Lectures on Multivariable Mathematics: Calculus in Higher Dimensions

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6. calculate partial derivatives,
7. calculate the derivative of vector valued functions, and
8. find expressions for tangent planes, vectors, and lines.

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Lines in 2D: Standard Expressions

Formats for lines in 2D:

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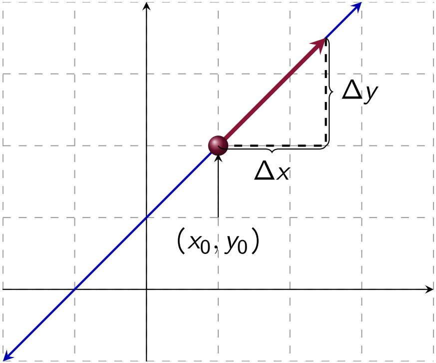


$m = \Delta y / \Delta x$

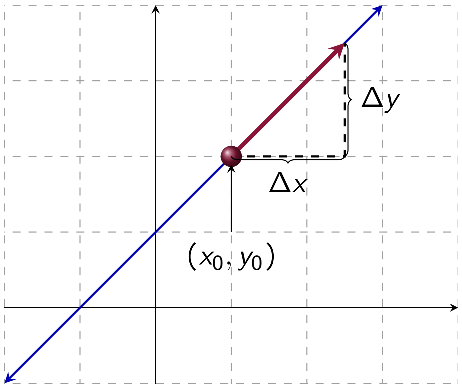
$b = (-\Delta y \cdot x_0)/\Delta x + y_0$

$A = -\Delta y,\ B = \Delta x$, and

$C = -\Delta y \cdot x_0 + \Delta x \cdot y_0$

Lines in 2D: Using Orthogonality

$$Ax + By = C$$



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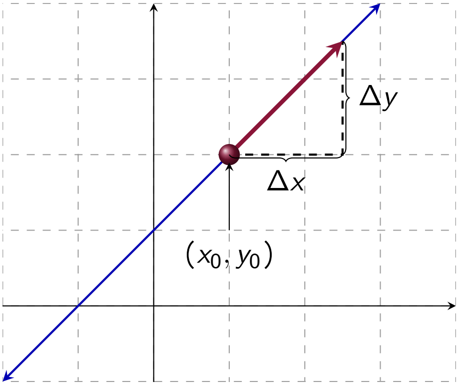

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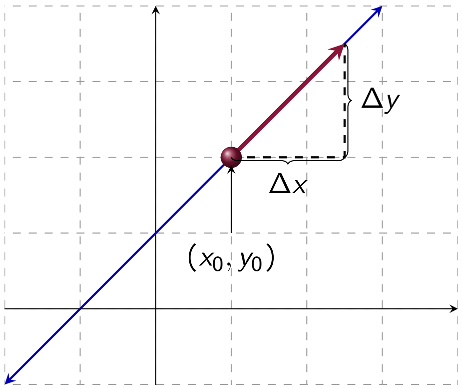
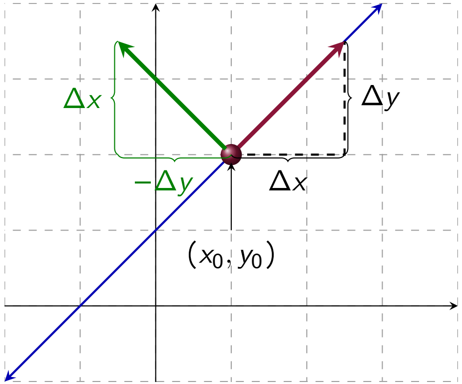
$$-\Delta y \cdot x + \Delta x \cdot y = -\Delta y \cdot x_0 + \Delta x \cdot y_0$$
$$-\Delta y \cdot x + \Delta x \cdot y + \Delta y \cdot x_0 - \Delta x \cdot y_0 = 0$$



Lines in 2D: Using Orthogonality

$$-\Delta y \cdot x + \Delta x \cdot y + \Delta y \cdot x_0 - \Delta x \cdot y_0 = 0$$
$$\langle -\Delta y, \Delta x \rangle \cdot \langle x - x_0, y - y_0 \rangle = 0$$



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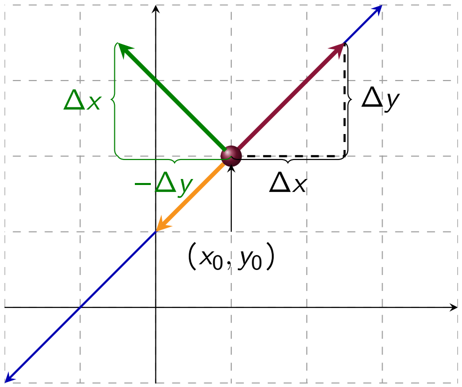

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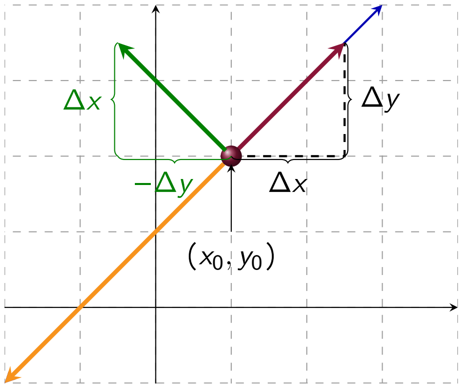

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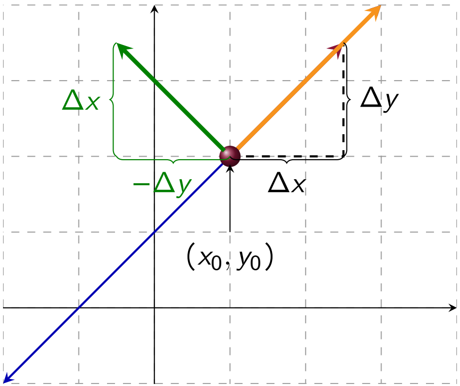

