Grammars and Pushdown Automata

Dr. Chuck Rocca
roccac@wcsu.edu
http://sites.wcsu.edu/roccac

WESTERN CONNECTICUT STATE UNIVERSITY
MACRICOSTAS SCHOOL OF ARTS & SCIENCES

Table of Contents

Table of Contents

A Simple Example

Grammar:

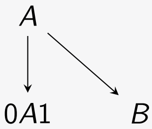
$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

A

A Simple Example

Grammar:

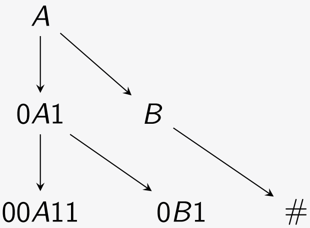
$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$



A Simple Example

Grammar:

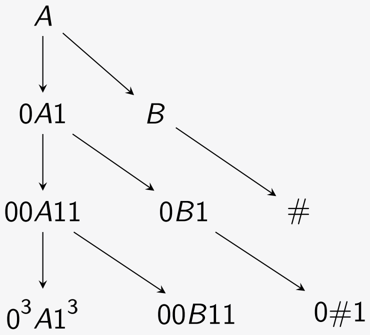
$$A \to 0A1$$
$$A \to B$$
$$B \to \#$$



A Simple Example

Grammar:

$$A \to 0A1$$
$$A \to B$$
$$B \to \#$$



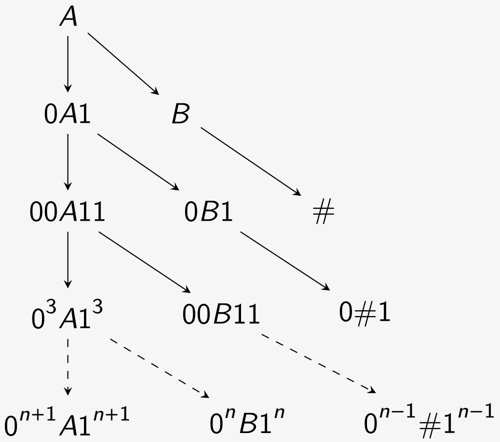
A Simple Example

Grammar:

$A \rightarrow 0A1$

$A \rightarrow B$

$B \rightarrow \#$



A Simple Example: Derivation and Parse Tree

Grammar:

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Derivation of 000#111:

$$A \Rightarrow$$

Parse Tree for 000#111:

A

A Simple Example: Derivation and Parse Tree

Grammar:

$$A \rightarrow 0A1$$

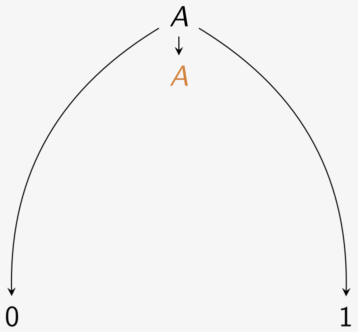
$$A \rightarrow B$$

$$B \rightarrow \#$$

Derivation of 000#111:

$$A \Rightarrow 0A1$$

Parse Tree for 000#111:



A Simple Example: Derivation and Parse Tree

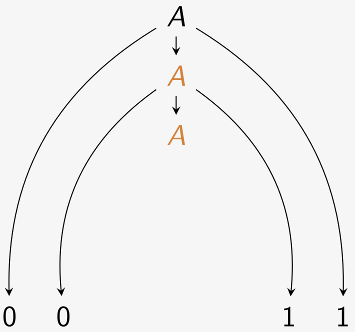
Grammar:

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Derivation of 000#111:

$$A \Rightarrow 0A1$$
$$\Rightarrow 0\,0A1\,1$$

Parse Tree for 000#111:



A Simple Example: Derivation and Parse Tree

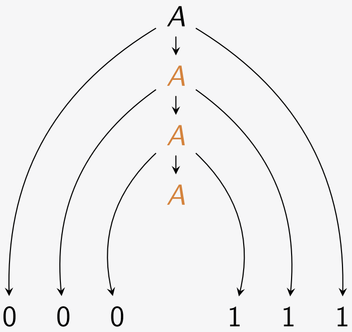
Grammar:

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Derivation of 000#111:

$$A \Rightarrow 0A1$$
$$\Rightarrow 0\,0A1\,1$$
$$\Rightarrow 00\,0A1\,11$$

Parse Tree for 000#111:



A Simple Example: Derivation and Parse Tree

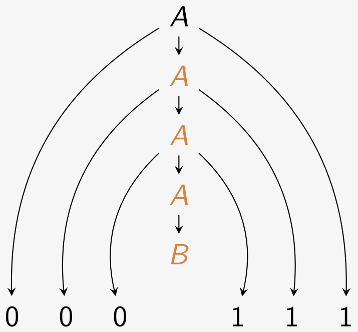
Grammar:

$$A \to 0A1$$
$$A \to B$$
$$B \to \#$$

Derivation of $000\#111$:

$$A \Rightarrow 0A1$$
$$\Rightarrow 0\,0A1\,1$$
$$\Rightarrow 00\,0A1\,11$$
$$\Rightarrow 000\,B\,111$$

Parse Tree for $000\#111$:



A Simple Example: Derivation and Parse Tree

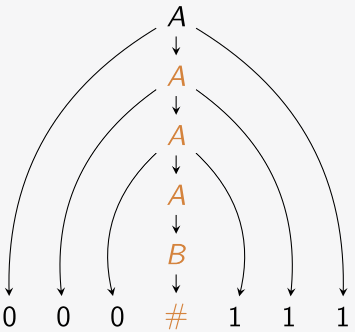
Grammar:

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Derivation of 000#111:

$$A \Rightarrow 0A1$$
$$\Rightarrow 0\,0A1\,1$$
$$\Rightarrow 00\,0A1\,11$$
$$\Rightarrow 000\,B\,111$$
$$\Rightarrow 000\,\#\,111$$

Parse Tree for 000#111:



Formal Definition

Definition (Context-Free Grammar)

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,

Grammar:

$$A \to 0A1$$
$$A \to B$$
$$B \to \#$$

Variables: $V = \{A, B\}$

Formal Definition

Definition (Context-Free Grammar)

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,

Grammar:

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Terminals: $\Sigma = \{0, 1, \#\}$

Formal Definition

Definition (Context-Free Grammar)

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule associating a variable to a string of variables and/or terminals, and

Grammar:

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Rules

Formal Definition

Definition (Context-Free Grammar)

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule associating a variable to a string of variables and/or terminals, and
4. $S \in V$ is a ***start variable***.

Grammar:

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Start Variable: $S = A$

Designing a CFG

$w * w^R$

Give a CFG what recognizes the language of all words beginning with palindromes using $\{0, 1\}$ with a random string of $\{a, b\}$ in between, e.g. 10110 (*abbabbaba*) 01101.

Arbitrary Word:

$$S_1 \rightarrow aS_1 | bS_1 | \varepsilon$$

Designing a CFG

$w * w^R$

Give a CFG what recognizes the language of all words beginning with palindromes using $\{0, 1\}$ with a random string of $\{a, b\}$ in between, e.g. 10110 $(abbabbaba)$ 01101.

