Binary Tree



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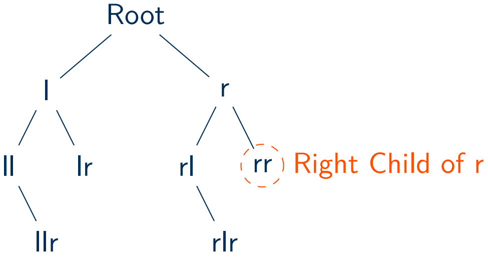
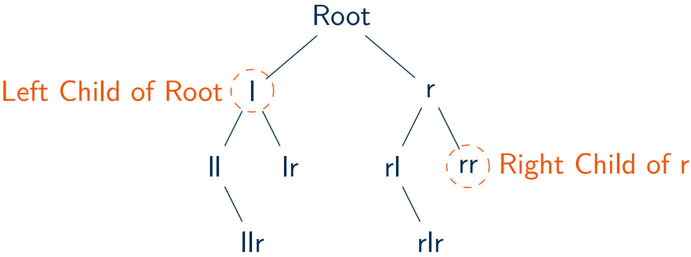


Right Child of r

Binary Tree



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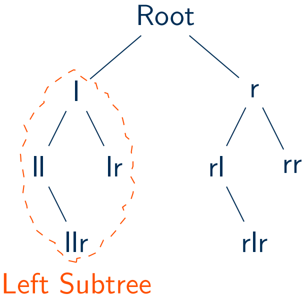

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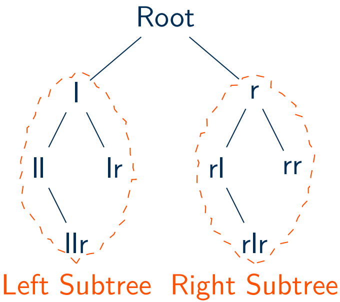
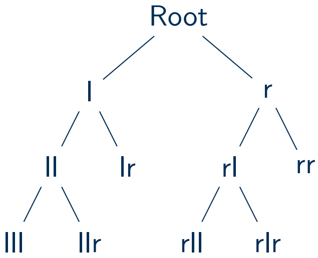


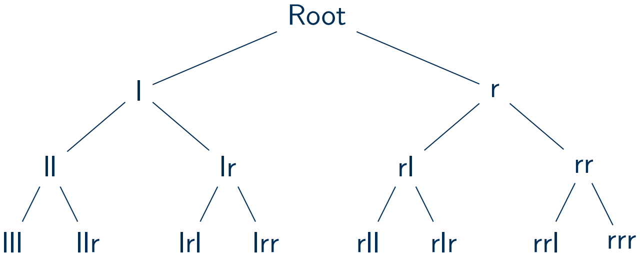
Full Binary Tree

Every *parent* has two *children*.



Complete Binary Tree

A *full binary tree* in which every *leaf* is at the same height.

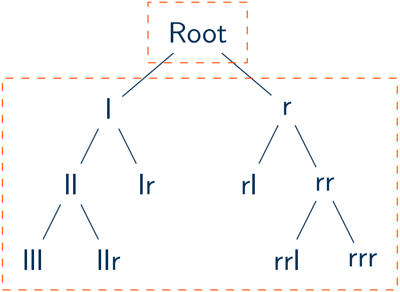


Theorems on Binary Trees

Theorem

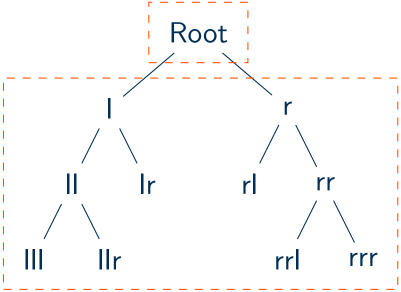
A full binary tree with k internal vertices has $2k+1$ total vertices and $k+1$ leaves.

Full Trees and Vertices



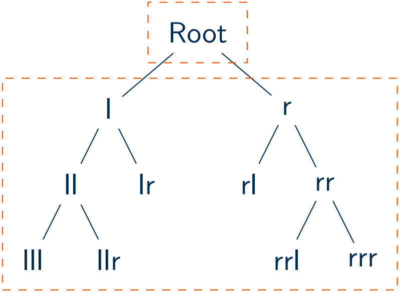
- All internal vertices have 2 children, $2k$ children.

