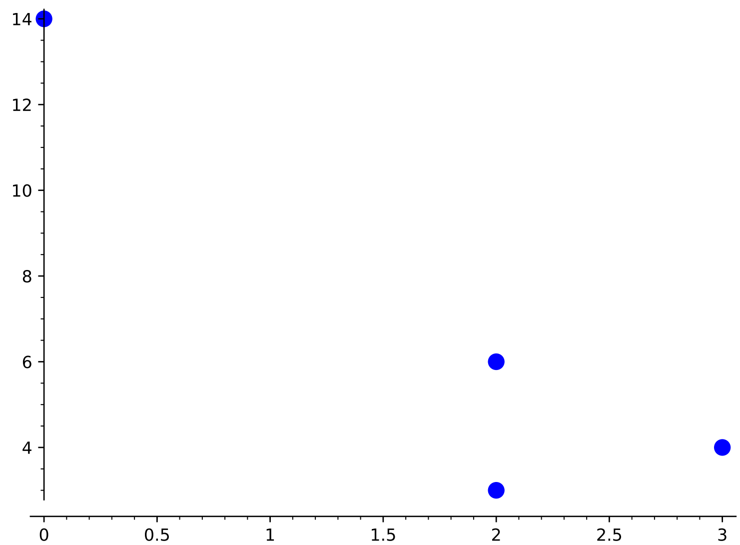
Suppose that we wish to find the line or parabola that best fits the list of data points $[[0,14],[2,3],[2,6],[3,4]]$.



We begin by forming a matrix whose first column is a vector of x values and whose second column is a vector of $1's$, and form a vector of the y values. Then we would like to find a vector $\vec{\beta}$ such that.

$$A\vec{\beta} = \vec{y} \equiv \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \vec{\beta} = \begin{pmatrix} 14 \\ 3 \\ 6 \\ 4 \end{pmatrix}.$$

Since this is an inconsistent system a solution won't exist so we need the *orthogonal projection* of $\vec{y}$ onto the column space of A. To do this we first need to find an *orthonormal basis* for the column space of A. The othogonal component of $\vec{x}$ (the first column of A) with respect to $\vec{1}$ (the second column of A) is

$$\vec{x'} = \vec{x} - proj_{\vec{1}}(\vec{x}) = \vec{x} - \left(\frac{\vec{1}\cdot\vec{x}}{\vec{1}\cdot\vec{x}}\vec{1}\right) = \begin{pmatrix} -\frac{7}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{5}{4} \end{pmatrix}$$

Then we need to *normalize* each of the vectors $\vec{x'}$ and $\vec{1}$ so that we get the matrix

$$U = \begin{pmatrix} -\frac{7}{38}\sqrt{19} & \frac{1}{2} \\ \frac{1}{38}\sqrt{19} & \frac{1}{2} \\ \frac{1}{38}\sqrt{19} & \frac{1}{2} \\ \frac{5}{38}\sqrt{19} & \frac{1}{2} \end{pmatrix}$$

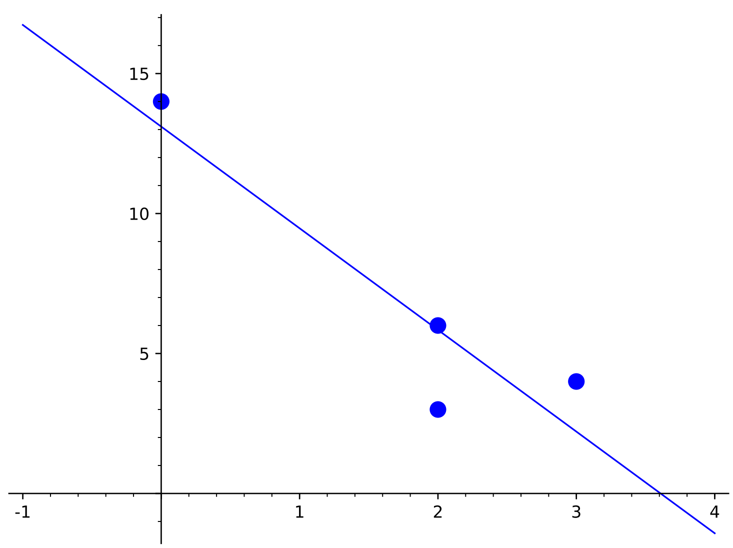
which has the some column space as A but whose columns form the desired *orthonormal basis*. Then the projection of $\vec{y}$ onto the column space will be

$$\begin{aligned} \widehat{y} &= proj_{u_1}\vec{y} + proj_{u_2}\vec{y} && (1) \\ &= U\,U^T\,\vec{y} && (2) \\ &= \begin{pmatrix} \frac{249}{19} & \frac{111}{19} & \frac{111}{19} & \frac{42}{19} \end{pmatrix}^T, && (3) \end{aligned}$$

where u_1 and u_2 are the columns of U. Then solving the equation $A\widehat{\beta} = \widehat{y}$ for $\widehat{\beta}$ we see that the coordinates for $\widehat{y}$ in terms of the basis given by the columns of A are

$$\widehat{\beta} = \begin{pmatrix} -\frac{69}{19} \\ \frac{249}{19} \end{pmatrix}.$$

