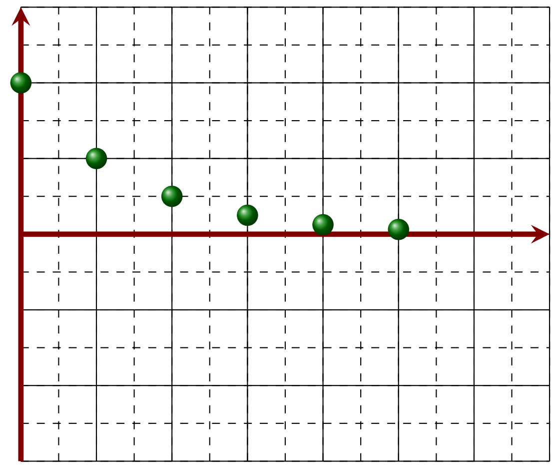
# 1 Sequences

**Exposition 1:** $a_n = 1/2^n$

- $a_0 = 1/2^0 = 1$
- $a_1 = 1/2^1 = 1/2$
- $a_2 = 1/2^2 = 1/4$
- $a_3 = 1/2^3 = 1/8$
- $a_4 = 1/2^4 = 1/16$
- $a_5 = 1/2^5 = 1/32$

Description: Decreasing



**Practice 1:** $b_n = 2^n$

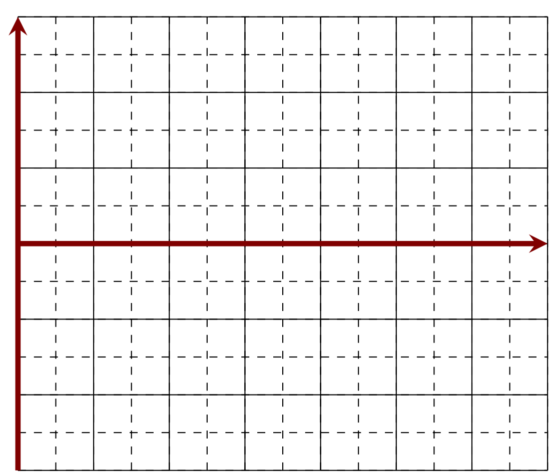
- $b_0 =$ ____________
- $b_1 =$ ____________
- $b_2 =$ ____________
- $b_3 =$ ____________
- $b_4 =$ ____________
- $b_5 =$ ____________

Description: ________________________

**Practice 2:** $c_n = (-2/3)^n$

- $c_0 =$ ____________
- $c_1 =$ ____________
- $c_2 =$ ____________
- $c_3 =$ ____________
- $c_4 =$ ____________
- $c_5 =$ ____________

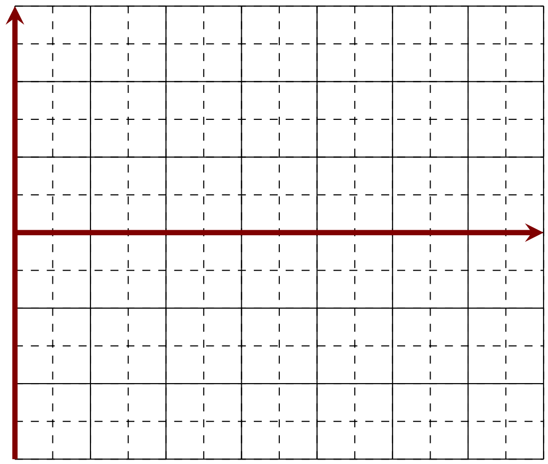
Description: ________________________



**Practice 3:** $d_n = 1 - 1/3^n$

- $d_0 =$ ____________
- $d_1 =$ ____________
- $d_2 =$ ____________
- $d_3 =$ ____________
- $d_4 =$ ____________
- $d_5 =$ ____________

Description: ________________________



**Practice 4:** $f_n = 1 + (-1/2)^n$

- $f_0 =$ ____________
- $f_1 =$ ____________
- $f_2 =$ ____________
- $f_3 =$ ____________
- $f_4 =$ ____________
- $f_5 =$ ____________

Description: ________________________

**Practice 5:** $h_n = (-1)^n + (-1/2)^n$

- $h_0 =$ ____________
- $h_1 =$ ____________
- $h_2 =$ ____________
- $h_3 =$ ____________
- $h_4 =$ ____________
- $h_5 =$ ____________