Full Trees and Vertices



- All internal vertices have 2 children, $2k$ children.
- Only one vertex has no parent, *Root*.

Full Trees and Vertices



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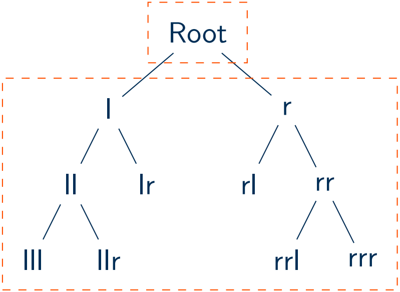
- *Total Vertices* $= 2k + 1$.

Full Trees and Vertices



- All internal vertices have 2 children, $2k$ children.
- Only one vertex has no parent, *Root*.
- $\mathit{Total\ Vertices} = 2k + 1$.
- Leaves aren't internal, $\mathit{Leaves} = 2k + 1 - k = k + 1$.

Theorems on Binary Trees

Theorem

A full binary tree with k internal vertices has $2k+1$ total vertices and $k+1$ leaves.

Theorem

Given a binary tree with height h and t leaves,

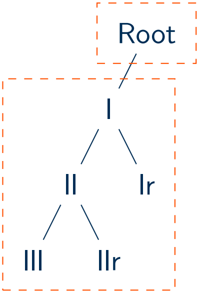
$$t \leq 2^h \text{ and } \log_2(t) \leq h.$$

Leaves and Height

Root

- $h = 0 \Rightarrow 2^h = 2^0 = 1 \geq t$

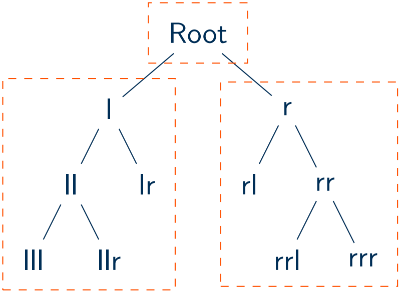
Leaves and Height



- $h = 0 \Rightarrow 2^h = 2^0 = 1 \geq t$
- $h = k + 1$ and *Root* has one child

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &\geq 2^k + 1 \\ &\geq t_l + 1 \\ &= t \end{aligned}$$

Leaves and Height



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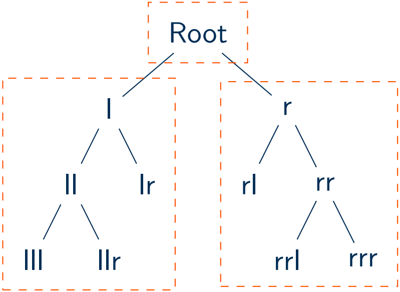
Leaves and Height



- $h = 0 \Rightarrow 2^h = 2^0 = 1 \geq t$
- $h = k + 1$ and *Root* has one child
- $h = k + 1$ and *Root* has two children
- $t \leq 2^h$ implies $\log_2(t) \leq h$

Theorems on Binary Trees

Theorem

A full binary tree with k *internal vertices has* $2k+1$ *total vertices and* $k+1$ *leaves.*

Theorem

Given a binary tree with height h *and* t *leaves,*

$$t \leq 2^h \text{ and } \log_2(t) \leq h.$$

Corollary

A complete binary tree of height h *has exactly* $t = 2^h$ *leaves.*

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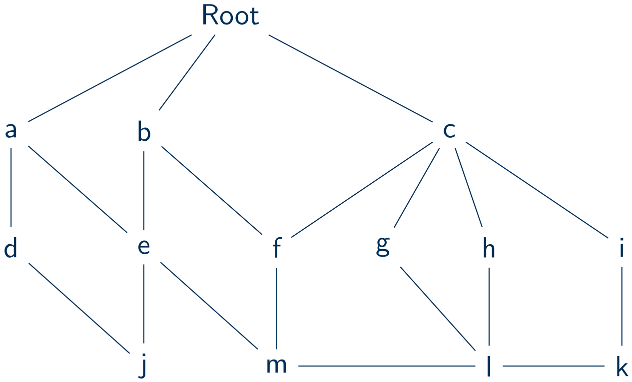
Spanning Tree Definition and Examples



