Exercises in Writing Up Mathematics

Contents

1 Directions

Mathematics is its own language with its own rules for syntax and grammar. The objective of this packet is to expose you to some of the, sometimes unwritten, rules for writing up mathematics so that you can be clearer, more precise, and better understood. The lessons learned here can be applied to any situation in which you need to communicate mathematics. Other helpful pointers are available from Dr. Francis Su, author and former president of the Mathematical Association of America, on his website here: https://math.hmc.edu/su/writing-math-well/.

In *section 2* we see examples of how to *flip* a question from an imperative or interrogative statement to a declarative one, that is change it from a question to a statement. Then *section 3* looks at some conventions for good mathematical writing. In both sections there are exercises you need to complete in order to practice these skills.

In *section 4* there are five completed sample problems where you are given the problem statement, scrap work, and final write up of the problem. Read through these carefully so that you understand how to transition from reading and solving a problem, to writing it up neatly.

Finally, in *section 5* and *section 6* there are problems for you to write up. For *section 5*, you are given the problems and the scrap work, then you need to write up the solutions neatly. For *section 6*, you are just given the problems and you must solve them and write them up. As you write these up, be sure to pay attention to all the tips you were given in *section 2* on flipping questions and *section 3* on mathematical conventions.

2 Flipping the Question

2.1 Examples

When you answer a question it should be clear what the question was; this does not mean that you just restate the question. Exercises are typically *interrogative* if they ask you a question or *imperative* if you are told to do something. When you respond to an exercise you should always *flip* those around to be *declarative*. For example:

- **Statement:** What is the intersection of the given functions?
- **Solution:** The intersection of the functions is found as follows ...
- **Statement:** Find the coordinate for the vertex of the parabola.
- **Solution:** The vertex of the parabola is ... which we found like so ...
- **Statement:** Simplify each of the following expressions.
- **Solution:** We simplified each expression as follows ...

Notice that in each case we *flipped* the exercise from a question or command directed toward us into a statement about the answer and how we found it. This doesn't necessitate long winded paragraphs, just a little change.

2.2 Exercises

For each of the following exercises *flip* the problem statement around and write the beginning of a solution.

1. **Statement:** What is the greatest common divisor for each pair of integers?
2. **Statement:** Locate the roots of the given equation.
3. **Statement:** For the following exercises, solve the equation.
4. **Statement:** What is the cube root of the given number?
5. **Statement:** What is the vertex of the given parabola?
6. **Statement:** What are the distinct prime factors of the given integers?

3 Comments on Mathematical Sentences

3.1 Examples

When we write up mathematics there are important conventions about how we use notation and vocabulary correctly, but there are less formal conventions about how to make the work readable. Here are some points to keep in mind as you write up mathematics.

- Write in grammatically correct complete sentences with proper capitalization and punctuation. This doesn't mean you can't use symbols, but it shouldn't be only symbols.
  - **Not Like This:** $f(x) = -2x + 3 \Rightarrow f(3) = -3,\ f(7) = -11,\ f(-3) = 9$
  - **Like This:** Given $f(x) = -2x + 3$ we can find $f(3) = -3$, $f(7) = -11$, and $f(-3) = 9$.
- Don't start a sentence with a symbol or number, it can make it harder to read and looks less professional.
  - **Not Like This:** $\forall x \in \mathbb{R} \ldots$
  - **Like This:** For all $x \in \mathbb{R} \ldots$
  - **Not Like This:** 257ft was the maximum height reached.
  - **Like This:** The maximum height reached was 257ft.
- Don't let an equation split over a line like a word with a hyphen, again it just makes it hard to read.
  - **Not Like This:** We can find solutions for a given the quadratic equation, $-4.9t^2+$ $20t + 2 = 0$, as follows ...
  - **Like This:** We can find solutions for a given the quadratic equation, $-4.9t^2 + 20t + 2 = 0$, as follows ...
- Large or tall expressions belong on their own line, these are called *displayed equations.*
  - **Not Like This:** To find the roots of a quadratic equation we can use $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.
  - **Like This:** To find the roots of a quadratic equation we can use

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- A sequence of equivalencies, equalities, or inequalities should be written as a *stacked equation* with the connectors lined up, not as a long string.
  - **Not Like This:** $\left(\frac{a^3}{b^2}\right)\left(ab^{-1}\right)^{-2} = \left(\frac{a^3}{b^2}\right)\left(\frac{a}{b}\right)^{-2} = \left(\frac{a^3}{b^2}\right)\left(\frac{b^2}{a^2}\right) = a$