Description: ________________________

**Practice 6:** $j_n = (-1)^n - (-1/2)^n$

- $j_0 =$ ____________
- $j_1 =$ ____________
- $j_2 =$ ____________
- $j_3 =$ ____________
- $j_4 =$ ____________

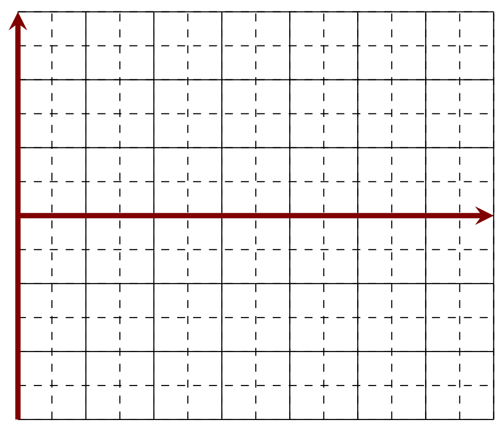
Description: ________________________

**Theorem 1.** *A bounded sequence has a* **convergent subsequence**. *(The converse is not true.)*

**Practice 7:** $k_n = 0$ **for even** $n$ **and** $2^n$ **for odd** $n$

- $k_0 =$ ____________
- $k_1 =$ ____________
- $k_2 =$ ____________
- $k_3 =$ ____________
- $k_4 =$ ____________

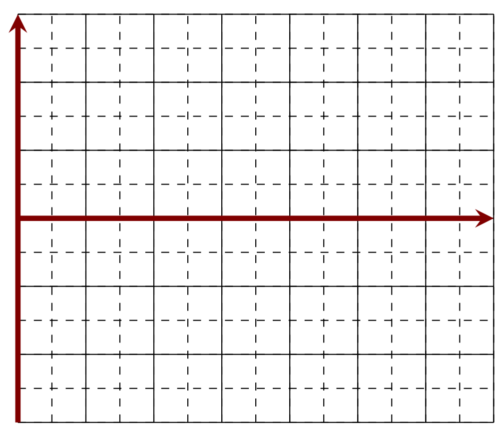
Description: ________________________



**Practice 8:** $g_n = (-\sqrt{2})^n$

- $g_0 =$ ____________
- $g_1 =$ ____________
- $g_2 =$ ____________
- $g_3 =$ ____________
- $g_4 =$ ____________

Description: ________________________



Note that we could also write $g_0 = 1$ and $g_n = -\sqrt{2} \cdot g_{n-1}$ for $g_n$, this is called a ***recursive definition*** for the sequence.

**Practice 9:** $r_0 = 3$ **and** $r_n = r_{n-1}/2$

- $r_0 =$ ______________
- $r_1 =$ ______________
- $r_2 =$ ______________
- $r_3 =$ ______________
- $r_4 =$ ______________
- $r_5 =$ ______________

Description: ____________________________

**Practice 10:** $s_0 = 0$ **and** $s_n = 2s_{n-1} + 1$

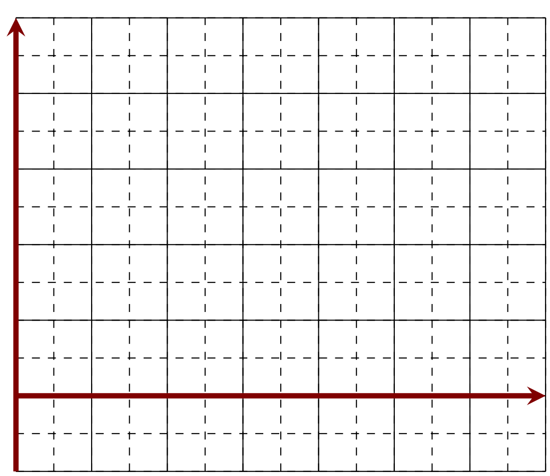
- $s_0 =$ ______________
- $s_1 =$ ______________
- $s_2 =$ ______________
- $s_3 =$ ______________
- $s_4 =$ ______________
- $s_5 =$ ______________

Description: ____________________________

**Practice 11:** $F_0 = 1$**,** $F_1 = 1$ **and** $F_n = F_{n-1} + F_{n-2}$

- $F_0 =$ ______________
- $F_1 =$ ______________
- $F_2 =$ ______________
- $F_3 =$ ______________
- $F_4 =$ ______________
- $F_5 =$ ______________