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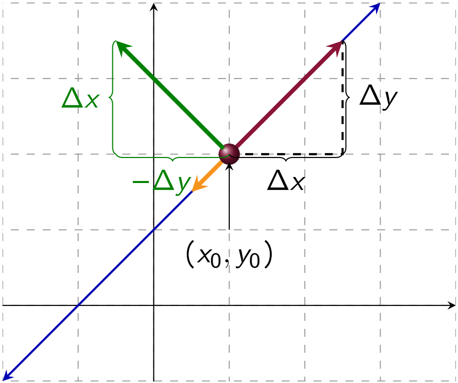

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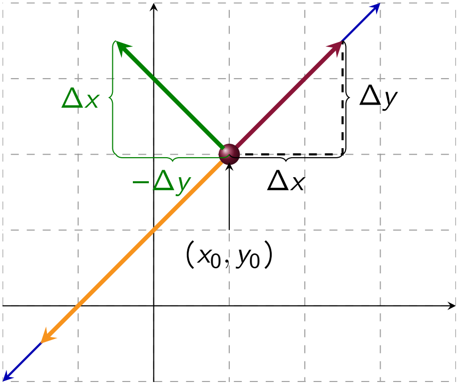

Lines in 2D: Another Example

Given $P = (-2, 3)$ and $Q = (4, -2)$ then

- Normal Vector: $\overrightarrow{N} = \langle -\Delta y, \Delta x \rangle = \langle -(-2-3), 4-(-2) \rangle = \langle 5, 6 \rangle$

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- General Form: $5x + 6y = 8$
- Parametric Form:

$$\langle x, y \rangle = \langle \Delta x, \Delta y \rangle t + \langle x_0, y_0 \rangle = \langle 6, -5 \rangle t + \langle 4, -2 \rangle$$

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2D to 3D

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- Vector Equation: $\overrightarrow{N} \cdot \langle x - x_0, y - y_0 \rangle = 0$
- Standard Form: $a(x - x_0) + b(y - y_0) = 0$
- General Form: $a \cdot x + b \cdot y = a \cdot x_0 + b \cdot y_0$
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2D to 3D

- Normal Vector: $\overrightarrow{N} = \langle a, b, c \rangle$
- Vector Equation: $\overrightarrow{N} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$ Plane
- Standard Form: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ Plane
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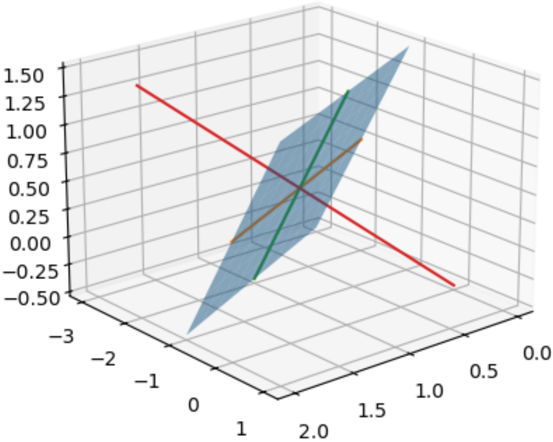
- Parametric Form: $\langle x, y, z \rangle = \langle \Delta x, \Delta y, \Delta z \rangle t + \langle x_0, y_0, z_0 \rangle$ Line
- Parametric Form: $\langle x, y, z \rangle = \overrightarrow{v}\, t + \overrightarrow{w}\, s + \langle x_0, y_0, z_0 \rangle$ Plane

Lines vs. Planes: Visually



Normal Vectors and "Planes" in $\mathbb{R}^n$

Given $\overrightarrow{N}, P \in \mathbb{R}^n$ and $\overrightarrow{x} = \langle x_1, x_2, \ldots, x_n \rangle$ an n-dimensional variable, then

$$\overrightarrow{N} \cdot (\overrightarrow{x} - P) = 0$$

is the "plane" orthogonal to $\overrightarrow{N}$. Which in general form is

$$N_1\, x_1 + N_2\, x_2 + \cdots + N_n\, x_n = B$$

where $B = N_1\, P_1 + N_2\, P_2 + \cdots + N_n\, P_n = \overrightarrow{N} \cdot P$. This is called a hyperplane.

Parametric Definitions and “Planes“ in $\mathbb{R}^n$

Given vectors $\overrightarrow{v}_i \in \mathbb{R}^n$ for $1 \leq i \leq k$ and a point $P \in \mathbb{R}^n$ then

$$\overrightarrow{y} = t_1 \overrightarrow{v}_1 + t_2 \overrightarrow{v}_2 + \cdots + t_k \overrightarrow{v}_k + P$$

where the t_i are one-dimensional variables is an affine space and can be viewed as an image in the since that we can write

$$\overrightarrow{y} = V\overrightarrow{t} + P = \begin{pmatrix} \overrightarrow{v}_1 & \overrightarrow{v}_2 & \cdots & \overrightarrow{v}_k \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{pmatrix} + P,$$

a composition of a linear transformation and a translation.
Note that if $k = n - 1$ then this will be a plane in the same sense as the previous slide with the normal vector $\overrightarrow{N}$, the basis for $V^{\perp}$, the orthogonal complement to V.

Linear vs. Affine Functions

Definition

A function $f(x)$ defined on a vector space is linear provided

$$f(ax + by) = af(x) + bf(y)$$

for all vectors x, y and scalars a, b. An affine function is the composition of a linear function and a translation.

Example

This is a linear function:

$$f(\vec{x}) = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \vec{x}$$

While this is an affine function:

$$g(\vec{x}) = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \vec{x} + \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

Plane Example: Parametric Form

Find the plane passing through the points $P = (0, 1, -2)$, $Q = (2, 0, 0)$, and $R = (1, 4, 7)$.