- **Like This:** We simplify the given expression like so:

$$\begin{aligned} \left(\frac{a^3}{b^2}\right)\left(ab^{-1}\right)^{-2} &= \left(\frac{a^3}{b^2}\right)\left(\frac{a}{b}\right)^{-2} \\ &= \left(\frac{a^3}{b^2}\right)\left(\frac{b^2}{a^2}\right) \\ &= a. \end{aligned}$$

Also, recall that more helpful pointers are available from Dr. Francis Su on his website here: https://math.hmc.edu/su/writing-math-well/.

3.2 Exercises

Retype each of the following using proper sentences and the conventions discussed above.

1. **Not Like This:** with $12 - 5(x+1) = x - 5$, $12 - 5(x+1) = x - 5 \Rightarrow 17 - 5x - 5 = x \Rightarrow 12 = 6x \Rightarrow x = 2$
2. **Not Like This:** $x \in (-\infty, -2/3) \cup (-2/3, \infty)$, $\frac{2}{3x+2}$ is well defined
3. **Not Like This:** $f(x) = x^2 - 2x + 2$, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, zeros at $x = (1-i), (1+i)$
4. **Not Like This:** $y = kx^2$, $36 = k(4^2) = 16k$, then $k = 36/16 = 9/4$, $y = \frac{9}{4}x^2$
5. **Not Like This:** $y = 7x^2 - 98x + 1$, then $x = 98/14 = 7$, $y = 7(49) - 98(7) + 1 = -342$, vertex at $(7, -342)$
6. **Not Like This:** $y_1 = 17x + 3$, $y_2 = -3x - 7$, $y_1 = y_2 \Rightarrow 17x + 3 = -3x - 7 \Rightarrow 20x = -10$, $x = -1/2$ and $y = -5.5$, $Point = (-1/2, -5.5)$

4 Complete Examples

In this section there are five complete problems showing the problem statements, the scrap work, and the final write up of the problems (which is the part you would actually hand in). Read through these carefully paying attention to the difference between scrap work and final solutions. Also, look at how the tips given in *Section 2* on flipping questions and *Section 3* on mathematical conventions were used in the final write ups.

4.1 Multipart Skill Problem

4.1.1 Problem Statement

For each set of the points find the slope of the line passing through them:

1. $P = (0, 2)$ and $Q = (5, -2)$
2. $P = (-1, 7)$ and $Q = (-3, 5)$
3. $P = (-5, -3)$ and $Q = (1, 7)$
4. $P = (3, 2)$ and $Q = (10, 2)$

4.1.2 Scrap Work

The scrap work for this exercise might look something like this:

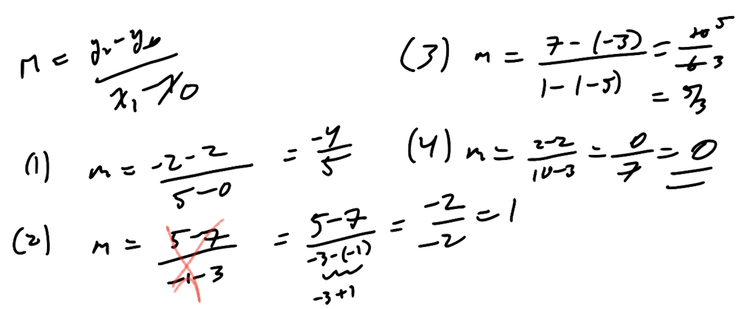


Figure 1: Scrap work for the *Points and Slopes Problem 4.1.1*.

4.1.3 Write Up to Hand In

For each pair of points we are given we find the slope of the line between them using the formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}:$$

1. For $P = (0, 2)$ and $Q = (5, -2)$, $m = \frac{\Delta y}{\Delta x} = \frac{-2-2}{5-0} = -\frac{4}{5}$.
2. For $P = (-1, 7)$ and $Q = (-3, 5)$, $m = \frac{\Delta y}{\Delta x} = \frac{5-7}{-3+1} = 1$.
3. For $P = (-5, -3)$ and $Q = (1, 7)$, $m = \frac{\Delta y}{\Delta x} = \frac{7+3}{1+5} = \frac{10}{6} = \frac{5}{3}$.
4. For $P = (3, 2)$ and $Q = (10, 2)$, $m = \frac{\Delta y}{\Delta x} = \frac{2-2}{10-3} = 0$.