Description: ____________________________



**Practice 12:** $a_0 = 1$ **and** $a_n = \frac{1}{3}a_{n-1}$

Find $a_5$:

$$a_5 = \frac{1}{3}a_4 \qquad a_n = \frac{1}{3}a_{n-1}$$

$$= \qquad a_n = \frac{1}{3}a_{n-1}$$

$$= \qquad a_n = \frac{1}{3}a_{n-1}$$

$$= \qquad a_n = \frac{1}{3}a_{n-1}$$

$$= \qquad a_n = \frac{1}{3}a_{n-1}$$

$$= \qquad a_0 = 1$$

Find $a_{10}$:

$$a_{10} = \frac{1}{3}a_9 \qquad a_n = \frac{1}{3}a_{n-1}$$

$$= \qquad a_n = \frac{1}{3}a_{n-1}$$

$$= \qquad a_n = \frac{1}{3}a_{n-1}$$

$$= \qquad a_n = \frac{1}{3}a_{n-1}$$

$$= \qquad a_n = \frac{1}{3}a_{n-1}$$

$$= \qquad a_5 = \underline{\qquad}$$

**Practice 13:** $b_0 = 1$ & $b_1 = 2$ **and** $b_n = b_{n-1} + 2b_{n-2}$

Find $b_5$:

$$b_5 = 2b_4 + 2b_3 \qquad b_n = b_{n-1} + 2b_{n-2}$$

$$= \qquad b_n = b_{n-1} + 2b_{n-2}$$

$$= \qquad b_n = b_{n-1} + 2b_{n-2}$$

$$= \qquad b_n = b_{n-1} + 2b_{n-2} \text{ \& } b_1 = 2$$

$$= \qquad b_0 = 1 \text{ \& } b_1 = 2$$

# 2 Sums

## 2.1 Summation/Product Notation

The 10 is the upper bound

The $\Sigma$ (sigma) means sum or add

Ellipses, $\cdots$, mean and so on

$$\sum_{i=0}^{10}(i^2+1) = (0^2+1)+(1^2+1)+(2^2+1)+\cdots+(10^2+1)$$

The 0 is the lower bound

The $i$ is the index

Each time we iterate we add another term by evaluating the expression for the new $i$

The 10 is the upper bound

The $\Pi$ (pi) means product or multiply

Ellipses, $\cdots$, mean and so on

$$\prod_{i=0}^{10}(i^2+1) = (0^2+1)\times(1^2+1)\times(2^2+1)\times\cdots\times(10^2+1)$$

The 0 is the lower bound

The $i$ is the index

Each time we iterate we multiply by another factor by evaluating the expression for the new $i$

## 2.2 Arithmetic Sums

**Exposition 2:** $S_n = \sum_{i=1}^{n} i$

$$S_n = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

- $S_1 =$ ____________
- $S_2 =$ ____________
- $S_3 =$ ____________
- $S_4 =$ ____________

For $n = 100$:

$$S_{100} = 1 + 2 + 3 + \cdots + 98 + 99 + 100$$
$$S_{100} = 100 + 99 + 98 + \cdots + 3 + 2 + 1$$

$2 \cdot S_{100} =$ ________________________________

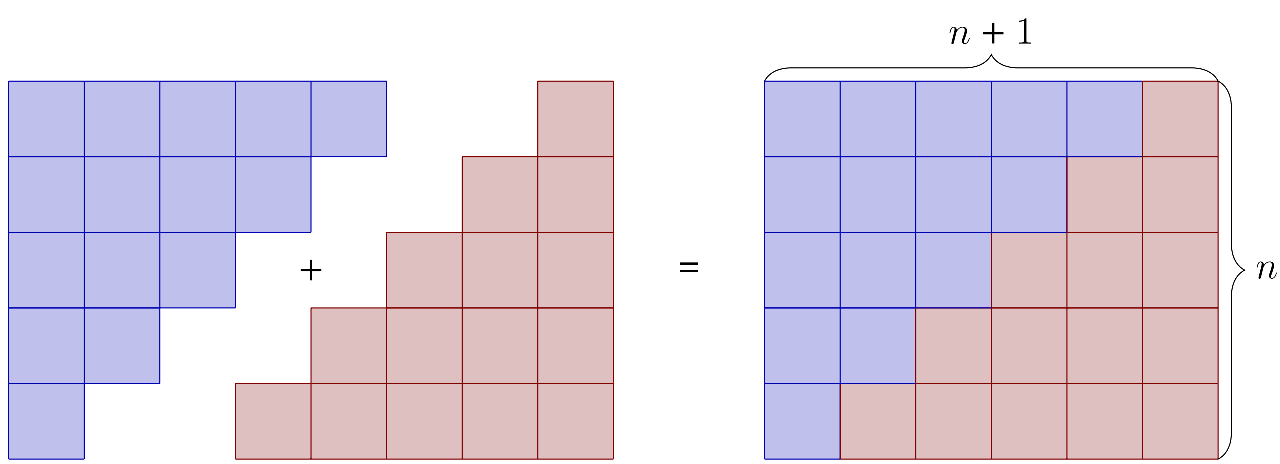
$S_{100} =$ ________________________________