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- $\overrightarrow{v} = Q - P = \langle 2, -1, 2 \rangle$ and $\overrightarrow{w} = R - P = \langle 1, 3, 9 \rangle$

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- $\vec{v} = Q - P = \langle 2, -1, 2 \rangle$ and $\vec{w} = R - P = \langle 1, 3, 9 \rangle$
- Then the plane is given by:

$$\vec{v} \cdot t + \vec{w} \cdot s + P = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} s + \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 9 & 2 \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ 9 & 2 & -2 \end{pmatrix} \begin{pmatrix} t \\ s \\ 1 \end{pmatrix}$$

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- In linear algebra we would say $\overrightarrow{v}$ and $\overrightarrow{w}$ form a subspace and then we shift that subspace by adding P.

Plane Example: Vector & Standard Form

Find the plane passing through the points $P = (0, 1, -2)$, $Q = (2, 0, 0)$, and $R = (1, 4, 7)$.

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- $\overrightarrow{v} = Q - P = \langle 2, -1, 2 \rangle$ and $\overrightarrow{w} = R - P = \langle 1, 3, 9 \rangle$
- Find the normal vector $\overrightarrow{N}$ using the **cross product**:

$$\begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & -1 & 2 \\ 1 & 3 & 9 \end{vmatrix} = (-1(9) - 2(3))\overrightarrow{i} - (2(9) - 2(1))\overrightarrow{j} + (2(3) - (-1)(1))\overrightarrow{k}$$

$$= \langle -15, -16, 7 \rangle$$

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- The plane is given by $\overrightarrow{N} \cdot \langle x, y - 1, z + 2\rangle = 0$ which is

$$-15x - 16y + 7z + 30 = 0.$$

Cross Product

Definition (Cross Product)

Given two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ in $\mathbb{R}$ the cross product $\overrightarrow{n} = \overrightarrow{v} \times \overrightarrow{w}$ is the vector

$$\overrightarrow{n} = \langle v_2 w_3 - v_3 w_2, -(v_1 w_3 - v_3 w_1), v_1 w_2 - v_2 w_3 \rangle$$

which is orthogonal to both $\overrightarrow{v}$ and $\overrightarrow{w}$. Note that $\overrightarrow{v} \times \overrightarrow{w} = -(\overrightarrow{w} \times \overrightarrow{v})$.

Example

With $\overrightarrow{v} = \langle 1, 0, 2 \rangle$ and $\overrightarrow{w} = \langle 1, -1, 0 \rangle$ the cross product is:

$$\overrightarrow{n} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 1 & -1 & 0 \end{vmatrix} = \langle (0 - (-2)), -(0 - 2), -1 - 0 \rangle = \langle 2, 2, -1 \rangle$$

Orthogonal Decomposition Theorem

Theorem (Orthogonal Decomposition Theorem)

Let W be a subspace of $\mathbb{R}^n$, then each $y \in \mathbb{R}^n$ can be written uniquely in the form $y = \hat{y} + z$ where $\hat{y} \in W$ and $z \in W^{\perp}$. In fact, if $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is an orthogonal basis for W, then

$$\hat{y} = \frac{y \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \cdots + \frac{y \cdot \vec{u}_k}{\vec{u}_k \cdot \vec{u}_k}\vec{u}_k, = \sum_{\vec{u}_i} proj_{\vec{u}_i} y$$

and $z = y - \hat{y}$.

Orthogonal Bases Reminder

With $\overrightarrow{v} = \langle 1, 0, 2\rangle$, $\overrightarrow{w} = \langle 1, -1, 0\rangle$, and $\overrightarrow{e}_1 = \langle 1, 0, 0\rangle$ we can create an orthogonal basis:

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$$\overrightarrow{p}_1 = \overrightarrow{w} - proj_{\overrightarrow{v}} \overrightarrow{w} = \frac{1}{5}\langle 4, -5, -2 \rangle$$

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$$\overrightarrow{p}_2 = \overrightarrow{e}_1 - \left(proj_{\overrightarrow{v}} \overrightarrow{e}_1 + proj_{\overrightarrow{p}_1} \overrightarrow{e}_1 \right)$$

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$$\begin{aligned}
\overrightarrow{p}_1 &= \overrightarrow{w} - proj_{\overrightarrow{v}}\overrightarrow{w} = \frac{1}{5}\langle 4, -5, -2\rangle \\
\overrightarrow{p}_2 &= \overrightarrow{e}_1 - \left(proj_{\overrightarrow{v}}\overrightarrow{e}_1 + proj_{\overrightarrow{p}_1}\overrightarrow{e}_1\right) \\
&= \frac{2}{9}\langle 2, 2, -1\rangle = \frac{2}{9}\overrightarrow{n}
\end{aligned}$$

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With $\vec{v} = \langle 1, 0, 2\rangle$, $\vec{w} = \langle 1, -1, 0\rangle$, and $\vec{e}_1 = \langle 1, 0, 0\rangle$ we can create an orthogonal basis:

$$\begin{aligned}\vec{p}_1 &= \vec{w} - proj_{\vec{v}}\vec{w} = \frac{1}{5}\langle 4, -5, -2\rangle \\ \vec{p}_2 &= \vec{e}_1 - (proj_{\vec{v}}\vec{e}_1 + proj_{\vec{p}_1}\vec{e}_1) \\ &= \frac{2}{9}\langle 2, 2, -1\rangle = \frac{2}{9}\vec{n}\end{aligned}$$

So we get a multiple of the same result as the cross product. But, this process works in arbitrary dimensions.