4.2 Algebra Problem

4.2.1 Problem Statement

Given $H(x) = (2x - 1)/(x - 2)$ simplify $H \circ H(x)$ to show that H is its own inverse.

4.2.2 Scrap Work

The scrap work for this exercise might look something like this:

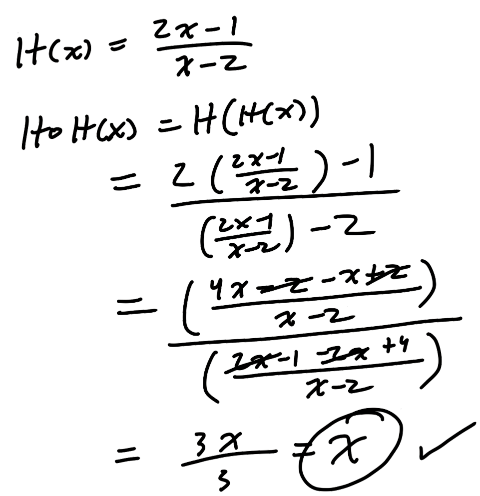


Figure 2: Scrap work for the *function composition problem 4.2.1*.

4.2.3 Write Up to Hand In

Given $H(x) = (2x - 1)/(x - 2)$, if we simplify $H \circ H(x)$ we get x, as we can see here:

$$\begin{aligned} H \circ H(x) &= H(H(x)) \\ &= \frac{2\left(\frac{2x-1}{x-2}\right) - 1}{\left(\frac{2x-1}{x-2}\right) - 2} \\ &= \frac{\left(\frac{4x-2-x+2}{x-2}\right)}{\left(\frac{2x-1-2x+4}{x-2}\right)} \\ &= \frac{3x}{3} = x. \end{aligned}$$

Therefore, $H(x)$ is its own inverse.

4.3 Word Problem

4.3.1 Problem Statement

Graph the absolute value function $y = |x+3|$ and the constant function $y = 5$. Observe the points of intersection and shade the x-axis representing the solution set to the inequality $|x+3| \geq 5$. Show your graph and write your final answer in interval notation.

4.3.2 Scrap Work

The scrap work for this exercise might look something like this:

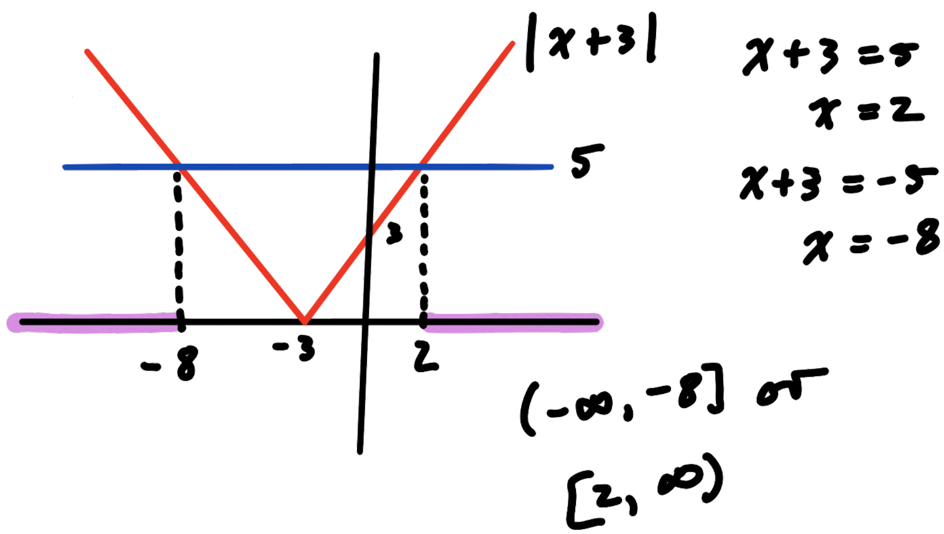
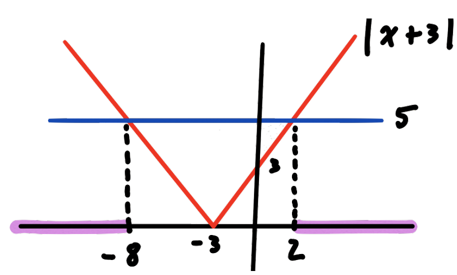


Figure 3: Scrap work for the *inequality problem 4.3.1*.