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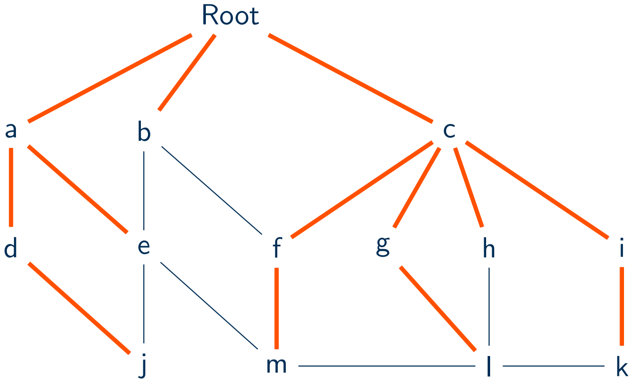

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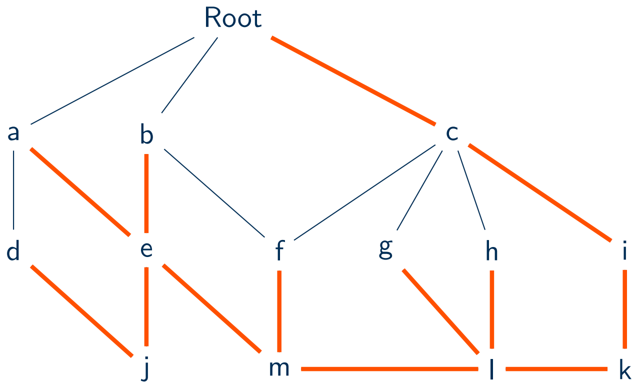

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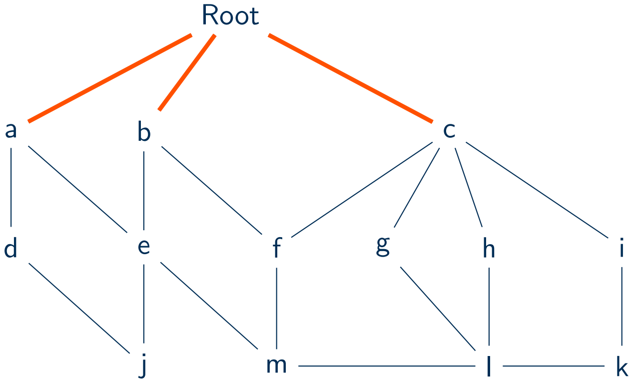

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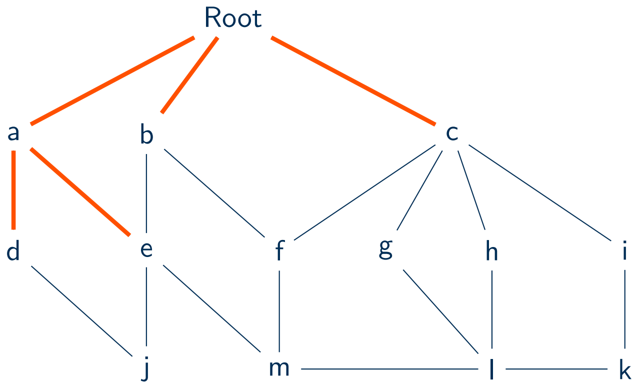

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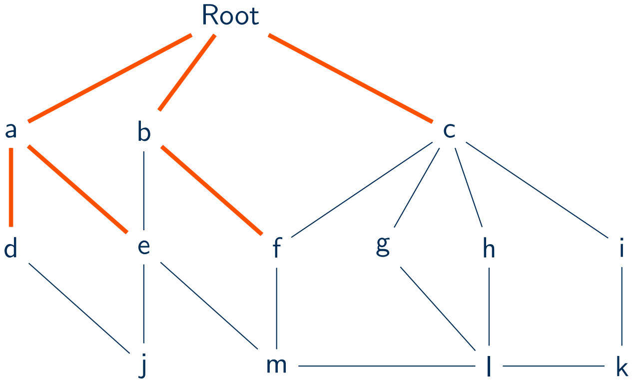