For $n = k$:

$$S_k = 1 + 2 + 3 + \cdots + (k-2) + (k-1) + k$$
$$S_k = k + (k-1) + (k-2) + \cdots + 3 + 2 + 1$$

$2 \cdot S_k =$ ________________________________

$S_k =$ ________________________________



**Practice 14:** $S_n = \sum_{i=1}^{n} 2 \cdot i$

$$S_n = \sum_{i=1}^{n} 2 \cdot i = 2 + 4 + 6 + \cdots + 2n$$

$S_5 =$ ____________________________

For $n = k$:

$$S_k = 2 + 4 + 6 + \cdots + 2(k-2) + 2(k-1) + 2k$$
$$S_k = 2k + 2(k-1) + 2(k-2) + \cdots + 6 + 4 + 2$$

$2 \cdot S_k =$ ____________________________

$S_k =$ ____________________________

$S_k =$ ____________________________

**Practice 15:** $S_n = \sum_{i=1}^{n} i + 3$

$$S_n = \sum_{i=1}^{n} i + 3 = 4 + 5 + 6 + \cdots + (n+3) = \sum_{j=4}^{n+3} j$$

$S_5 =$ ____________________________

For $n = k$:

$$S_k = 4 + 5 + 6 + \cdots + (k+1) + 2(k+2) + (k+3)$$
$$S_k = (k+3) + (k+2) + (k+1) + \cdots + 6 + 5 + 4$$

$2 \cdot S_k =$ ____________________________

$S_k =$ ____________________________

$S_k =$ ____________________________

**Exposition 3:** $S_n = \sum_{i=1}^{n} ai + b$

$$S_n = \sum_{i=1}^{n} ai + b = (a+b) + (2a+b) + (3a+b) + \cdots + (n \cdot a + b) = n \cdot b + a \sum_{i=1}^{n} i$$

For $n = k$:

$$S_k = (a+b) + (2a+b) + (3a+b) + \cdots + (k \cdot a + b)$$
$$S_k = (k \cdot a + b) + \cdots + (3a+b) + (2a+b) + (a+b)$$

$2 \cdot S_k =$ ______________________________

$S_k =$ ______________________________

$S_k =$ ______________________________

## 2.3 Geometric Sums

**Exposition 4:** $S_n = \sum_{i=0}^{n} 2^i$

$$S_n = \sum_{i=0}^{n} 2^i = 1 + 2 + 4 + 8 + \cdots + 2^n$$

$$S_n = \sum_{i=0}^{3} 2^i = \underline{\hspace{8cm}}$$


For $n = 10$:

$$\begin{aligned} S_{10} &= 1 + 2 + 4 + 8 + \cdots + 2^9 + 2^{10} \\ 2 \cdot S_{10} &= 2 + 4 + 8 + 16 + \cdots + 2^{10} + 2^{11} \end{aligned}$$

$$2 \cdot S_{10} - S_{10} = \underline{\hspace{8cm}}$$

$$S_{10} = \underline{\hspace{8cm}}$$

For $n = k$:

$$\begin{aligned} S_k &= 1 + 2 + 4 + \cdots + 2^{k-1} + 2^k \\ 2 \cdot S_k &= 2 + 4 + 8 + \cdots + 2^k + 2^{k+1} \end{aligned}$$