Arbitrary Word:

$$S_1 \to aS_1 | bS_1 | \varepsilon$$

Even Length Palindrome, ww^R:

$$S_2 \to 0X0 | 1X1$$
$$X \to S_2 | \varepsilon$$

Designing a CFG

$w * w^R$

Give a CFG what recognizes the language of all words beginning with palindromes using $\{0, 1\}$ with a random string of $\{a, b\}$ in between, e.g. 10110 (*abbabbaba*) 01101.

Arbitrary Word:

$$S_1 \rightarrow aS_1 | bS_1 | \varepsilon$$

Even Length Palindrome, ww^R:

$$S_2 \rightarrow 0X0 | 1X1$$
$$X \rightarrow S_2 | \varepsilon$$

Combining Them, $w * w^R$:

$$S \rightarrow S_2$$
$$S_2 \rightarrow 0X0 | 1X1$$
$$X \rightarrow S_2 | (S_1)$$
$$S_1 \rightarrow aS_1 | bS_1 | \varepsilon$$

A Couple Variations

Combining Grammars

$$S_1 \to aS_1 | bS_1 | \varepsilon$$

$$\begin{aligned} S_2 &\to 0X0 | 1X1 \\ X &\to S_2 | \varepsilon \end{aligned}$$

$$\begin{aligned} S &\to S_2 \\ S_2 &\to 0X0 | 1X1 \\ X &\to S_2 | (S_1) \\ S_1 &\to aS_1 | bS_1 | \varepsilon \end{aligned}$$

A Couple Variations

Combining Grammars

$$S_1 \to aS_1 | bS_1 | \varepsilon$$

$$\begin{aligned} S_2 &\to 0X0 | 1X1 \\ X &\to S_2 | \varepsilon \end{aligned}$$

$$\begin{aligned} S &\to S_2 \\ S_2 &\to 0X0 | 1X1 \\ X &\to S_2 | (S_1) \\ S_1 &\to aS_1 | bS_1 | \varepsilon \end{aligned}$$

$$\begin{aligned} S &\to S_1 | S_2 \\ S_2 &\to 0X0 | 1X1 \\ X &\to S_2 | \varepsilon \\ S_1 &\to aS_1 | bS_1 | \varepsilon \end{aligned}$$

A Couple Variations

Combining Grammars

$$S_1 \rightarrow aS_1 | bS_1 | \varepsilon$$

$$\begin{aligned} S_2 &\rightarrow 0X0 | 1X1 \\ X &\rightarrow S_2 | \varepsilon \end{aligned}$$

$$\begin{aligned} S &\rightarrow S_2 \\ S_2 &\rightarrow 0X0|1X1 \\ X &\rightarrow S_2|(S_1) \\ S_1 &\rightarrow aS_1|bS_1|\varepsilon \end{aligned}$$

$$\begin{aligned} S &\rightarrow S_1|S_2 \\ S_2 &\rightarrow 0X0|1X1 \\ X &\rightarrow S_2|\varepsilon \\ S_1 &\rightarrow aS_1|bS_1|\varepsilon \end{aligned}$$

$$\begin{aligned} S &\rightarrow S_1 S_2 \\ S_2 &\rightarrow 0X0|1X1 \\ X &\rightarrow S_2|\varepsilon \\ S_1 &\rightarrow aS_1|bS_1|\varepsilon \end{aligned}$$

A Couple Variations

Combining Grammars

$$S_1 \to aS_1 | bS_1 | \varepsilon$$

$$\begin{aligned} S_2 &\to 0X0 | 1X1 \\ X &\to S_2 | \varepsilon \end{aligned}$$

$$\begin{aligned} S &\to S_2 \\ S_2 &\to 0X0 | 1X1 \\ X &\to S_2 | (S_1) \\ S_1 &\to aS_1 | bS_1 | \varepsilon \end{aligned}$$

$$\begin{aligned} S &\to S_1 | S_2 \\ S_2 &\to 0X0 | 1X1 \\ X &\to S_2 | \varepsilon \\ S_1 &\to aS_1 | bS_1 | \varepsilon \end{aligned}$$

$$\begin{aligned} S &\to S_1 S_2 \\ S_2 &\to 0X0 | 1X1 \\ X &\to S_2 | \varepsilon \\ S_1 &\to aS_1 | bS_1 | \varepsilon \end{aligned}$$

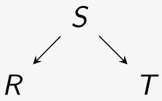
$$\begin{aligned} S &\to S_2 S_2 \\ S_2 &\to 0X0 | 1X1 | S \\ X &\to S_2 | \varepsilon \end{aligned}$$

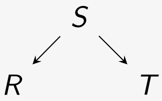
Example with Ambiguity

Given the grammar:

$$S \rightarrow RT$$
$$R \rightarrow aR|Rb|\varepsilon$$
$$T \rightarrow Tb|\varepsilon$$

there are multiple ways to derive the string *aab*. This is an ***ambiguous*** grammar.

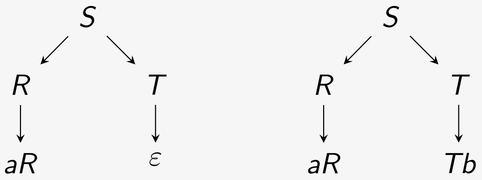




Example with Ambiguity

Given the grammar:

$$S \rightarrow RT$$
$$R \rightarrow aR|Rb|\varepsilon$$
$$T \rightarrow Tb|\varepsilon$$



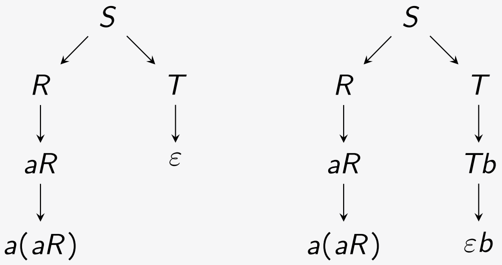
there are multiple ways to derive the string *aab*. This is an ***ambiguous*** grammar.

Example with Ambiguity

Given the grammar:

$$S \rightarrow RT$$
$$R \rightarrow aR|Rb|\varepsilon$$
$$T \rightarrow Tb|\varepsilon$$

there are multiple ways to derive the string *aab*. This is an ***ambiguous*** grammar.

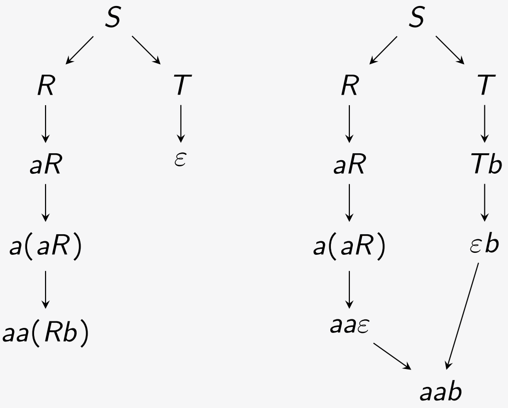


Example with Ambiguity

Given the grammar:

$$S \rightarrow RT$$
$$R \rightarrow aR|Rb|\varepsilon$$
$$T \rightarrow Tb|\varepsilon$$

there are multiple ways to derive the string *aab*. This is an ***ambiguous*** grammar.