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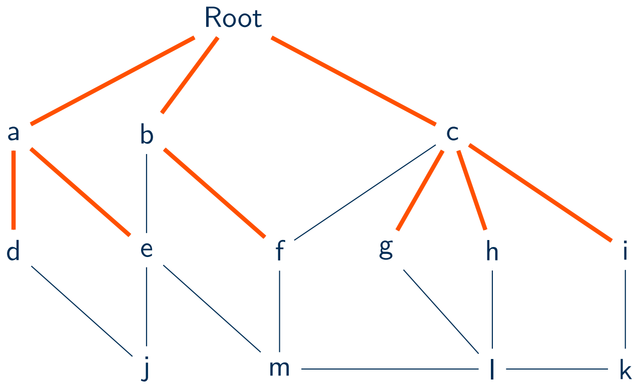

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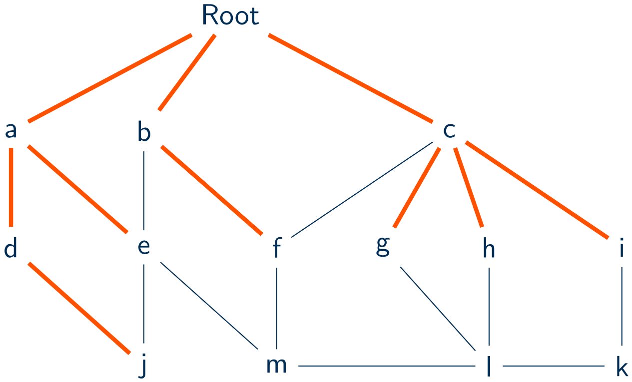

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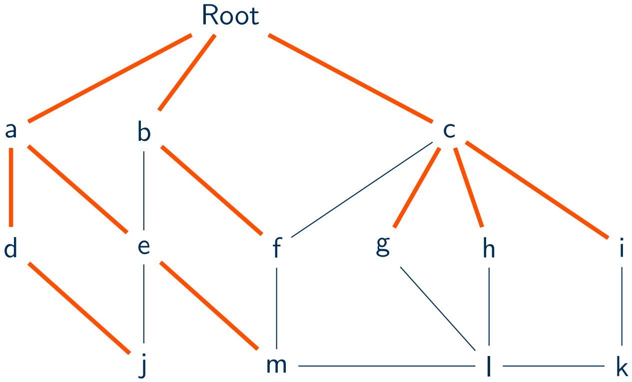

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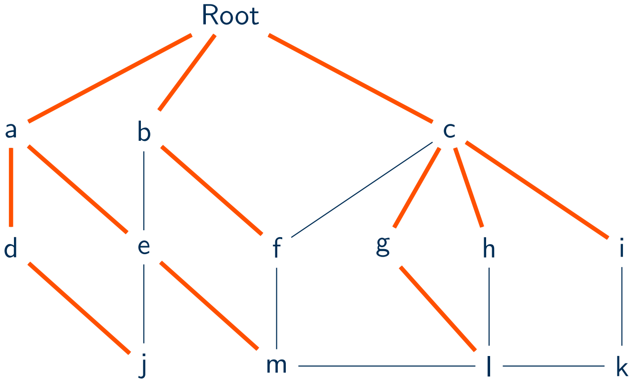

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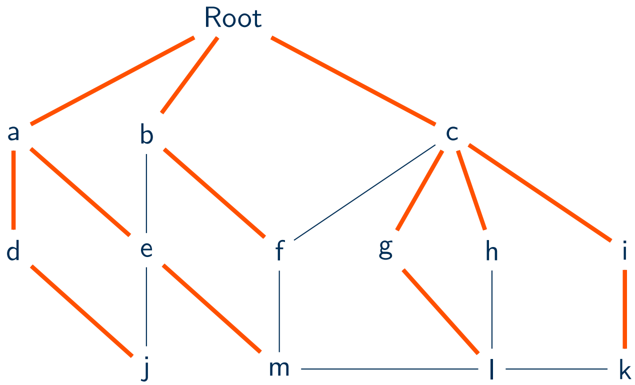
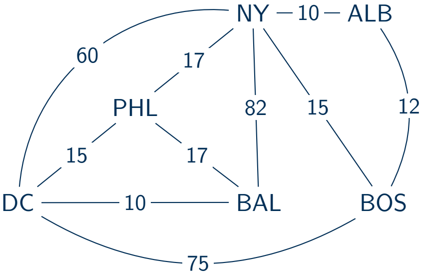