$$2 \cdot S_k - S_k = \underline{\hspace{8cm}}$$

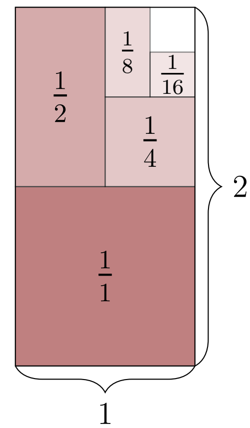
$$S_k = \underline{\hspace{8cm}}$$

$$S_k = \underline{\hspace{8cm}}$$

**Practice 16:** $S_n = \sum_{i=0}^{n} (1/2)^i$

$$S_n = \sum_{i=0}^{n} (1/2)^i = 1 + 1/2 + 1/4 + 1/8 + \cdots + (1/2)^n$$

$$S_n = \sum_{i=0}^{4} (1/2)^i = \underline{\hspace{6cm}}$$




For $n = k$:

$$S_k = 1/1 + 1/2 + 1/4 + \cdots + 1/2^{k-1} + 1/2^k$$
$$(1/2) \cdot S_k = 1/2 + 1/4 + 1/8 + \cdots + 1/2^k + 1/2^{k+1}$$

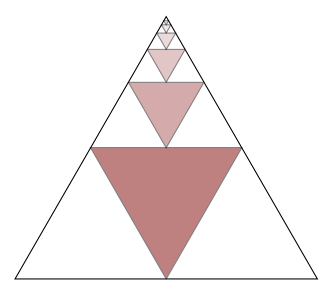
$$(1/2) \cdot S_k - S_k = \underline{\hspace{6cm}}$$

$$S_k = \underline{\hspace{6cm}}$$

**Practice 17:** $S_n = \sum_{i=0}^{n} (1/4)^i$

$$S_n = \sum_{i=0}^{n} (1/4)^i = 1 + 1/4 + 1/16 + 1/64 + \cdots + (1/4)^n$$

$$S_n = \sum_{i=0}^{4} (1/4)^i = \underline{\hspace{6cm}}$$



For $n = k$:

$$S_k = 1/1 + 1/4 + 1/16 + \cdots + 1/4^{k-1} + 1/4^k$$
$$(1/4) \cdot S_k = 1/4 + 1/16 + 1/64 + \cdots + 1/4^k + 1/4^{k+1}$$

$$(1/4) \cdot S_k - S_k = \underline{\hspace{6cm}}$$

$$S_k = \underline{\hspace{6cm}}$$

**Practice 18:** $S_n = \sum_{i=0}^{n} a \cdot r^i$

$$S_n = \sum_{i=0}^{n} a \cdot r^i = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots + a \cdot r^n$$

$$S_n = \sum_{i=0}^{4} a \cdot r^i = \underline{\hspace{6cm}}$$

---

For $n = k$:

$$\begin{aligned} S_k &= a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots + a \cdot r^n \\ r \cdot S_k &= a \cdot r + a \cdot r^2 + a \cdot r^3 + a \cdot r^4 + \cdots + a \cdot r^{n+1} \end{aligned}$$

$$r \cdot S_k - S_k = \underline{\hspace{6cm}}$$

$$S_k = \underline{\hspace{6cm}}$$

$$S_k = \underline{\hspace{6cm}}$$

# 3 Using and Finding Formulas

**Exposition 5: Sum Formulas**

Arithmetic Sum

$$\sum_{i=1}^{n} a \cdot i + b = n \cdot b + a\frac{n(n+1)}{2}$$

Geometric Sum

$$\sum_{i=0}^{n} a \cdot r^{i} = a\left(\frac{r^{n+1}-1}{r-1}\right)$$

**Practice 19: Using Formulas, an Arithmetic Example**

Find the sum of:

$$\sum_{i=5}^{31} i = 5 + 6 + 7 + \cdots + 30 + 31$$

Solution 1 (Re-Indexing):

$$\begin{aligned}
\sum_{i=5}^{31} i &= 5 + 6 + 7 + \cdots + 30 + 31 \\
&= (1+4) + (2+4) + (3+4) + \cdots + (26+4) + (27+4) \\
&= \sum_{j=1}^{27} (j+4) && \text{let } j = i - 4 \\
&= 27 \cdot 4 + \frac{27 \cdot 28}{2} \\
&= 108 + 378 \\
&= 486
\end{aligned}$$