Example with Ambiguity

Given the grammar:

$$S \rightarrow RT$$
$$R \rightarrow aR|Rb|\varepsilon$$
$$T \rightarrow Tb|\varepsilon$$

there are multiple ways to derive the string aab. This is an ***ambiguous*** grammar.

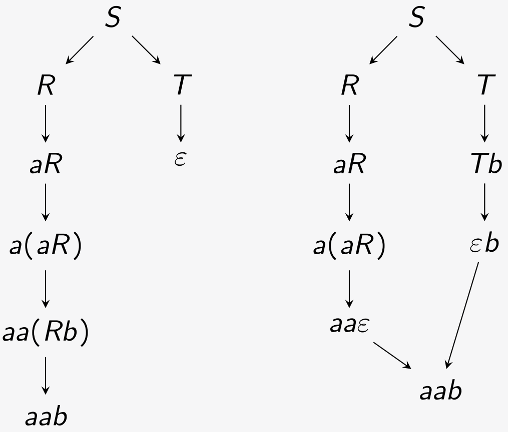


Table of Contents

Chomsky Normal Form

Definition

A context-free grammar is in ***Chomsky Normal Form*** if every rule is of the form

$$A \to BC \text{ or}$$
$$A \to a$$

where a is a terminal, A, B, and C are variables, B and C are not start variables, and only a start variable my point to the empty string ε

Converting to Chomsky Normal Form

New Start:
Replace

$$S \rightarrow A|b$$
$$A \rightarrow aS|a$$

With

$$S_0 \rightarrow S$$
$$S \rightarrow A|b$$
$$A \rightarrow aS|a$$

Converting to Chomsky Normal Form

New Start:
Replace

$$S \to A|b$$
$$A \to aS|a$$

With

$$S_0 \to S$$
$$S \to A|b$$
$$A \to aS|a$$

No ε:
Replace

$$A \to XBY$$
$$B \to \varepsilon$$

With

$$A \to XBY$$
$$A \to XY$$

Converting to Chomsky Normal Form

New Start:
Replace

$$S \to A|b$$
$$A \to aS|a$$

With

$$S_0 \to S$$
$$S \to A|b$$
$$A \to aS|a$$

No ε:
Replace

$$A \to XBY$$
$$B \to \varepsilon$$

With

$$A \to XBY$$
$$A \to XY$$

No Units:
Replace

$$A \to B$$
$$B \to XY|a$$

With

$$A \to XY|a$$
$$B \to XY|a$$

Converting to Chomsky Normal Form

New Start:
Replace

$$S \to A|b$$
$$A \to aS|a$$

With

$$S_0 \to S$$
$$S \to A|b$$
$$A \to aS|a$$

No ε:
Replace

$$A \to XBY$$
$$B \to \varepsilon$$

With

$$A \to XBY$$
$$A \to XY$$

No Units:
Replace

$$A \to B$$
$$B \to XY|a$$

With

$$A \to XY|a$$
$$B \to XY|a$$

New Variables:
Replace:

$$A \to BXY$$
$$B \to bX$$

With:

$$A \to BU$$
$$U \to XY$$
$$B \to VX$$
$$V \to b$$

Chomsky Example Steps 1-3

$$
\begin{aligned}
S_0 &\rightarrow 0X0|1X1 \\
X &\rightarrow S_0|(S_1)|\varepsilon \\
S_1 &\rightarrow aS_1|bS_1|\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
S &\rightarrow S_0 \\
S_0 &\rightarrow 0X0|1X1 \\
X &\rightarrow S_0|(S_1)|\ \varepsilon \\
S_1 &\rightarrow aS_1|bS_1|\ \varepsilon
\end{aligned}
$$

$$
\begin{aligned}
S &\rightarrow S_0 \\
S_0 &\rightarrow 0X0|1X1 \\
S_0 &\rightarrow 00|11 \\
X &\rightarrow S_0|(S_1)|() \\
S_1 &\rightarrow aS_1|bS_1 \\
S_1 &\rightarrow a|b
\end{aligned}
$$

Chomsky Example Steps 3-5

$$
\begin{aligned}
S &\rightarrow S_0 \\
S_0 &\rightarrow 0X0|1X1 \\
S_0 &\rightarrow 00|11 \\
X &\rightarrow S_0|(S_1)|() \\
S_1 &\rightarrow aS_1|bS_1|a|b
\end{aligned}
$$

$$
\begin{aligned}
S &\rightarrow S_0 \\
S_0 &\rightarrow 0W_0|1W_1 \\
W_0 &\rightarrow X0 \\
W_1 &\rightarrow X1 \\
S_0 &\rightarrow 00|11 \\
X &\rightarrow S_0|(S_1)|() \\
S_1 &\rightarrow S_1S_1|a|b
\end{aligned}
$$

$$
\begin{aligned}
S &\rightarrow S_0 \\
S_0 &\rightarrow 0W_0|1W_1 \\
W_0 &\rightarrow X0 \\
W_1 &\rightarrow X1 \\
S_0 &\rightarrow 00|11 \\
X &\rightarrow S_0|(S_1)|() \\
S_1 &\rightarrow S_1S_1|a|b
\end{aligned}
$$

Chomsky Example Steps 6-7

$$
\begin{aligned}
S &\rightarrow S_0 \\
S_0 &\rightarrow U_0 W_0 \,|\, U_1 W_1 \\
W_0 &\rightarrow X U_0 \\
W_1 &\rightarrow X U_1 \\
S_0 &\rightarrow U_0 U_0 \,|\, U_1 U_1 \\
U_0 &\rightarrow 0 \\
U_1 &\rightarrow 1 \\
X &\rightarrow S_0 \,|\, (S_1) \,|\, () \\
S_1 &\rightarrow S_1 S_1 \,|\, a \,|\, b
\end{aligned}
$$

$$
\begin{aligned}
S &\rightarrow U_0 W_0 \,|\, U_1 W_1 \,|\, U_0 U_0 \,|\, U_1 U_1 \\
S_0 &\rightarrow U_0 W_0 \,|\, U_1 W_1 \,|\, U_0 U_0 \,|\, U_1 U_1 \\
W_0 &\rightarrow X U_0 \\
W_1 &\rightarrow X U_1 \\
U_0 &\rightarrow 0 \\
U_1 &\rightarrow 1 \\
X &\rightarrow U_0 W_0 \,|\, U_1 W_1 \,|\, U_0 U_0 \,|\, U_1 U_1 \\
X &\rightarrow (S_1 S_1) \,|\, (a) \,|\, (b) \,|\, () \\
S_1 &\rightarrow S_1 S_1 \,|\, a \,|\, b
\end{aligned}
$$