Kruskal's Minimal Tree Algorithm



- Find the unused edge with the lowest value
- If it doesn't create a circuit add it to the tree
- Repeat until there are $n - 1$ edges

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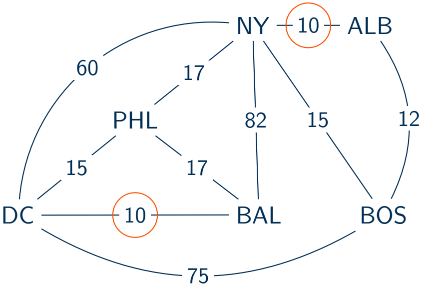

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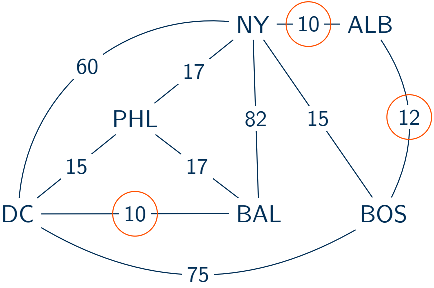

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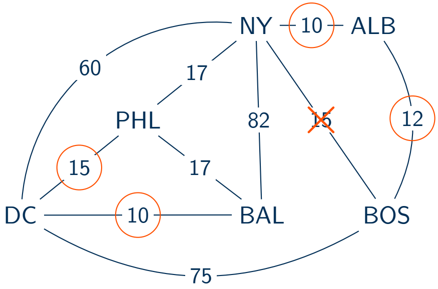

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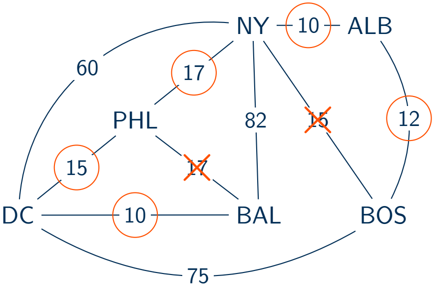

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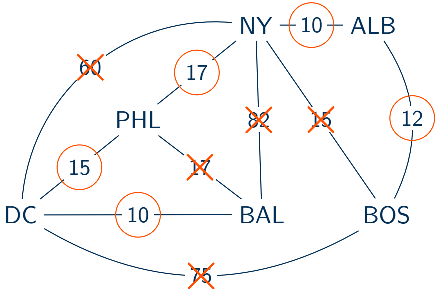

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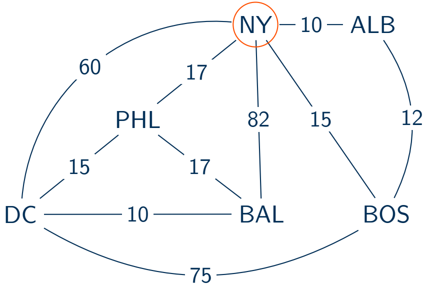
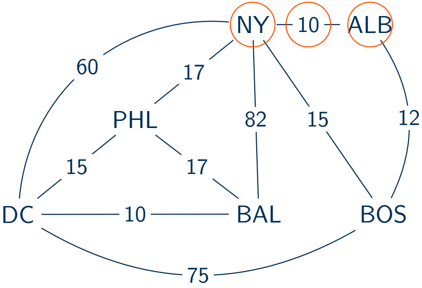
Prim's Minimal Tree Algorithm



- Pick a starting vertex to add to the tree
- Add the edge that has least weight and connects to ***exactly*** one vertex already in the tree
- Add the vertex on the other end of the edge to the tree
- Repeat $n - 1$ times

Prim's Minimal Tree Algorithm



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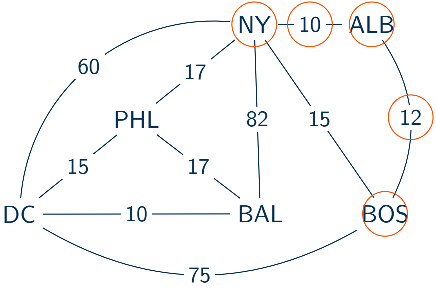