Solution 2 (Adding Zero):

$$\begin{aligned}
\sum_{i=5}^{31} i &= 5 + 6 + 7 + \cdots + 30 + 31 \\
&= (1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots + 30 + 31) - (1 + 2 + 3 + 4) \\
&= \left(\sum_{i=1}^{31} i\right) - \left(\sum_{i=1}^{4} i\right) \\
&= \left(\frac{31 \cdot 32}{2}\right) - \left(\frac{4 \cdot 5}{2}\right) \\
&= 496 - 10 \\
&= 486
\end{aligned}$$

**Practice 20: Using Formulas, a Geometric Example**

Find the sum of:

$$\sum_{i=10}^{106}\left(\frac{3}{7}\right)^{i} = \left(\frac{3}{7}\right)^{10} + \left(\frac{3}{7}\right)^{11} + \cdots + \left(\frac{3}{7}\right)^{106}$$

Solution 1 (Re-Indexing):

$$\sum_{i=10}^{106}\left(\frac{3}{7}\right)^{i} = \left(\frac{3}{7}\right)^{10} + \left(\frac{3}{7}\right)^{11} + \cdots + \left(\frac{3}{7}\right)^{106}$$

$=$

$=$

$=$

$=$

Solution 2 (Adding Zero):

$$\sum_{i=10}^{106}\left(\frac{3}{7}\right)^{i} = \left(\frac{3}{7}\right)^{10} + \left(\frac{3}{7}\right)^{11} + \cdots + \left(\frac{3}{7}\right)^{106}$$

$=$

$=$

$=$

$=$

**Exposition 6: Iteration Algorithm**

***Iteration***: Given $a_0 = b$, $a_n = c \cdot a_{n-1} + d$, and an expression with $a_k$:

1. While $k > 0$:
   (a) Replace $a_k$ with $(c \cdot a_{k-1} + d)$
   (b) Collect like terms
   (c) Decrement $k$
2. When $k = 0$ replace $a_0$ with $b$
3. Simplify final expression with appropriate formula
4. Generalize

Apply ***Iteration*** to $a_0 = 2$, $a_n = a_{n-1} + 2$, starting at $a_3$ ($k = 3$):

| | |
|---|---|
| $a_3 = (a_2 + 2)$ | apply steps 1a & 1b |
| $= ((a_1 + 2) + 2)$ | apply step 1a |
| $= (a_1 + 2 \cdot 2)$ | apply step 1b |
| $= ((a_0 + 2) + 2 \cdot 2)$ | apply step 1a |
| $= (a_0 + 3 \cdot 2)$ | apply step 1b |
| $= (2 + 3 \cdot 2)$ | apply step 2 |
| $= 4 \cdot (2)$ | apply step 3-ish |
| $a_n = (n + 1) \cdot (2)$ | apply step 4 |

**Exposition 7: Apply Iteration to $a_0 = 4$ and $a_n = 6 \cdot a_{n-1} + 4$**

| | |
|---|---|
| $a_4 = (6 \cdot a_3 + 4)$ | step 1a: $a_k = 6 \cdot a_{k-1} + 4$ |
| $= 6 \cdot (6 \cdot a_2 + 4) + 4$ | step 1a: $a_k = 6 \cdot a_{k-1} + 4$ |
| $= 6^2 \cdot a_2 + 6 \cdot 4 + 4$ | step 1b |
| $= 6^2 \cdot (6 \cdot a_1 + 4) + 6 \cdot 4 + 4$ | step 1a: $a_k = 6 \cdot a_{k-1} + 4$ |
| $= 6^3 \cdot a_1 + 6^2 \cdot 4 + 6 \cdot 4 + 4$ | step 1b |
| $= 6^3 \cdot (6 \cdot a_0 + 4) + 6^2 \cdot 4 + 6 \cdot 4 + 4$ | step 1a: $a_k = 6 \cdot a_{k-1} + 4$ |
| $= 6^4 \cdot a_0 + 6^3 \cdot 4 + 6^2 \cdot 4 + 6 \cdot 4 + 4$ | step 1b |
| $= 6^4 \cdot 4 + 6^3 \cdot 4 + 6^2 \cdot 4 + 6 \cdot 4 + 4$ | step 2: $a_0 = 4$ |
| $= \sum_{i=0}^{4} 4 \cdot 6^i = 4\left(\frac{6^5 - 1}{6 - 1}\right)$ | step 3 |
| $a_k = \sum_{i=0}^{k} 4 \cdot 6^i = 4\left(\frac{6^{k+1} - 1}{6 - 1}\right)$ | step 4 |