4.3.3 Write Up to Hand In

In the graph of $y = |x+3|$ and $y = 5$, the inequality $|x+3| \geq 5$ is true to the left and right of the intersection points; when x is in the interval $(-\infty, -8]$ or $[2, \infty)$. We find the intersection points by solving $|x+3| = 5$ which implies $x + 3 = 5$ or -5. Therefore we get $x = 2$ or -8.



4.4 Projectile Problem

4.4.1 Problem Statement

The height of a ball thrown in the air is given by the equation

$$s(t) = -4.9t^2 + 20t + 2 \; meters,$$

were t measures time in seconds. Using this find the time at which the ball reaches its maximum height and the time at which it lands on the ground.

4.4.2 Scrap Work

The scrap work for this exercise might look something like this:

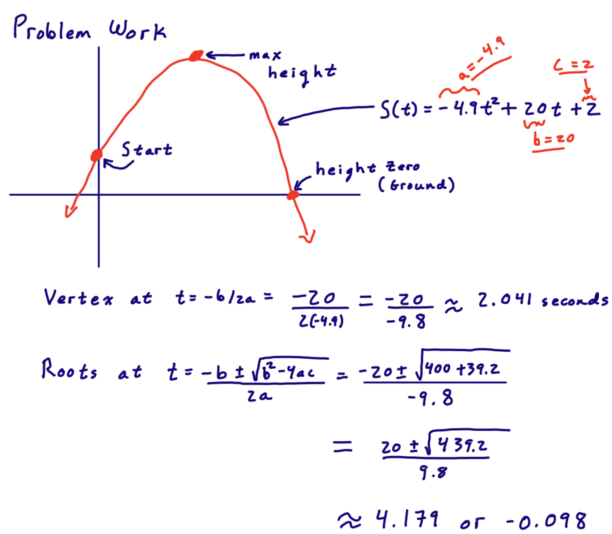


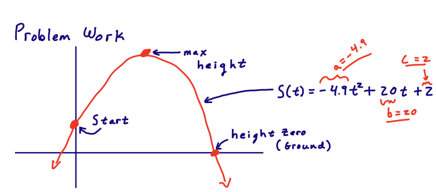
Figure 4: Scrap work for the *projectile problem 4.4.1*.

4.4.3 Write Up to Hand In

The ball whose height is given by $s(t) = -4.9t^2 + 20t + 2 \; meters$ reaches its maximum height at $t \approx 2.041$ seconds and lands on the ground after $t \approx 4.179$ seconds.

The path of the ball is a parabola, the maximum height is at the vertex when

$$t = \frac{-b}{2a} = \frac{-20}{-9.8} \approx 2.041 \; sec.$$



The ball lands on the ground when the height is 0 meters; the roots of the parabola. Using the quadratic formula we get

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-20 \pm \sqrt{400 + 39.2}}{-9.8} \approx 4.179 \text{ or } -0.098.$$

Only the positive value makes sense so $t \approx 4.179 \; sec.$

4.5 Proportion Problem

4.5.1 Problem Statement

The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm)) and the cube of the diameter. If the shaft of a certain material is 3 inches in diameter, then it can transmit 45 hp at 100 rpm; what must the diameter be in order to transmit 60 hp at 150 rpm?