Another Plane Example

Given $\overrightarrow{v} = \langle 1, 0, 2\rangle$, $\overrightarrow{w} = \langle 1, -1, 0\rangle$, and a base point $P = (1, -2, 3)$ we can write the equation of a plane passing through P and containing

$$P + \overrightarrow{v} = (2, -2, 5) \text{ and } P + \overrightarrow{w} = (2, -3, 3)$$

in parametric form as:

$$f(t, s) = \overrightarrow{v} \cdot t + \overrightarrow{w} \cdot s + P = \begin{pmatrix} \overrightarrow{v} & \overrightarrow{w} \end{pmatrix} \begin{pmatrix} t \\ s \end{pmatrix} + P = \begin{pmatrix} \overrightarrow{v} & \overrightarrow{w} & P \end{pmatrix} \begin{pmatrix} t \\ s \\ 1 \end{pmatrix}$$

where $t, s \in \mathbb{R}$ or in vector form as:

$$\overrightarrow{n} \cdot \langle x - 1, y + 2, z - 3\rangle = 0.$$

Lines in 3d

We can write lines in three dimensions in parametric form like so:

$$\langle x, y, z\rangle = \langle \Delta x, \Delta y, \Delta z\rangle t + \langle x_0, y_0, z_0\rangle$$

so that each coordinate is its own function

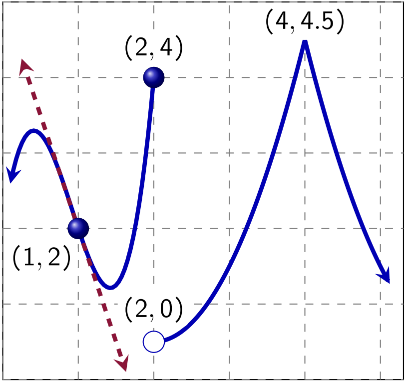
$$\begin{aligned} \langle x, y, z\rangle &= \langle \Delta x, \Delta y, \Delta z\rangle t + \langle x_0, y_0, z_0\rangle \\ x &= \Delta x \cdot t + x_0, \\ y &= \Delta y \cdot t + y_0, \text{ and} \\ z &= \Delta z \cdot t + z_0. \end{aligned}$$

Or, solving for t we get the symmetric form of the line like so:

$$t = \frac{x - x_0}{\Delta x} = \frac{y - y_0}{\Delta y} = \frac{z - z_0}{\Delta z}$$

Table of Contents

Motivation from Two Dimensions



1. $\lim_{x \to 1} f(x) =$
2. $\lim_{x \to 2^-} f(x) =$
3. $\lim_{x \to 2^+} f(x) =$
4. $\lim_{x \to 2} f(x) =$
5. $f'(1) \approx$
6. $f'(4) \approx$
7. Tangent Line at $x = 1$:

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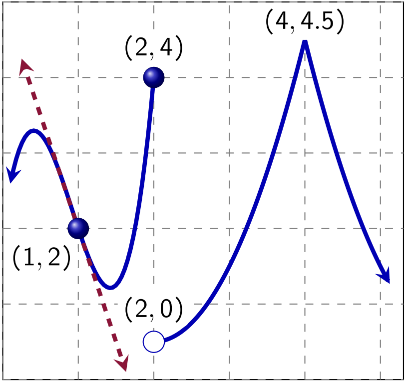

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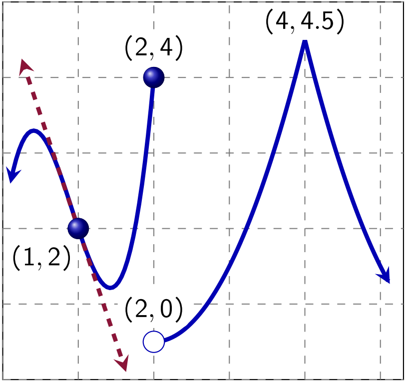

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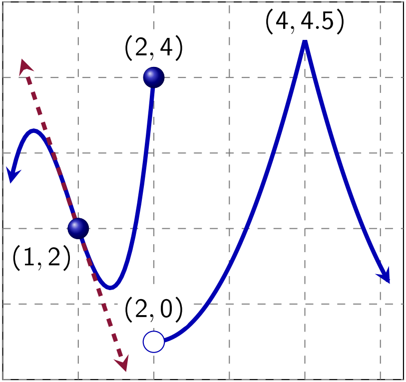

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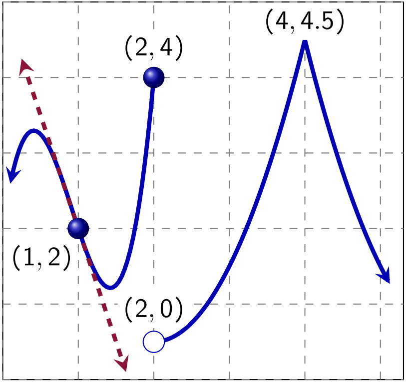

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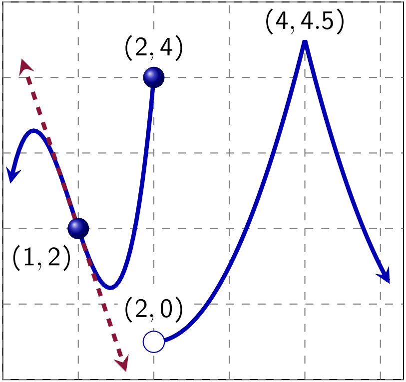

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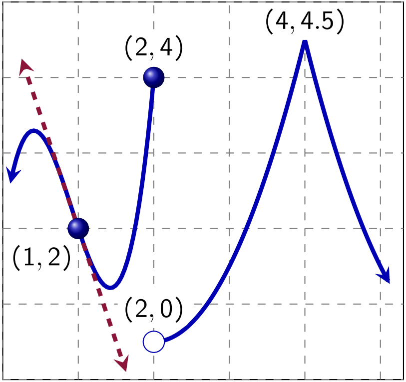

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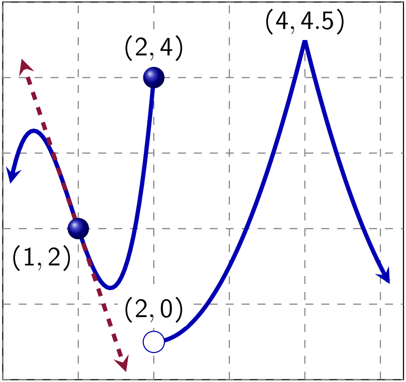