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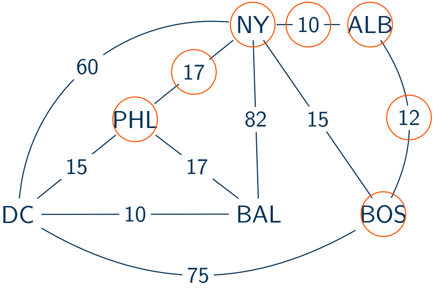

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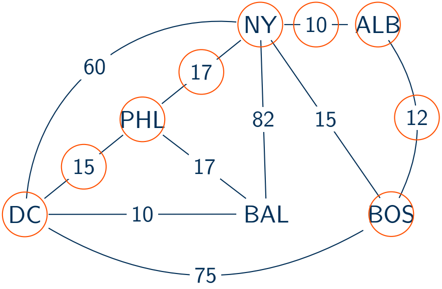
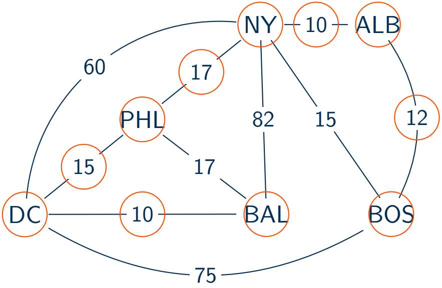
- Repeat $n - 1$ times

Prim's Minimal Tree Algorithm



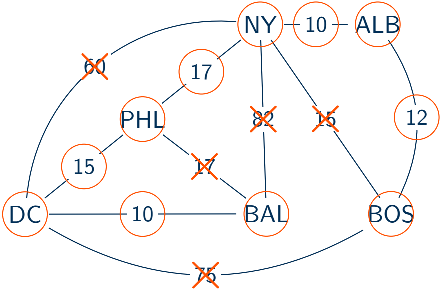
- Pick a starting vertex to add to the tree
- Add the edge that has least weight and connects to ***exactly*** one vertex already in the tree
- Add the vertex on the other end of the edge to the tree
- Repeat $n - 1$ times

Prim's Minimal Tree Algorithm



- Pick a starting vertex to add to the tree
- Add the edge that has least weight and connects to ***exactly*** one vertex already in the tree
- Add the vertex on the other end of the edge to the tree
- Repeat $n - 1$ times

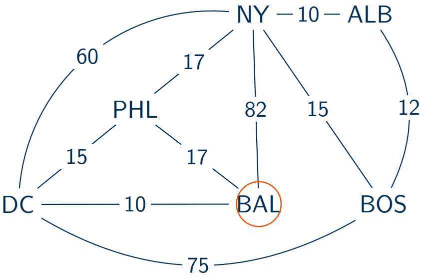
Prim's Minimal Tree Algorithm



- Pick a starting vertex to add to the tree
- Add the edge that has least weight and connects to ***exactly*** one vertex already in the tree
- Add the vertex on the other end of the edge to the tree
- Repeat $n - 1$ times

Dijkstra's "Shortest" Path Algorithm

Baltimore to Boston



Round 1

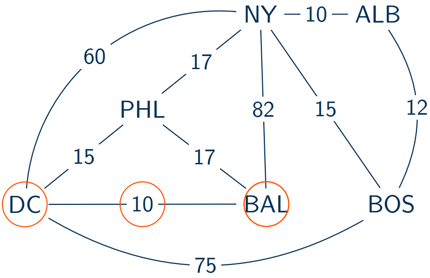
$v = BAL$

$F = \{DC, PHL, NY\}$

City	Old Label	New Label
ALB	$(\infty, -)$	$(\infty, -)$
BAL	$(0, -)$	$(0, -)$
BOS	$(\infty, -)$	$(\infty, -)$
DC	$(\infty, -)$	$(10, BAL)$
NY	$(\infty, -)$	$(82, BAL)$
PHL	$(\infty, -)$	$(17, BAL)$

