Notes : October 11^{th}, 2018

> Find the basis for the row, column, and null spaces of the matrix:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 7 & 1 & 0 \\ 7 & 16 & 2 & 1 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

$$\xrightarrow[R_3 - 7R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 0 & -1 & -1 & 3 \end{bmatrix} \xrightarrow[R_4 + R_2,\ R_1 - 2R_2]{R_3 - 2R_2} \begin{bmatrix} 1 & 0 & -2 & 7 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

"Pivot positions"

- Basis for Row A: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 1 \\ 0 \end{pmatrix} \right\}$ (dimension = 2)
- Basis for Col A: $\left\{ \begin{pmatrix} 1 \\ 3 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 16 \\ -1 \end{pmatrix} \right\}$ (dimension = 2)
- Basis for Nul A = $\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$ (dimension = 2)

* Because there are four entries, four dimensions are needed to represent this.

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\begin{bmatrix} 1 & 0 & -2 & 7 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x &= 2z - 7 \\ y &= -z + 3 \\ z &= z \\ w &= w \end{aligned}$$

- The basis for the row, column, and null spaces of the matrix A are,

$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 1 \\ 0 \end{pmatrix} \right\}$, $\left\{ \begin{pmatrix} 1 \\ 3 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 16 \\ -1 \end{pmatrix} \right\}$, and $\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$ respectively.

notes continued:

> Is the vector $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 1 \end{pmatrix}$ in the span of the Null Space?

think Linear Combination

$$a\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \\ 1 \end{pmatrix}$$

When $a = 4$ and $b = 1$, the vector $\vec{v}$ is in the span of the Null Space.

> Is the vector $\vec{u} = \begin{pmatrix} -1 \\ 15 \\ 0 \\ 7 \end{pmatrix}$ in the Row space of A?

$$a\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 3 \\ 7 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 15 \\ 0 \\ 7 \end{pmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 2 & 7 & 15 \\ 1 & 3 & -1 \end{array}\right]$$

$R_4 \sim R_1$
$R_3 \sim R_2$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 2 & 7 & 15 \\ 1 & 3 & -1 \end{array}\right]$$

* No, vector $\vec{u}$ is <u>not</u> in the row space of A.
  - There would have to be a zero coefficient.

> Is $\vec{u}$ in the span of the Null Space and Row space combined?

- Yes, $\vec{u}$ is in the span of the Null Space and Row space combined because the null and row space combined is 4 space and we have a four dimensional vector.

$$a\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 3 \\ 7 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 15 \\ 0 \\ 7 \end{pmatrix}$$

* Make an augmented matrix
* Row reduce

$\longrightarrow$

notes continued:

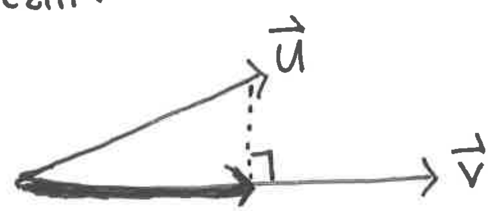
$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & -7 & -1 \\ 2 & 7 & -1 & 3 & 15 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 7 \end{array}\right] \xrightarrow[R_2 \sim R_3]{R_4 \sim R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & -7 & 15 \\ 2 & 7 & -1 & 3 & -1 \end{array}\right] \xrightarrow[R_4 - 2R_1 - 7R_2]{R_3 - R_1 - 3R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -8 & -8 \\ 0 & 0 & -8 & 1 & 1 \end{array}\right] \xrightarrow{R_4 - 8R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -8 & -8 \\ 0 & 0 & 0 & 65 & 65 \end{array}\right] \xrightarrow[\frac{1}{65}R_4]{-R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 7 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 & 8 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - 8R_4]{R_1 - R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad \text{solution: } \begin{pmatrix} 6 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

* note:

Because the null space and row space are perpendicular, recall:



$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} (\vec{v})$$

$\vec{u} = \langle -1 \;\; 15 \;\; 0 \;\; 7 \rangle$

$\text{proj}_{r_1} \vec{u} = \frac{(36)}{6} \cdot \vec{r}_1 = 6\vec{r}_1 \Rightarrow$ Best approximate of r_1.

$\text{proj}_{r_2} \vec{u} = \frac{102}{59} \vec{r}_2 =$ Best approximate of r_2.

$6\vec{r}_1 + \frac{102}{59} \vec{r}_2$

$$\begin{aligned} \vec{u} - 6\vec{r}_1 = &\ \langle -1 \;\; 15 \;\; 0 \;\; 7 \rangle \\ &\ \underline{-6\langle 1 \;\; 2 \;\; 0 \;\; 1 \rangle} \\ &\ \langle -7 \;\; 3 \;\; 0 \;\; 1 \rangle \end{aligned}$$