Chomsky Example Steps 6-7

$$
\begin{aligned}
S &\to S_0 \\
S_0 &\to U_0 W_0 | U_1 W_1 \\
W_0 &\to X U_0 \\
W_1 &\to X U_1 \\
S_0 &\to U_0 U_0 | U_1 U_1 \\
U_0 &\to 0 \\
U_1 &\to 1 \\
X &\to S_0 | (S_1) | () \\
S_1 &\to S_1 S_1 | a | b
\end{aligned}
$$

$$
\begin{aligned}
S &\to U_0 W_0 | U_1 W_1 | U_0 U_0 | U_1 U_1 \\
S_0 &\to U_0 W_0 | U_1 W_1 | U_0 U_0 | U_1 U_1 \\
W_0 &\to X U_0 \\
W_1 &\to X U_1 \\
U_0 &\to 0 \\
U_1 &\to 1 \\
X &\to U_0 W_0 | U_1 W_1 | U_0 U_0 | U_1 U_1 \\
X &\to (S_1 S_1) | (a) | (b) | () \\
S_1 &\to S_1 S_1 | a | b
\end{aligned}
$$

How should this be split up:

$$X \to (S_1 S_1) | (a) | (b) | () \ ?$$

And, what rows can we delete without changing the grammar?

Converting to Chomsky Normal Form

New Start:
Replace

$$S \to A|b$$
$$A \to aS|a$$

With

$$S_0 \to S$$
$$S \to A|b$$
$$A \to aS|a$$

No ε:
Replace

$$A \to XBY$$
$$B \to \varepsilon$$

With

$$A \to XBY$$
$$A \to XY$$

No Units:
Replace

$$A \to B$$
$$B \to XY|a$$

With

$$A \to XY|a$$
$$B \to XY|a$$

New Variables:
Replace:

$$A \to BXY$$
$$B \to bX$$

With:

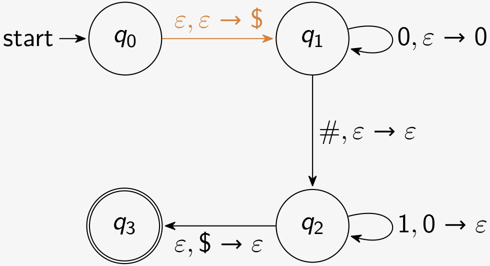
$$A \to BU$$
$$U \to XY$$
$$B \to VX$$
$$V \to b$$

Practice Grammar:

$$S \to XY \ \& \ X \to aXb|\varepsilon \ \& \ Y \to Yc|c|\varepsilon$$

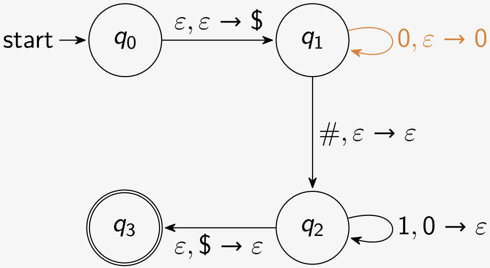
Table of Contents

A Simple Example: $L = \{0^n\#1^n | n \in \mathbb{N}\}$



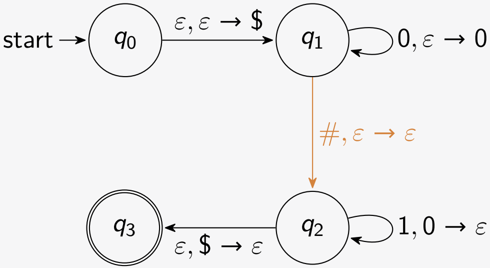
\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

A Simple Example: $L = \{0^n \# 1^n | n \in \mathbb{N}\}$



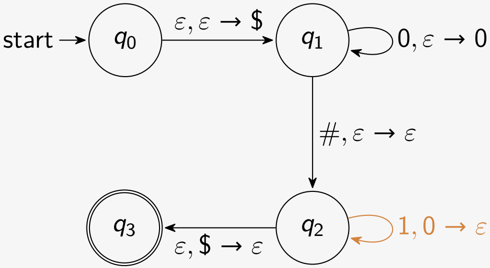
\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

A Simple Example: $L = \{0^n \# 1^n | n \in \mathbb{N}\}$



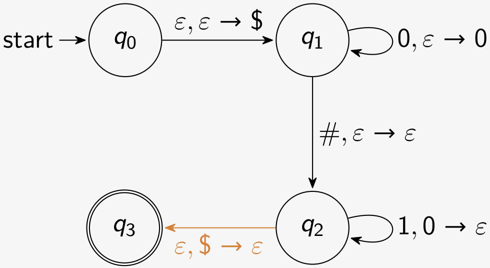
\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

A Simple Example: $L = \{0^n \# 1^n | n \in \mathbb{N}\}$



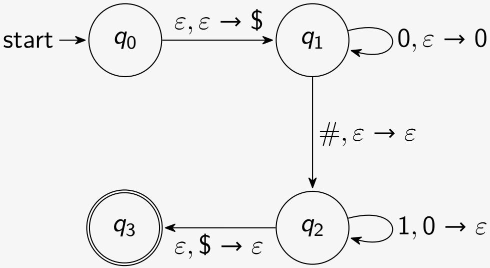
\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

A Simple Example: $L = \{0^n \# 1^n | n \in \mathbb{N}\}$



\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

A Simple Example: $L = \{0^n \# 1^n | n \in \mathbb{N}\}$



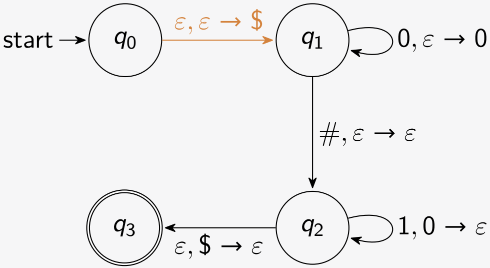
\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

Input Tape:

000#111

Stack: ␣

A Simple Example: $L = \{0^n\#1^n | n \in \mathbb{N}\}$



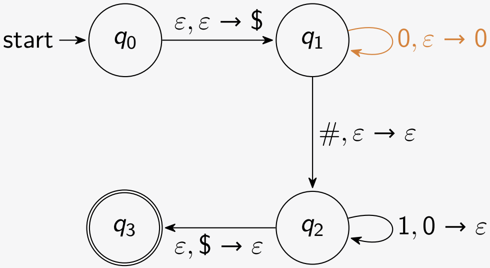
\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

Input Tape:

000#111

Stack: \$

A Simple Example: $L = \{0^n\#1^n | n \in \mathbb{N}\}$



\$ – Empty Stack Symbol
$a, \varepsilon \rightarrow \varepsilon$ – read
$\varepsilon, \varepsilon \rightarrow b$ – push
$\varepsilon, b \rightarrow \varepsilon$ – pop
$a, \varepsilon \rightarrow b$ – read and push
$a, b \rightarrow \varepsilon$ – read and pop
$a, b \rightarrow c$ – read, pop, and push

Input Tape:

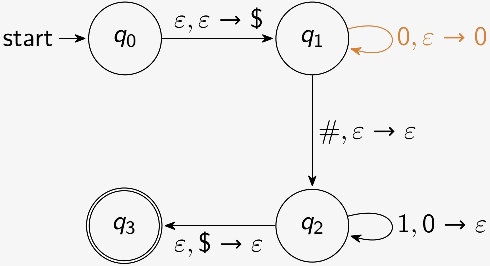
000#111

Stack:

0

\$

A Simple Example: $L = \{0^n\#1^n | n \in \mathbb{N}\}$



$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

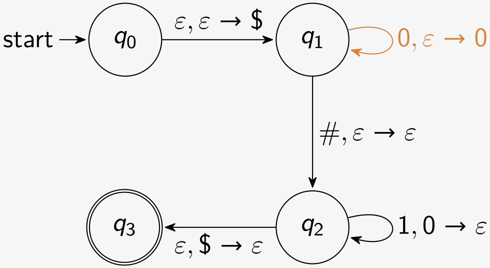
Input Tape:

000#111

Stack:

0
0
$

A Simple Example: $L = \{0^n\#1^n | n \in \mathbb{N}\}$



\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

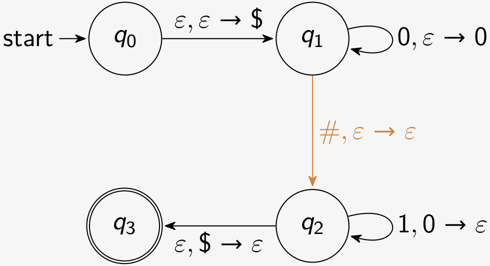
Input Tape:

000#111

Stack:

0
0
0
\$

A Simple Example: $L = \{0^n \# 1^n | n \in \mathbb{N}\}$



\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

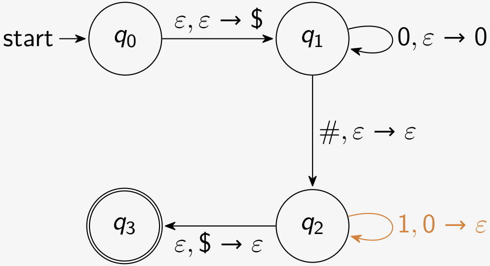
Input Tape:

000#111

Stack:

0
0
0
\$

A Simple Example: $L = \{0^n \# 1^n | n \in \mathbb{N}\}$



\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

Input Tape:

000#111

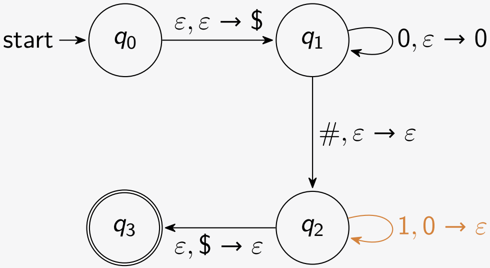
Stack:

0

0

\$

A Simple Example: $L = \{0^n\#1^n | n \in \mathbb{N}\}$



\$ – Empty Stack Symbol
$a, \varepsilon \rightarrow \varepsilon$ – read
$\varepsilon, \varepsilon \rightarrow b$ – push
$\varepsilon, b \rightarrow \varepsilon$ – pop
$a, \varepsilon \rightarrow b$ – read and push
$a, b \rightarrow \varepsilon$ – read and pop
$a, b \rightarrow c$ – read, pop, and push

Input Tape:

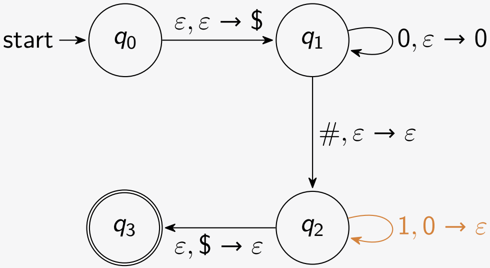
000#111

Stack:

0

\$

A Simple Example: $L = \{0^n\#1^n | n \in \mathbb{N}\}$



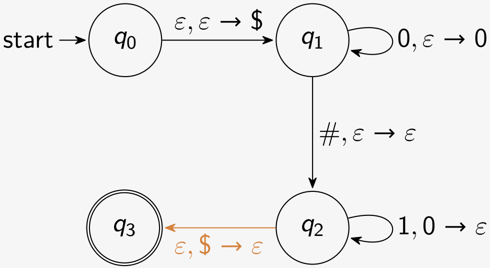
\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

Input Tape:

000#111

Stack: \$

A Simple Example: $L = \{0^n\#1^n | n \in \mathbb{N}\}$



\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

Input Tape:

000#111

Stack: ⊔

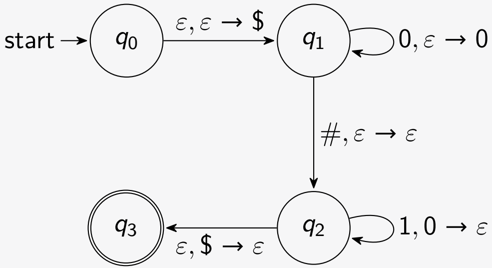
Formal Definition

Definition (Pushdown Automaton)

A **pushdown automaton** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are all finite sets, and

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathscr{P}(Q \times \Gamma_\varepsilon)$ is the transition function,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

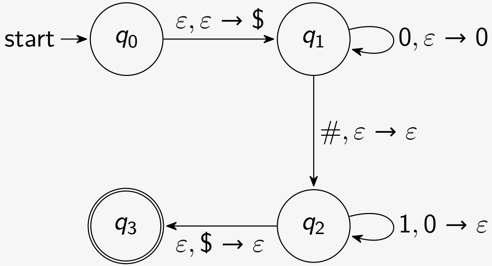
Second Look at $L = \{0^n\#1^n | n \in \mathbb{N}\}$



\$ – Empty Stack Symbol
$a, \varepsilon \rightarrow \varepsilon$ – read
$\varepsilon, \varepsilon \rightarrow b$ – push
$\varepsilon, b \rightarrow \varepsilon$ – pop
$a, \varepsilon \rightarrow b$ – read and push
$a, b \rightarrow \varepsilon$ – read and pop
$a, b \rightarrow c$ – read, pop, and push

- $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$

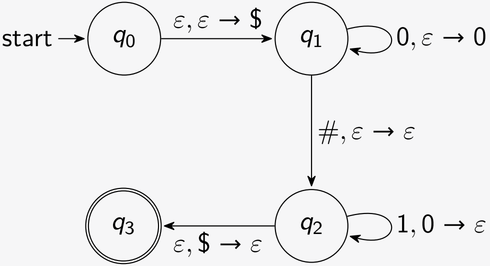
Second Look at $L = \{0^n\#1^n | n \in \mathbb{N}\}$



\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

- $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$
- $\delta(q_1, 0, \varepsilon) = \{(q_1, 0)\}$

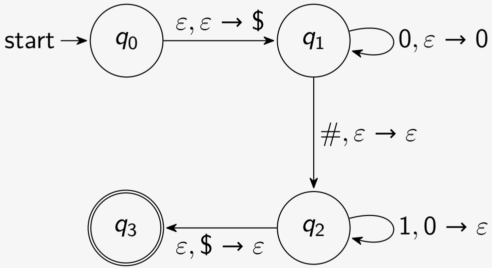
Second Look at $L = \{0^n\#1^n | n \in \mathbb{N}\}$