Dijkstra's "Shortest" Path Algorithm

Baltimore to Boston



Round 2

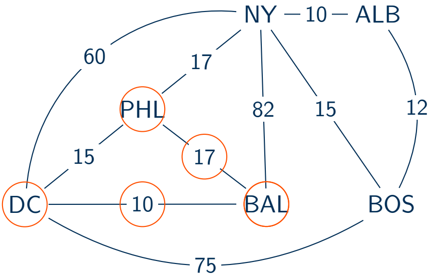
$v = DC$

$F = \{PHL, NY, BOS\}$

City	Old Label	New Label
ALB	$(\infty, -)$	$(\infty, -)$
BAL	$(0, -)$	$(0, -)$
BOS	$(\infty, -)$	$(85, DC)$
DC	$(10, BAL)$	$(10, BAL)$
NY	$(82, BAL)$	$(70, DC)$
PHL	$(17, BAL)$	$(17, BAL)$

Dijkstra's "Shortest" Path Algorithm

Baltimore to Boston



Round 3

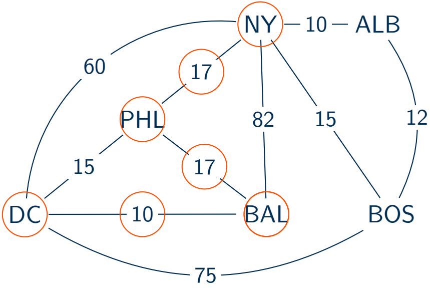
$v = PHL$

$F = \{NY, BOS\}$

City	Old Label	New Label
ALB	$(\infty, -)$	$(\infty, -)$
BAL	$(0, -)$	$(0, -)$
BOS	$(85, DC)$	$(85, DC)$
DC	$(10, BAL)$	$(10, BAL)$
NY	$(70, DC)$	$(34, PHL)$
PHL	$(17, BAL)$	$(17, BAL)$

Dijkstra's "Shortest" Path Algorithm

Baltimore to Boston



Round 4

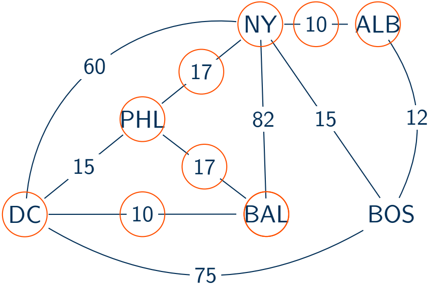
$v = NY$

$F = \{ALB, BOS\}$

City	Old Label	New Label
ALB	$(\infty, -)$	$(44, NY)$
BAL	$(0, -)$	$(0, -)$
BOS	$(85, DC)$	$(49, NY)$
DC	$(10, BAL)$	$(10, BAL)$
NY	$(34, PHL)$	$(34, PHL)$
PHL	$(17, BAL)$	$(17, BAL)$

Dijkstra's "Shortest" Path Algorithm

Baltimore to Boston



Round 5

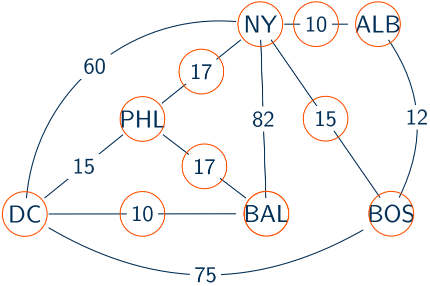
$v = ALB$

$F = \{BOS\}$

City	Old Label	New Label
ALB	$(44, NY)$	$(44, NY)$
BAL	$(0, -)$	$(0, -)$
BOS	$(49, NY)$	$(49, NY)$
DC	$(10, BAL)$	$(10, BAL)$
NY	$(34, PHL)$	$(34, PHL)$
PHL	$(17, BAL)$	$(17, BAL)$