4.5.2 Scrap Work

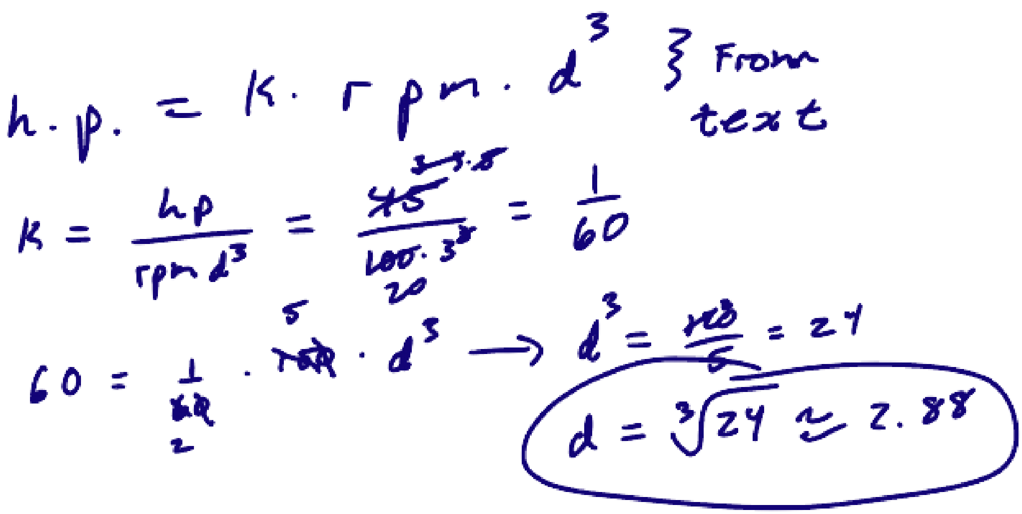


Figure 5: Scrap work for the *proportion problem 4.5.1*.

4.5.3 Write Up to Hand In

Horse power transmitted by a shaft varies jointly with speed and the cube of its diameter: $hp = k \cdot rpm \cdot (d^3)$. A $3\,inch$ diameter shaft transmitting $45\,hp$ at $100\,rpm$ gives a constant

$$k = \frac{hp}{rpm \cdot (d^3)} = \frac{45}{100 \cdot (3^3)} = \frac{1}{60}.$$

Therefore, to generate $60\,hp$ at $150\,rpm$ we need a shaft with diameter

$$\begin{aligned} d = \sqrt[3]{\frac{hp}{k \cdot rpm}} &= \sqrt[3]{\frac{60}{(1/60) \cdot 150}} \\ &= \sqrt[3]{24} \\ &= 2\sqrt[3]{3}\,inch \\ &\approx 2.88\,inch. \end{aligned}$$

5 Incomplete Problems

In this section there are two incomplete problems. You are provided with the problem statements and the scrap work for the problems; your assignment is to write them up neatly using the material you learned earlier. Be sure to pay attention to all the tips you were given in *Section 2* on flipping questions and *Section 3* on mathematical conventions.

5.1 Exercise 1:

5.1.1 Problem Statement

Find the distance between the line $y = \frac{1}{3}x + 1$ and the point $P = (5, 1)$.

5.1.2 Scrap Work

The scrap work for this exercise might look something like this:

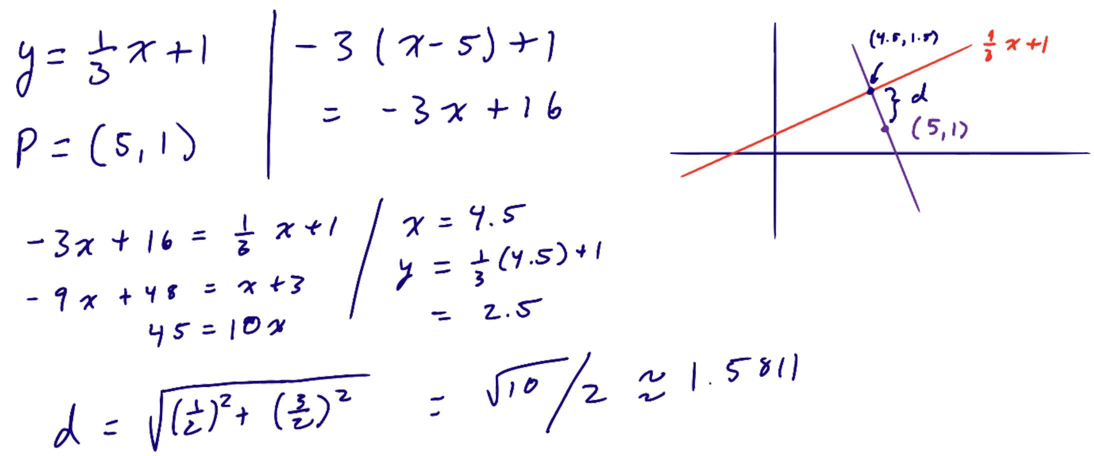


Figure 6: Scrap work for the *distance problem 5.1.1*.