$\$$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

- $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$
- $\delta(q_1, 0, \varepsilon) = \{(q_1, 0)\}$
- $\delta(q_1, \#, \varepsilon) = \{(q_2, \varepsilon)\}$

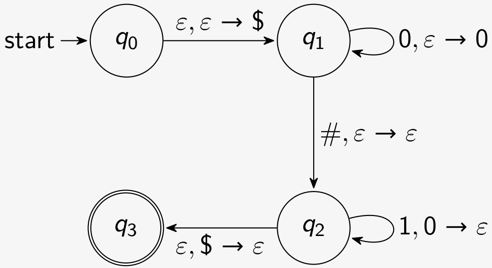
Second Look at $L = \{0^n \# 1^n | n \in \mathbb{N}\}$



\$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

- $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$
- $\delta(q_1, 0, \varepsilon) = \{(q_1, 0)\}$
- $\delta(q_1, \#, \varepsilon) = \{(q_2, \varepsilon)\}$
- $\delta(q_2, 1, 0) = \{(q_2, \varepsilon)\}$

Second Look at $L = \{0^n\#1^n | n \in \mathbb{N}\}$



$\$$ – Empty Stack Symbol
$a, \varepsilon \to \varepsilon$ – read
$\varepsilon, \varepsilon \to b$ – push
$\varepsilon, b \to \varepsilon$ – pop
$a, \varepsilon \to b$ – read and push
$a, b \to \varepsilon$ – read and pop
$a, b \to c$ – read, pop, and push

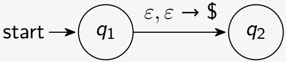
- $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$
- $\delta(q_1, 0, \varepsilon) = \{(q_1, 0)\}$
- $\delta(q_1, \#, \varepsilon) = \{(q_2, \varepsilon)\}$
- $\delta(q_2, 1, 0) = \{(q_2, \varepsilon)\}$
- $\delta(q_2, \varepsilon, \$) = \{(q_3, \varepsilon)\}$

A More Complicated Example:

$$\left\{a^i b^j c^k \middle| i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\right\}$$

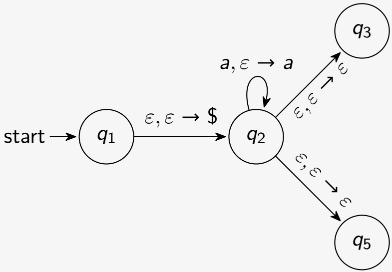
A More Complicated Example:

$$\left\{a^i b^j c^k \middle| i, j, k \geq 0 \textit{ and } i = j \textit{ or } i = k\right\}$$



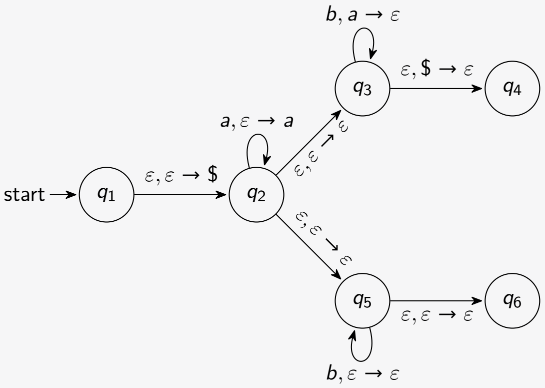
A More Complicated Example:

$$\left\{a^i b^j c^k \middle| i, j, k \geq 0 \textit{ and } i = j \textit{ or } i = k\right\}$$



A More Complicated Example:

$$\left\{a^i b^j c^k \,\middle|\, i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\right\}$$



A More Complicated Example:

$$\left\{a^i b^j c^k \middle| i, j, k \geq 0 \textit{ and } i = j \textit{ or } i = k\right\}$$

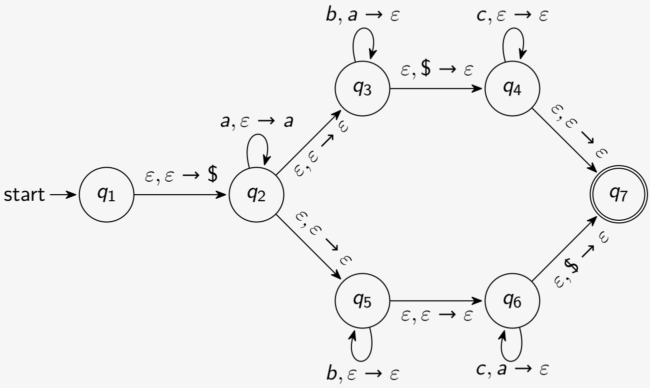


Table of Contents

CFL implies PDA

Context Free
Grammar

$$X \to XX$$
$$X \to aXb | bXa$$
$$X \to \varepsilon$$

CFL implies PDA

Context Free Grammar

$$X \rightarrow XX$$
$$X \rightarrow aXb|bXa$$
$$X \rightarrow \varepsilon$$

OR

$$X \rightarrow XX|aB|bA$$
$$A \rightarrow Xa|a$$
$$B \rightarrow Xb|b$$
$$X \rightarrow \varepsilon$$