Dijkstra's "Shortest" Path Algorithm

Baltimore to Boston



Round 6

$v = BOS$

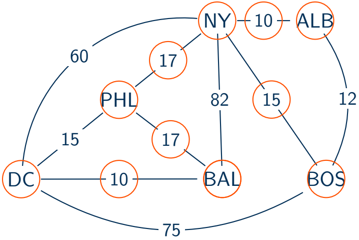
$F = \{\}$

City	Old Label	New Label
ALB	$(44, NY)$	$(44, NY)$
BAL	$(0, -)$	$(0, -)$
BOS	$(49, NY)$	$(49, NY)$
DC	$(10, BAL)$	$(10, BAL)$
NY	$(34, PHL)$	$(34, PHL)$
PHL	$(17, BAL)$	$(17, BAL)$

$$L(BOS) = (49, NY) \leftarrow L(NY) = (34, PHL) \leftarrow L(PHL) = (17, BAL)$$

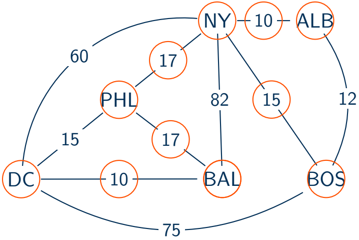
Dijkstra's "Shortest" Path Algorithm

- Start with vertex v at the start point



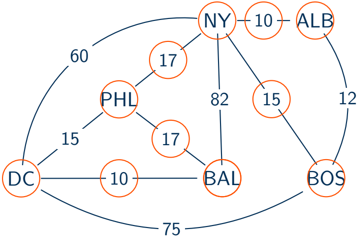
Dijkstra's "Shortest" Path Algorithm

- Start with vertex *v* at the start point
- Update the *label values* for vertices adjacent to *v*



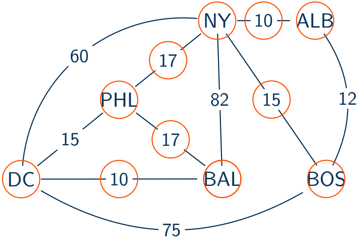
Dijkstra's "Shortest" Path Algorithm

- Start with vertex *v* at the start point
- Update the *label values* for vertices adjacent to *v*
- Update the *fringe F* by removing *v* and adding vertices adjacent to *v*



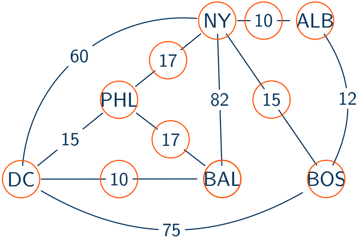
Dijkstra's "Shortest" Path Algorithm

- Start with vertex v at the start point
- Update the *label values* for vertices adjacent to v
- Update the *fringe* F by removing v and adding vertices adjacent to v
- Update v to be the vertex in F with the lowest label value



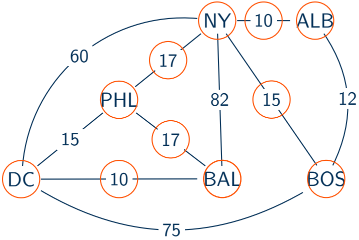
Dijkstra's "Shortest" Path Algorithm

- Start with vertex v at the start point
- Update the *label values* for vertices adjacent to v
- Update the *fringe* F by removing v and adding vertices adjacent to v
- Update v to be the vertex in F with the lowest label value
- Add the new v to the tree *along with the edge that let it achieve its minimal label value*

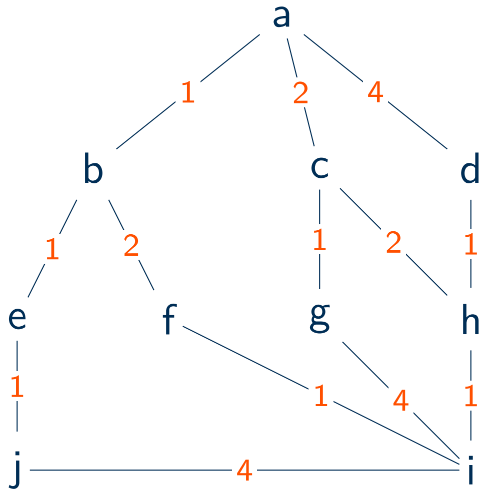


Dijkstra's "Shortest" Path Algorithm

- Start with vertex v at the start point
- Update the *label values* for vertices adjacent to v
- Update the *fringe* F by removing v and adding vertices adjacent to v
- Update v to be the vertex in F with the lowest label value
- Add the new v to the tree *along with the edge that let it achieve its minimal label value*
- Repeat until you reach the destination



One More Example



Trees

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