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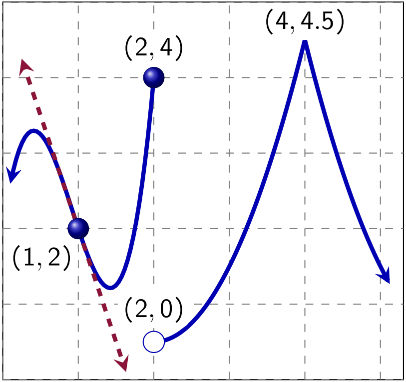

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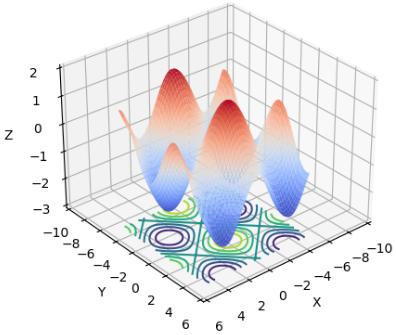
7. Tangent Line at $x = 1$:

$$y = -3(x - 1) + 2$$

Limits in 3D: First One Variable Then Another

$$\lim_{y\to\frac{\pi}{2}} \lim_{x\to\pi} f(x,y) = \lim_{y\to\frac{\pi}{2}} -1+\sin(y)$$
$$= -1+1=0$$

$f(x,y) = \cos(x) + \sin(y)$

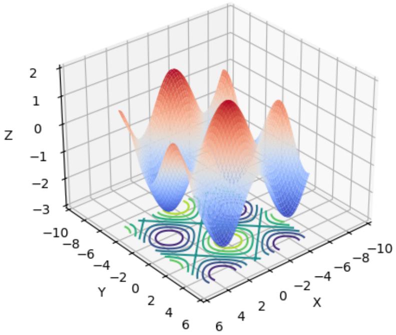


Limits in 3D: First One Variable Then Another

$$\lim_{y\to\frac{\pi}{2}} \lim_{x\to\pi} f(x,y) = \lim_{y\to\frac{\pi}{2}} -1+\sin(y)$$
$$= -1+1 = 0$$

$$\lim_{x\to\pi} \lim_{y\to\frac{\pi}{2}} f(x,y) = \lim_{y\to\frac{\pi}{2}} \cos(x)+1$$
$$= -1+1 = 0$$

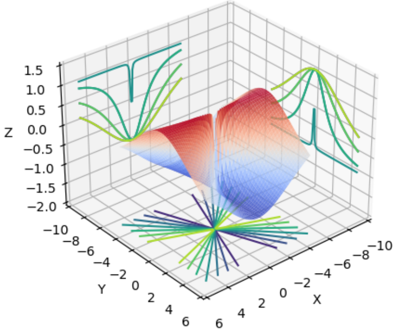
$f(x,y) = \cos(x) + \sin(y)$



Limits in 3D: Along Curves

- $y = 0$: $\lim_{(x,y)\to(0,0)} f(x,y) =$

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$



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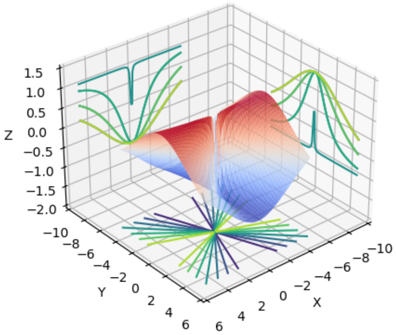

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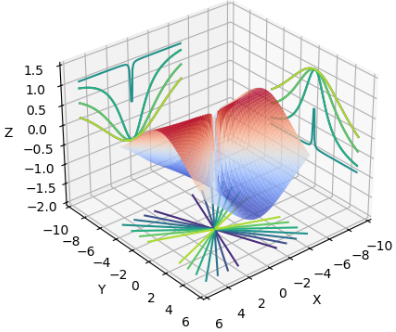

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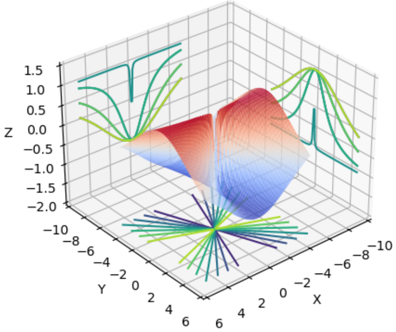

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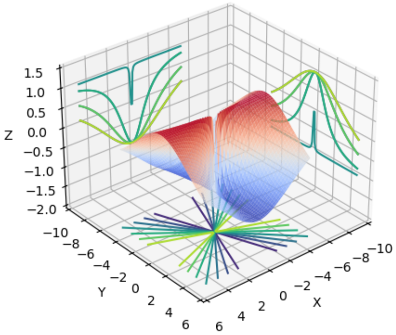

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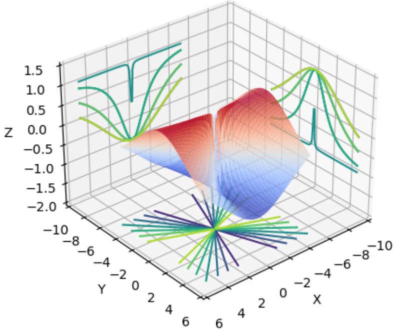

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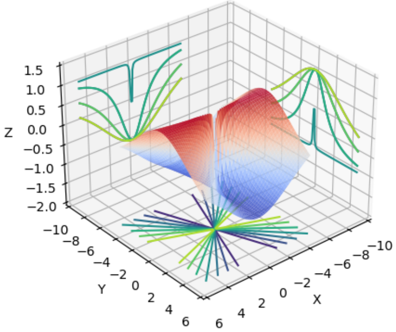