CFL implies PDA

Context Free Grammar

$X \rightarrow XX$

$X \rightarrow aXb|bXa$

$X \rightarrow \varepsilon$

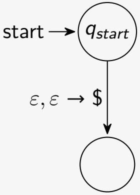
OR

$X \rightarrow XX|aB|bA$

$A \rightarrow Xa|a$

$B \rightarrow Xb|b$

$X \rightarrow \varepsilon$



CFL implies PDA

Context Free Grammar

$X \rightarrow XX$

$X \rightarrow aXb|bXa$

$X \rightarrow \varepsilon$

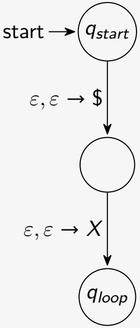
OR

$X \rightarrow XX|aB|bA$

$A \rightarrow Xa|a$

$B \rightarrow Xb|b$

$X \rightarrow \varepsilon$



CFL implies PDA

Context Free Grammar

$X \to XX$

$X \to aXb|bXa$

$X \to \varepsilon$

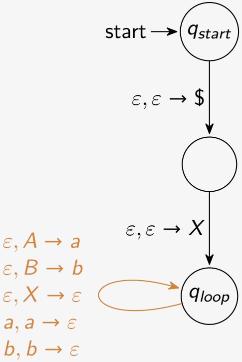
OR

$X \to XX|aB|bA$

$A \to Xa|a$

$B \to Xb|b$

$X \to \varepsilon$



CFL implies PDA

Context Free Grammar

$X \to XX$

$X \to aXb | bXa$

$X \to \varepsilon$

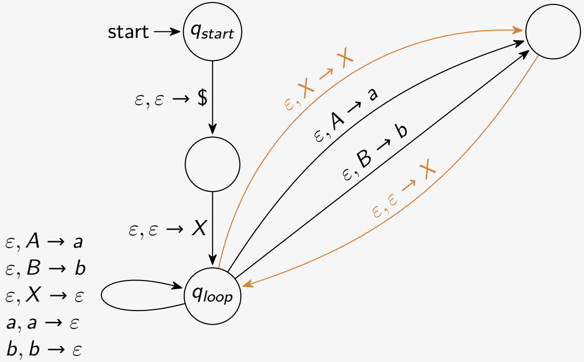
OR

$X \to XX | aB | bA$

$A \to Xa | a$

$B \to Xb | b$

$X \to \varepsilon$



CFL implies PDA

Context Free Grammar

$X \to XX$

$X \to aXb | bXa$

$X \to \varepsilon$

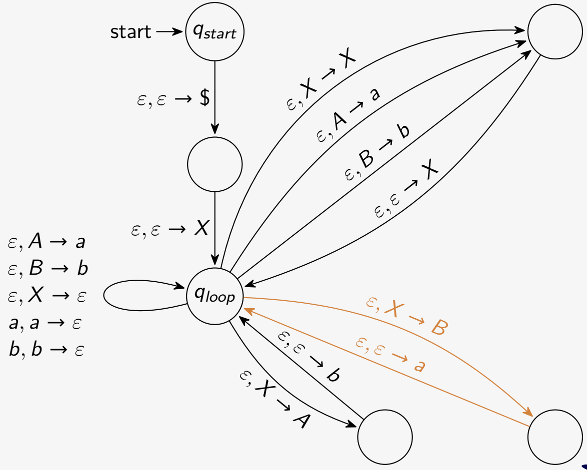
OR

$X \to XX | aB | bA$

$A \to Xa | a$

$B \to Xb | b$

$X \to \varepsilon$



CFL implies PDA

Context Free Grammar

$X \to XX$

$X \to aXb|bXa$

$X \to \varepsilon$

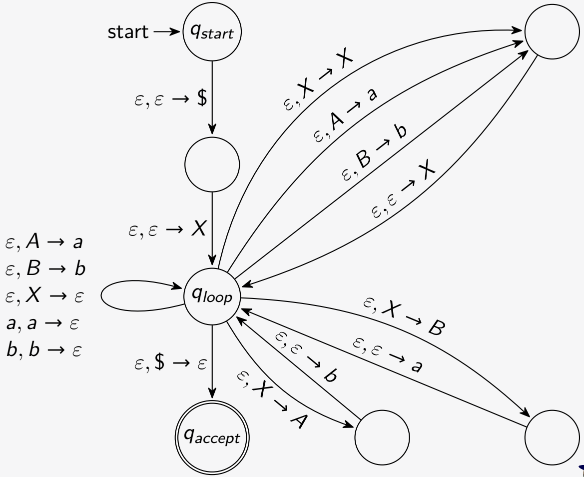
OR

$X \to XX|aB|bA$

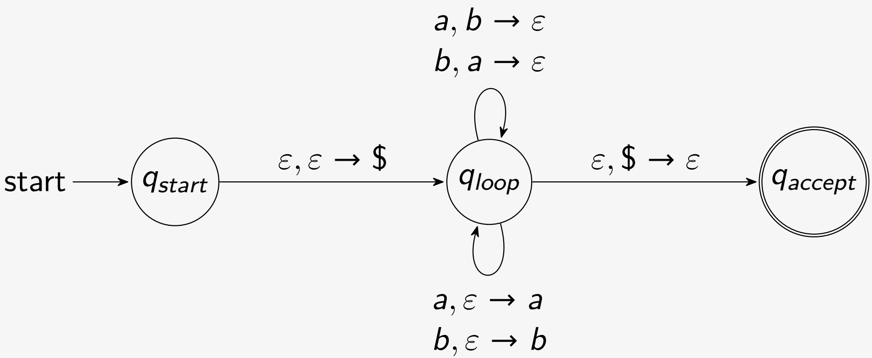
$A \to Xa|a$

$B \to Xb|b$

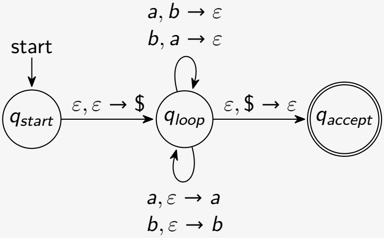
$X \to \varepsilon$



PDA implies CFL (Part 1)

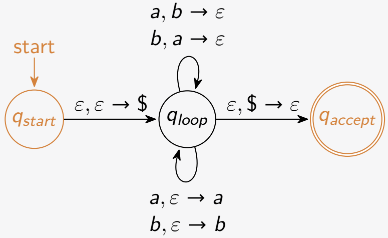


PDA implies CFL (Part 2)



1. Add $S \to A_{start,accept}$ as the start variable.
2. If $(r, u) \in \delta(p, a, \varepsilon) \wedge (q, \varepsilon) \in \delta(s, b, u)$ then add $A_{pq} \to aA_{rs}b$.
3. For each p, q, r add $A_{pq} \to A_{pr}A_{rq}$.
4. For each p add $A_{pp} \to \varepsilon$.

PDA implies CFL (Part 2)

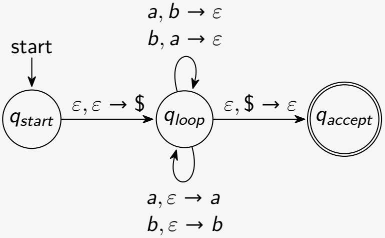


New Rules:

- $S \to R_{SA}$
- $R_{SS} \to \varepsilon$
- $R_{LL} \to \varepsilon$
- $R_{AA} \to \varepsilon$

1. Add $S \to A_{start,accept}$ as the start variable.
2. If $(r, u) \in \delta(p, a, \varepsilon) \wedge (q, \varepsilon) \in \delta(s, b, u)$ then add $A_{pq} \to aA_{rs}b$.
3. For each p, q, r add $A_{pq} \to A_{pr}A_{rq}$.
4. For each p add $A_{pp} \to \varepsilon$.

PDA implies CFL (Part 2)

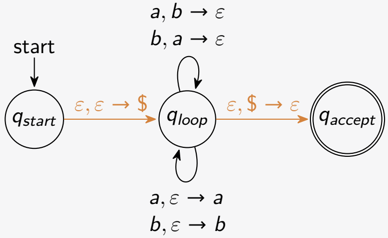


1. Add $S \to A_{start,accept}$ as the start variable.
2. If $(r, u) \in \delta(p, a, \varepsilon) \wedge (q, \varepsilon) \in \delta(s, b, u)$ then add $A_{pq} \to aA_{rs}b$.
3. For each p, q, r add $A_{pq} \to A_{pr}A_{rq}$.
4. For each p add $A_{pp} \to \varepsilon$.