This means that the line of best fit is $l(x) = -\frac{69}{19}\,x + \frac{249}{19}$ which looks like



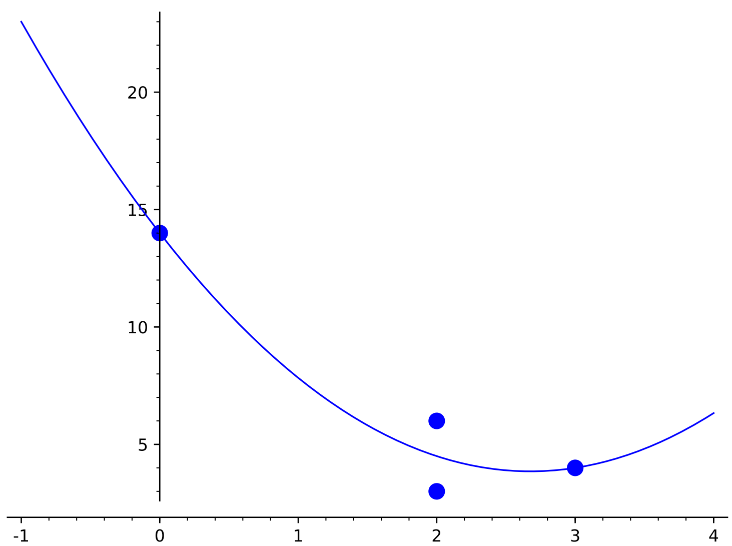
In a similar manner we can fit a parabola to the data points by projecting $\vec{y}$ onto the column space of

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix},$$

which is the same as the column space of the matrix

$$U = \begin{pmatrix} \sqrt{\frac{2}{19}} & -\frac{7}{38}\sqrt{19} & \frac{1}{2} \\ -\frac{3}{2}\sqrt{\frac{2}{19}} & \frac{1}{38}\sqrt{19} & \frac{1}{2} \\ -\frac{3}{2}\sqrt{\frac{2}{19}} & \frac{1}{38}\sqrt{19} & \frac{1}{2} \\ 2\sqrt{\frac{2}{19}} & \frac{5}{38}\sqrt{19} & \frac{1}{2} \end{pmatrix},$$

whose columns form an orthonormal basis. Doing this we will get the parapola $p(x) = \frac{17}{12}x^2 - \frac{91}{12}x + 14$ which looks like



But, of course, there is a clever way to do this. Since the $\widehat{y}$ is in the column space of A we know that $\vec{y}-\widehat{y}$ is in the *orthogonal complement* $col(A)^{\perp}$. This means that

$$A^T(\vec{y}-\widehat{y}) = \vec{0},$$

which given, $A\widehat{\beta} = \widehat{y}$, can be written as

$$A^T(\vec{y}-A\widehat{\beta}) = 0,$$

or with a little work

$$\widehat{\beta} = (A^TA)^{-1}A^T\vec{y}.$$

Substituting in the given values for A and $\vec{y}$ we get

$$\begin{aligned}
\widehat{\beta} &= (A^TA)^{-1}A^T\vec{y} && (4)\\
&= \left(\begin{pmatrix} 0 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ 3 \\ 6 \\ 4 \end{pmatrix} && (5)\\
&= \begin{pmatrix} \frac{4}{19} & -\frac{7}{19} \\ -\frac{7}{19} & \frac{17}{19} \end{pmatrix} \begin{pmatrix} 0 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 14 \\ 3 \\ 6 \\ 4 \end{pmatrix} && (6)\\
&= \begin{pmatrix} -\frac{7}{19} & \frac{1}{19} & \frac{1}{19} & \frac{5}{19} \\ \frac{17}{19} & \frac{3}{19} & \frac{3}{19} & -\frac{4}{19} \end{pmatrix} \begin{pmatrix} 14 \\ 3 \\ 6 \\ 4 \end{pmatrix} && (7)\\
&= \begin{pmatrix} -\frac{69}{19} \\ \frac{249}{19} \end{pmatrix} && (8)
\end{aligned}$$