**Practice 21: Iteration Technique with $b_0 = 2$ and $b_n = -2 \cdot b_{n-1} + 2$**

Use iteration to show that

$$b_n = 2\left(\frac{(-2)^{n+1} - 1}{-2 - 1}\right).$$

remember to always replace $b_n$ with $-2 \cdot b_{n-1} + 2$ if $k \neq 0$ and 2 if it does.

$$\begin{aligned} b_4 &= \\ \\ &= \\ \\ &= \\ \\ &= \\ \\ &= \\ \\ &= \\ \\ &= \\ \\ &= \end{aligned}$$

**Exposition 8: Try Iteration with $F_1 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$**

$$
\begin{aligned}
F_5 &= F_4 + F_3 & F_n &= F_{n-1} + F_{n-2} \\
&= (F_3 + F_2) + (F_2 + F_1) & F_n &= F_{n-1} + F_{n-2} \\
&= F_3 + 2 \cdot F_2 + F_1 & & \\
&= (F_2 + F_1) + 2 \cdot (F_1 + F_0) + F_1 & F_n &= F_{n-1} + F_{n-2} \\
&= F_2 + 4 \cdot F_1 + 2 \cdot F_0 & & \\
&= (F_1 + F_0) + 4 \cdot F_1 + 2 \cdot F_0 & F_n &= F_{n-1} + F_{n-2} \\
&= 5 \cdot F_1 + 3 \cdot F_0 & &
\end{aligned}
$$

But $F_4 = 5$ and $F_3 = 3$ so this doesn't really tell us anything.

**Theorem 2** (Distinct Roots Version). *Given $r_0$, $r_1$, and $r_n = A \cdot r_{n-1} + B \cdot r_{n-2}$, if the roots of*

$$x^2 - Ax - B = 0,$$

*are distinct values $s_0$ and $s_1$, then $r_n = C \cdot s_0^n + D \cdot s_1^n$ for appropriate values of $C$ and $D$.*

**Exposition 9: Characteristic Polynomials with $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$**

We have $F_n = F_{n-1} + F_{n-2}$ so that $A = B = 1$ and we need the roots of $x^2 - x - 1$ which are

$$x = \frac{1 \pm \sqrt{5}}{2}$$

and by theorem 2

$$F_n = C\left(\frac{1 + \sqrt{5}}{2}\right)^n + D\left(\frac{1 - \sqrt{5}}{2}\right)^n.$$

Then

$$
\begin{aligned}
&F_0 = C + D = 1 \text{ and} \\
&F_1 = C\left(\frac{1 + \sqrt{5}}{2}\right) + D\left(\frac{1 - \sqrt{5}}{2}\right) = 1.
\end{aligned}
$$

Solving for $C$ and $D$ we get

$$C = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right) \text{ and } D = -\frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right).$$

Therefore the closed formula for the Fibonacci sequence is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^{n+1}$$

**Practice 22: Characteristic Polynomials with $G_0 = 2$, $G_1 = 3$ and $G_n = 4 \cdot G_{n-1} + 5 \cdot G_{n-2}$**

Using theorem 2 as before, we have $G_n = 4 \cdot G_{n-1} + 5 \cdot G_{n-2}$ so that $A = 4$, $B = 5$ and we find the roots of $x^2 - 4x - 5$ which are $x = 5$ and $x = -1$. Then

**Theorem 3** (Single Root Version). *Given $r_0$, $r_1$, and $r_n = A \cdot r_{n-1} + B \cdot r_{n-2}$, if the only root of*

$$x^2 - Ax - B = 0,$$

*is the value $s_0$, then $r_n = C \cdot s_0^n + D \cdot n \cdot s_0^n$ for appropriate values of $C$ and $D$.*

**Practice 23: Characteristic Polynomials with $H_0 = 5$, $H_1 = 4$ and $H_n = 4 \cdot H_{n-1} - 4 \cdot H_{n-2}$**

Using theorem 3, we have $H_n = 4 \cdot H_{n-1} - 4 \cdot H_{n-2}$ so that $A = 4$, $B = -4$ and we need the roots of $x^2 - 4x + 4$ which are $x = 2$ and $x = 2$. Then