New Rules:

- $S \to R_{SA}$
- $R_{SS} \to \varepsilon | R_{SS}R_{SS}$
- $R_{LL} \to \varepsilon | R_{LL}R_{LL}$
- $R_{AA} \to \varepsilon | R_{AA}R_{AA}$
- $R_{SA} \to R_{SL}R_{LA}$
- $R_{SL} \to R_{SS}R_{SL} | R_{SL}R_{LL}$
- $R_{LA} \to R_{LL}R_{LA} | R_{LA}R_{AA}$

PDA implies CFL (Part 2)

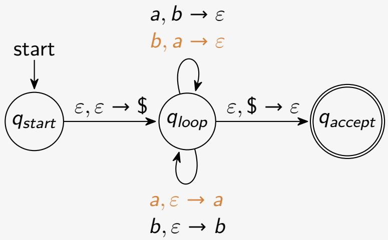


1. Add $S \to A_{start,accept}$ as the start variable.
2. If $(r, u) \in \delta(p, a, \varepsilon) \wedge (q, \varepsilon) \in \delta(s, b, u)$ then add $A_{pq} \to aA_{rs}b$.
3. For each p, q, r add $A_{pq} \to A_{pr}A_{rq}$.
4. For each p add $A_{pp} \to \varepsilon$.

New Rules:

- $S \to R_{SA}$
- $R_{SS} \to \varepsilon | R_{SS}R_{SS}$
- $R_{LL} \to \varepsilon | R_{LL}R_{LL}$
- $R_{AA} \to \varepsilon | R_{AA}R_{AA}$
- $R_{SA} \to R_{SL}R_{LA} | \varepsilon R_{LL} \varepsilon$
- $R_{SL} \to R_{SS}R_{SL} | R_{SL}R_{LL}$
- $R_{LA} \to R_{LL}R_{LA} | R_{LA}R_{AA}$

PDA implies CFL (Part 2)

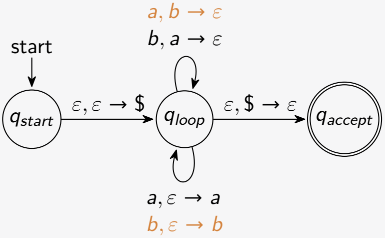


1. Add $S \to A_{start,accept}$ as the start variable.
2. If $(r, u) \in \delta(p, a, \varepsilon) \wedge (q, \varepsilon) \in \delta(s, b, u)$ then add $A_{pq} \to aA_{rs}b$.
3. For each p, q, r add $A_{pq} \to A_{pr}A_{rq}$.
4. For each p add $A_{pp} \to \varepsilon$.

New Rules:

- $S \to R_{SA}$
- $R_{SS} \to \varepsilon | R_{SS}R_{SS}$
- $R_{LL} \to \varepsilon | R_{LL}R_{LL} | aR_{LL}b$
- $R_{AA} \to \varepsilon | R_{AA}R_{AA}$
- $R_{SA} \to R_{SL}R_{LA} | \varepsilon R_{LL}\varepsilon$
- $R_{SL} \to R_{SS}R_{SL} | R_{SL}R_{LL}$
- $R_{LA} \to R_{LL}R_{LA} | R_{LA}R_{AA}$

PDA implies CFL (Part 2)

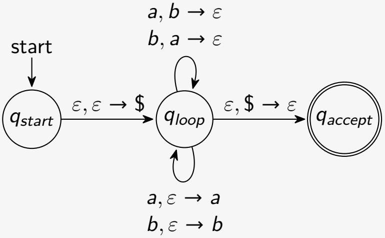


1. Add $S \to A_{start,accept}$ as the start variable.
2. If $(r,u) \in \delta(p,a,\varepsilon) \wedge (q,\varepsilon) \in \delta(s,b,u)$ then add $A_{pq} \to aA_{rs}b$.
3. For each p, q, r add $A_{pq} \to A_{pr}A_{rq}$.
4. For each p add $A_{pp} \to \varepsilon$.

New Rules:

- $S \to R_{SA}$
- $R_{SS} \to \varepsilon | R_{SS}R_{SS}$
- $R_{LL} \to \varepsilon | R_{LL}R_{LL} | aR_{LL}b | bR_{LL}a$
- $R_{AA} \to \varepsilon | R_{AA}R_{AA}$
- $R_{SA} \to R_{SL}R_{LA} | \varepsilon R_{LL} \varepsilon$
- $R_{SL} \to R_{SS}R_{SL} | R_{SL}R_{LL}$
- $R_{LA} \to R_{LL}R_{LA} | R_{LA}R_{AA}$

PDA implies CFL (Part 2)



1. Add $S \to A_{start,accept}$ as the start variable.
2. If $(r, u) \in \delta(p, a, \varepsilon) \wedge (q, \varepsilon) \in \delta(s, b, u)$ then add $A_{pq} \to aA_{rs}b$.
3. For each p, q, r add $A_{pq} \to A_{pr}A_{rq}$.
4. For each p add $A_{pp} \to \varepsilon$.

New Rules:

- $S \to R_{SA}$
- $R_{SS} \to \varepsilon | R_{SS}R_{SS}$
- $R_{LL} \to \varepsilon | R_{LL}R_{LL} | aR_{LL}b | bR_{LL}a$
- $R_{AA} \to \varepsilon | R_{AA}R_{AA}$
- $R_{SA} \to R_{SL}R_{LA} | \varepsilon R_{LL} \varepsilon$
- $R_{SL} \to R_{SS}R_{SL} | R_{SL}R_{LL}$
- $R_{LA} \to R_{LL}R_{LA} | R_{LA}R_{AA}$

Compared to:

$$X \to XX | aXb | bXa | \varepsilon.$$

Practice

1. $S \to A_{start,accept}$
2. Add $A_{pq} \to aA_{rs}b.$, if $(r, u) \in \delta(p, a, \varepsilon)$ and $(q, \varepsilon) \in \delta(s, b, u)$
3. Add $A_{pq} \to A_{pr}A_{rq}$.
4. Add $A_{pp} \to \varepsilon$.

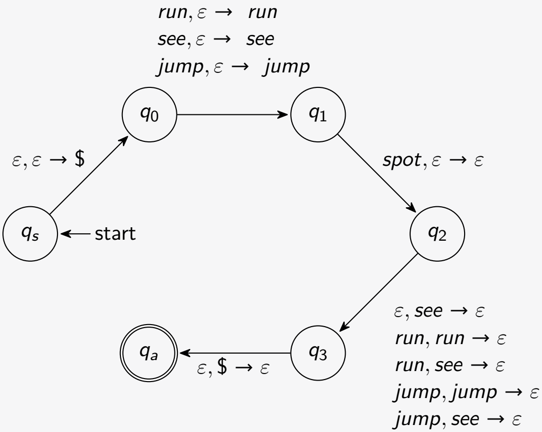


Table of Contents

Next Class

- Non-Context-Free Languages

Next Class

- Non-Context-Free Languages
- Pumping Lemma for CFL

Next Class

- Non-Context-Free Languages
- Pumping Lemma for CFL
- Deterministic CFL

Grammars and Pushdown Automata

Dr. Chuck Rocca
roccac@wcsu.edu
http://sites.wcsu.edu/roccac

WESTERN CONNECTICUT STATE UNIVERSITY
MACRICOSTAS SCHOOL OF ARTS & SCIENCES