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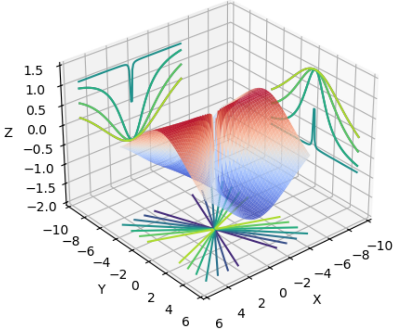

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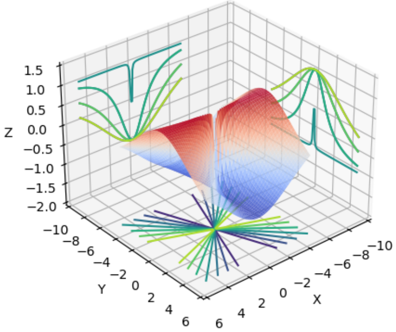
- $x = \pm ry, \ r \neq 0$:

$$\lim_{(x,y)\to(0,0)} f(x,y) =$$

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$



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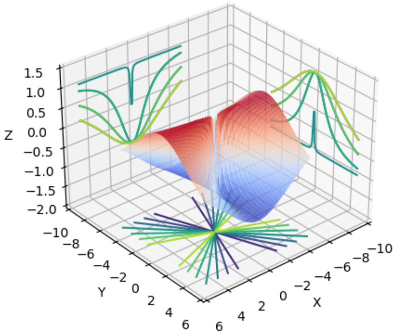
- $x = \pm 2y$: $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{3}{5}$
- $x = \pm ry, \;\; r \neq 0$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{r^2 - 1}{r^2 + 1}$$

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$



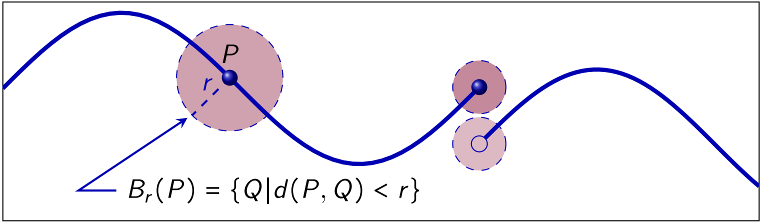
Limits in 3D: Definition

Definition

Given a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and a point $P = (x_0, y_0)$, we say that the limit as $Q \in \mathbb{R}^2$ approaches P is L, written

$$\lim_{Q \to P} f(Q) = L,$$

when given $\epsilon > 0$ there exits $\delta > 0$, such that if $Q \in B_\delta(P)$, then $|f(Q) - L| < \epsilon$.



One More Example

$$z = 2 - \frac{1}{2}\sqrt[3]{x^2 + y^2}$$

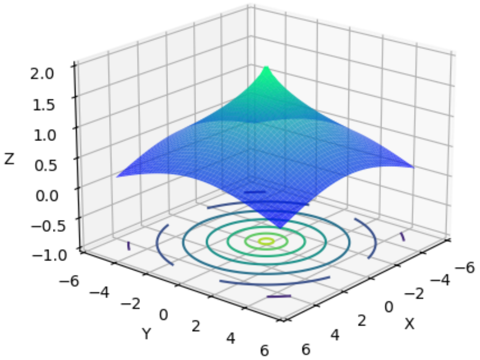


Table of Contents

Difference Quotients and Limits

$$f(x,y) = \frac{1}{10}(x^2 - y^2)$$

$$\begin{aligned}
\frac{\partial}{\partial x} f(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\
&= \frac{1}{10} \lim_{h\to 0} \frac{((x+h)^2 - y^2) - (x^2 - y^2)}{h} \\
&= \frac{1}{10} \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \frac{2}{10} x
\end{aligned}$$

Difference Quotients and Limits

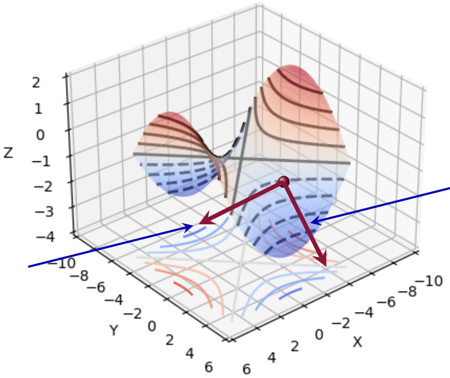
$$f(x,y) = \frac{1}{10}(x^2 - y^2)$$

$$\begin{aligned}
\frac{\partial}{\partial y} f(x,y) &= \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} \\
&= \frac{1}{10} \lim_{h \to 0} \frac{(x^2 - (y+h)^2) - (x^2 - y^2)}{h} \\
&= \frac{1}{10} \lim_{h \to 0} \frac{-(y+h)^2 + y^2}{h} = -\frac{2}{10} y
\end{aligned}$$

Difference Quotients and Limits

$$f(x,y) = \frac{1}{10}(x^2 - y^2)$$

$\partial/\partial x$ is the slope parallel to the x-axis



$\partial/\partial y$ is the slope parallel to the y-axis

Derivative With Respect to x Then x Again

Consider $g(x, y) = x^2y^2$:

$$\frac{\partial^2}{\partial x \partial x} g(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} g(x, y)\right) = \frac{\partial}{\partial x}\left(\lim_{h \to 0} \frac{g(x+h, y) - g(x, y)}{h}\right)$$

$$= \frac{\partial}{\partial x}\left(\lim_{h \to 0} \frac{y^2((x+h)^2 - x^2)}{h}\right) = \frac{\partial}{\partial x}\left(2xy^2\right)$$

$$= \lim_{h \to 0} \frac{2y^2((x+h) - x)}{h} = 2y^2$$

Derivative With Respect to x Then y

Consider $g(x,y) = x^2y^2$ again:

$$\begin{aligned}
\frac{\partial^2}{\partial y \partial x} g(x,y) &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} g(x,y)\right) = \frac{\partial}{\partial y}\left(\lim_{h\to 0} \frac{g(x+h,y) - g(x,y)}{h}\right) \\
&= \frac{\partial}{\partial y}\left(\lim_{h\to 0} \frac{y^2((x+h)^2 - x^2)}{h}\right) = \frac{\partial}{\partial y}\left(2xy^2\right) \\
&= \lim_{h\to 0} \frac{2x(y+h)^2 - 2xy^2}{h} = \lim_{h\to 0} \frac{2x((y+h)^2 - y^2)}{h} \\
&= 4xy
\end{aligned}$$