5.1.3 Your Assignment

Use the scrap work above to write up a solution to the *distance problem (5.1.1)*. Be sure to pay attention to all the tips you were given in *Section 2* on flipping questions and *Section 3* on mathematical conventions.

5.2 Exercise 2:

Theorem (Rational Root Theorem)**.** *If* $x = p/q$ *is a* rational zero *of*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and p/q *is in lowest terms, then* p *is a factor of* a_0 *and* q *is a factor of* a_n*.*

5.2.1 Problem Statement

Use the *Rational Root Theorem* to help you solve the polynomial equation

$$3x^3 + 11x^2 + 8x - 4 = 0.$$

5.2.2 Scrap Work

The scrap work for this exercise might look something like this:

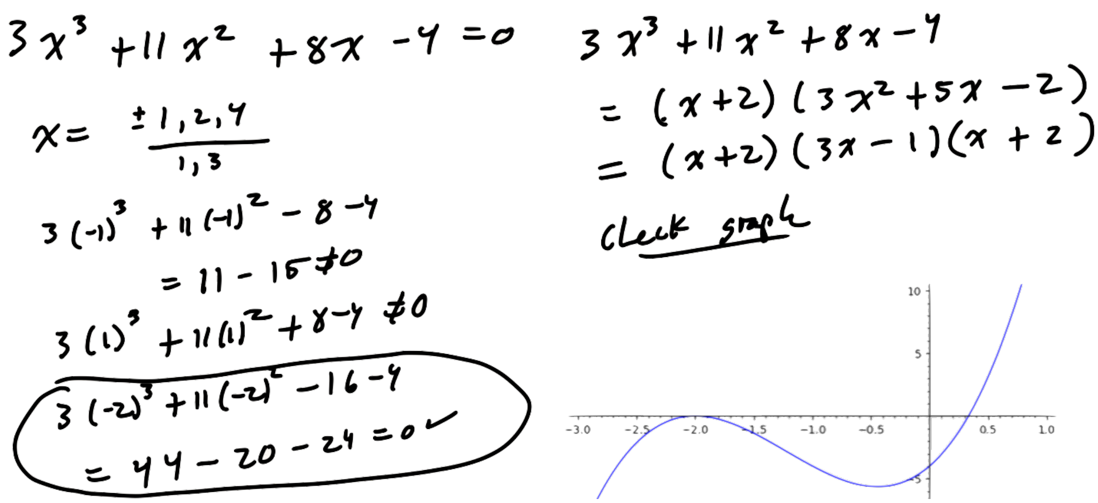


Figure 7: Scrap work for the *rational zeros problem 5.2.1*.

5.2.3 Your Assignment

Use the scrap work above to write up a solution to the *rational zeros problem (5.2.1)*. Be sure to pay attention to all the tips you were given in *Section 2* on flipping questions and *Section 3* on mathematical conventions.

6 New Problems

In this section there are two problems. You are provided with the problem statements and your assignment is to solve them and then write them up neatly using the material you learned earlier. Be sure to pay attention to all the tips you were given in *Section 2* on flipping questions and *Section 3* on mathematical conventions.

6.1 Exercise 1:

6.1.1 Problem Statement

A savings account earning $r\%$ interest per year, with interest compounded each month, will have a value of

$$P_t = P_0 \left(1 + \frac{r}{12}\right)^{12t}$$

after t years where P_0 is the initial money deposited in the account. Assuming $r = 0.02$ (i.e. $r = 2\%$), determine how long it would take to double an initial deposit of $P_0 = \$1000$. Does the doubling time depend on the value of P_0 and/or r, and why?

6.1.2 Your Assignment

Solve the given *interest problem (6.1.1)* and then write up your solution neatly. Be sure to pay attention to all the tips you were given in *Section 2* on flipping questions and *Section 3* on mathematical conventions. Submit both your scrap work and your final write up.

6.2 Exercise 2:

6.2.1 Problem Statement

The weight, w, of an object above the surface of the earth varies inversely with the square of the distance, d, from the center of the earth, that is $w = k/d^2$. If a person weighs 150 pounds when they are on the surface of the earth (3,960 miles from center), find their weight if they are 20 miles above the surface.

6.2.2 Your Assignment

Solve the given *weight problem (6.2.1)* and then write up your solution neatly. Be sure to pay attention to all the tips you were given in *Section 2* on flipping questions and *Section 3* on mathematical conventions. Submit both your scrap work and your final write up.