Partials in General

Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with variables $x_1, x_2, \ldots, x_n$, when we take the derivative of f with respect to x_i we treat the other variables as constants. So given $h(x, y) = x^3 - 7xy^2 + y^7$:

$$\frac{\partial}{\partial x} h =$$

$$\frac{\partial}{\partial y} h =$$

$$\frac{\partial}{\partial x\, \partial x} h =$$

$$\frac{\partial}{\partial y\, \partial y} h =$$

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Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with variables $x_1, x_2, \ldots, x_n$, when we take the derivative of f with respect to x_i we treat the other variables as constants. So given $h(x, y) = x^3 - 7xy^2 + y^7$:

$$\frac{\partial}{\partial x} h = 3x^2 - 7y^2$$

$$\frac{\partial}{\partial y} h =$$

$$\frac{\partial}{\partial x\,\partial x} h =$$

$$\frac{\partial}{\partial y\,\partial y} h =$$

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Partials in General

Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with variables $x_1, x_2, \ldots, x_n$, when we take the derivative of f with respect to x_i we treat the other variables as constants. So given $h(x, y) = x^3 - 7xy^2 + y^7$:

$$\frac{\partial}{\partial x} h = 3x^2 - 7y^2$$

$$\frac{\partial}{\partial y} h = -14xy + 7y^6$$

$$\frac{\partial}{\partial x\,\partial x} h =$$

$$\frac{\partial}{\partial y\,\partial y} h =$$

$$\frac{\partial}{\partial y\,\partial x} h =$$

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Partials in General

Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with variables $x_1, x_2, \ldots, x_n$, when we take the derivative of f with respect to x_i we treat the other variables as constants. So given $h(x, y) = x^3 - 7xy^2 + y^7$:

$$\frac{\partial}{\partial x} h = 3x^2 - 7y^2$$

$$\frac{\partial}{\partial y} h = -14xy + 7y^6$$

$$\frac{\partial}{\partial x\,\partial x} h = 6x$$

$$\frac{\partial}{\partial y\,\partial y} h =$$

$$\frac{\partial}{\partial y\,\partial x} h =$$

$$\frac{\partial}{\partial x\,\partial y} h =$$

Partials in General

Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with variables $x_1, x_2, \ldots, x_n$, when we take the derivative of f with respect to x_i we treat the other variables as constants. So given $h(x, y) = x^3 - 7xy^2 + y^7$:

$$\frac{\partial}{\partial x} h = 3x^2 - 7y^2$$

$$\frac{\partial}{\partial y} h = -14xy + 7y^6$$

$$\frac{\partial}{\partial x\, \partial x} h = 6x$$

$$\frac{\partial}{\partial y\, \partial y} h = -14x + 42y$$

$$\frac{\partial}{\partial y\, \partial x} h =$$

$$\frac{\partial}{\partial x\, \partial y} h =$$

Partials in General

Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with variables $x_1, x_2, \ldots, x_n$, when we take the derivative of f with respect to x_i we treat the other variables as constants. So given $h(x, y) = x^3 - 7xy^2 + y^7$:

$$\frac{\partial}{\partial x} h = 3x^2 - 7y^2 \qquad \frac{\partial}{\partial y\, \partial y} h = -14x + 42y$$

$$\frac{\partial}{\partial y} h = -14xy + 7y^6 \qquad \frac{\partial}{\partial y\, \partial x} h = -14y$$

$$\frac{\partial}{\partial x\, \partial x} h = 6x \qquad \frac{\partial}{\partial x\, \partial y} h =$$

Partials in General

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Equality of Mixed Partials

Theorem (Clairaut's Theorem)

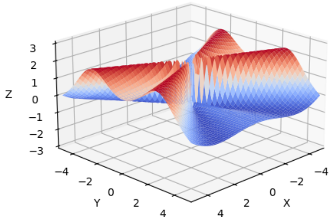
Suppose $f(x, y)$ is defined on an open disk D that contains a point (a, b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy} = f_{yx}$.

Unequal Mixed Partials[1]

Expressions for mixed partials may appear equal, this doesn't insure continuity or equality at every point:

$$\frac{\partial^2}{\partial y \partial x} \frac{xy\left(x^2 - y^2\right)}{x^2 + y^2} = \frac{\partial^2}{\partial x \partial y} \frac{xy\left(x^2 - y^2\right)}{x^2 + y^2} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{x^6 + 3x^4y^2 + 3x^2y^4 + y^6}$$

Approaching $(0, 0)$ from different directions the mixed partial reaches different values:



[1] See work at https://math.hawaii.edu/~ramsey/MixedPartialDerivatives

Partial Derivatives and The Chain Rule:

Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and x and y are also function of a variable t:

$$\frac{d}{dt}f(x,y) = \frac{\partial}{\partial x}f(x,y)\frac{dx}{dt} + \frac{\partial}{\partial y}f(x,y)\frac{dy}{dt}.$$

For example $f(x,y) = x^2y^2$, $x = \cos(t)$ and $y = \sin(t)$ then:

$$\frac{d}{dt}f(x,y) = 2xy^2(-\sin(t)) + 2yx^2\cos(t).$$

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Partial Derivatives and The Chain Rule:

Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and x and y are also function two variables t and s:

$$\frac{\partial}{\partial t} f(x,y) = \frac{\partial}{\partial x} f(x,y) \frac{\partial x}{\partial t} + \frac{\partial}{\partial y} f(x,y) \frac{\partial y}{\partial t} = \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

$$\frac{\partial}{\partial s} f(x,y) = \frac{\partial}{\partial x} f(x,y) \frac{\partial x}{\partial s} + \frac{\partial}{\partial y} f(x,y) \frac{\partial y}{\partial s} = \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix}$$

Partial Derivatives and The Chain Rule:

Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with variables x_j, $1 \leq j \leq n$, which are functions of variables t_i, $1 \leq i \leq m$ for some $m \in \mathbb{N}$:

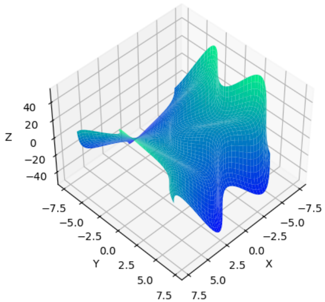
$$\frac{\partial}{\partial t_i} f(x,y) = \frac{\partial}{\partial x_1} f(x,y) \frac{\partial x_1}{\partial t_i} + \frac{\partial}{\partial x_2} f(x,y) \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial}{\partial x_n} f(x,y) \frac{\partial x_n}{\partial t_i}$$

$$= \begin{pmatrix} f_{x_1} & f_{x_2} & \cdots & f_{x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial t_i} \\ \frac{\partial x_2}{\partial t_i} \\ \vdots \\ \frac{\partial x_n}{\partial t_i} \end{pmatrix}$$

Partial Derivatives and The Chain Rule

Let $f(x,y) = x^2 - y^2$ with $x = t + \cos(s)$ and $y = s + \sin(t)$:

$$\frac{d}{d(t,s)} f(x,y) = \begin{pmatrix} f_t & f_s \end{pmatrix} = \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} x_t & x_s \\ y_t & y_s \end{pmatrix}$$



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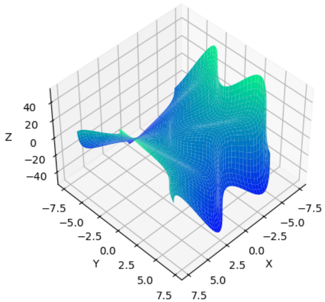
$$= \begin{pmatrix} 2x & -2y \end{pmatrix} \begin{pmatrix} 1 & -\sin(s) \\ \cos(t) & 1 \end{pmatrix}$$



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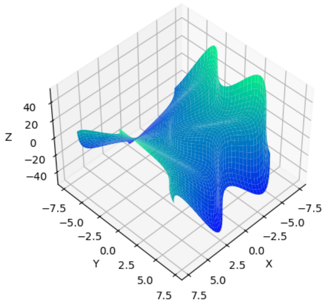

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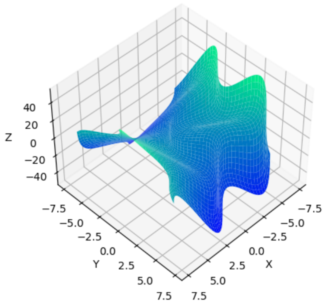


At $(t,s) = (0,\pi)$ we get $(x,y) = (-1,\pi)$, $f(x,y) = 1 - \pi^2$, and the derivatives with respect to t and s are

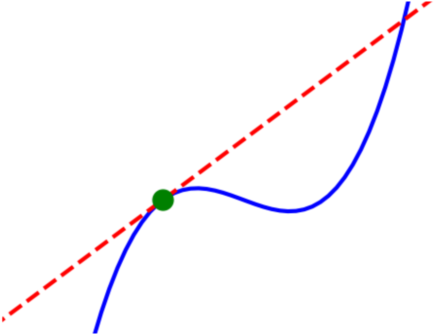
$$\frac{d}{d(t,s)} f(x,y) = \begin{pmatrix} f_t & f_s \end{pmatrix} = \begin{pmatrix} -2 - 2\pi & -2\pi \end{pmatrix}$$

Table of Contents

Tangent Lines Revisited

Given a function $f(x)$ the tangent line at $x = x_0$ is given by

$$y = f'(x_0)(x - x_0) + f(x_0)$$



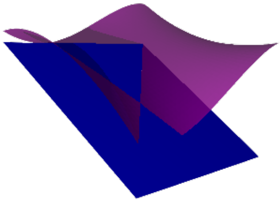
Tangent Planes to a Surface

Let $f(x, y) = x^3 - xy^2$ then

$$\frac{\partial}{\partial x} f(x, y) = 3x^2 - y^2 \text{ and } \frac{\partial}{\partial y} f(x, y) = -2xy.$$

Then at $(x, y) = (1, 1)$ the slope in the x-direction is 2 and in the y-direction is -2. Then the tangent plane will be given by

$$\begin{aligned} z &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0 + y_0) \\ &= 2(x - 1) - 2(y - 1). \end{aligned}$$

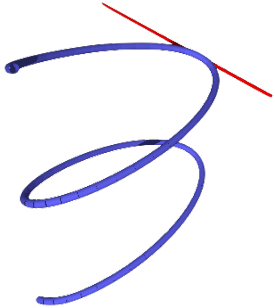


Tangent Vector and Line to a Curve

Given $f(t) = \langle \cos(2\pi t), \sin(2\pi t), t \rangle$, then

$$\frac{d}{dt} f(t) = \langle -2\pi \sin(2\pi t), 2\pi \cos(2\pi t), 1 \rangle.$$

The tangent line to the curve at $t_0 = 1/2$ is then given by



$$\begin{aligned} l(t) &= \langle x_t(t_0), y_t(t_0), z_t(t_0) \rangle t + f(t_0) \\ &= \langle 0, -2\pi, 1 \rangle t + \langle -1, 0, 1/2 \rangle \\ &= \langle -1, -2\pi t, t + 1/2 \rangle \end{aligned}$$

Lectures on Multivariable Mathematics: Calculus in Higher Dimensions

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