Lectures on Multivariable Mathematics: Symmetric Matrices, Quadratic Forms, and Optimization

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Objectives

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6. define principle axes of a quadratic form and their relation to the eigenvectors,
7. construct Cholesky factorization for a positive definite matrix, and
8. demonstrate the relation between constrained optimization problems and eigenvalues.

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Symmetric Matrices

Given:

$$A = \begin{pmatrix} -2 & -6 & -10 \\ -6 & 7 & -5 \\ -10 & -5 & 14 \end{pmatrix}$$

We can find $A = PDP^{-1}$ with

$$D = \begin{pmatrix} 10 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 20 \end{pmatrix}$$

and

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A Definition & Theorem

Definition (Symmetric Matrix)

A matrix A is symmetric if and only if $A = A^T$.

Theorem

If A is symmetric then eigenvectors with different eigenvalues are orthogonal.

Orthogonal Eigenvectors

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Proof.

Begin with $\vec{v}_1, \vec{v}_2$ eigenvectors with distinct values λ_1, λ_2.

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$$\lambda_1 \vec{v}_1 \cdot \vec{v_2} = (\lambda_1 \vec{v}_1)^T \vec{v}_2 = (A\vec{v_1})^T \vec{v}_2$$

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$$= \overrightarrow{v}_1^T A^T \overrightarrow{v}_2 = \overrightarrow{v}_1^T A \overrightarrow{v}_2$$

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But $\lambda_1 \neq \lambda_2$ so $\vec{v}_1 \cdot \vec{v}_2 = 0$, i.e. they are orthogonal. □

Spectral Theorem for Symmetric Matrices

Theorem (Spectral Decomposition Theorem)

An $n \times n$ symmetric matrix A has the following properties:

- *A has n real eigenvalues counting multiplicities*
- *The dimension of the eigenspace for each eigenvalue λ is the multiplicity of λ as a root of the characteristic equation*
- *Eigenspaces are mutually orthogonal*
- *A is orthogonally diagonalizable*

Comment on Inner and Outer Products

Let $\vec{u} = \langle 1, -2, 0 \rangle$ then

$$\vec{u} \cdot \vec{u} = \vec{u}^{\,T} \vec{u} = \begin{pmatrix} 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = 1 + 4 + 0 = 5$$

is the inner product of $\vec{u}$ with its self, while

$$\vec{u}\,\vec{u}^{\,T} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the outer product of $\vec{u}$ with its self.

Spectral Decomposition: Theory

$$\begin{aligned}
A &= PDP^{-1} \\
&= PDP^{T} = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \vec{u}_1^{\,T} \\ \vec{u}_2^{\,T} \\ \vec{u}_3^{\,T} \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1 \vec{u}_1 & \lambda_2 \vec{u}_2 & \lambda_3 \vec{u}_3 \end{pmatrix} \begin{pmatrix} \vec{u}_1^{\,T} \\ \vec{u}_2^{\,T} \\ \vec{u}_3^{\,T} \end{pmatrix} \\
&= \lambda_1 \vec{u}_1 \vec{u}_1^{\,T} + \lambda_2 \vec{u}_2 \vec{u}_2^{\,T} + \lambda_3 \vec{u}_3 \vec{u}_3^{\,T}
\end{aligned}$$

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$$\lambda_1 \vec{u}_1 \vec{u}_1^T = \frac{10}{5} \begin{pmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_2 \vec{u}_2 \vec{u}_2^T = -\frac{10}{6} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\lambda_3 \vec{u}_3 \vec{u}_3^T = \frac{20}{30} \begin{pmatrix} 4 & 2 & -10 \\ 2 & 1 & -5 \\ -10 & -5 & 25 \end{pmatrix}$$

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Note, $\vec{u}_i \vec{u}_i^T \vec{x}$ is the orthogonal projection of $\vec{x}$ onto the subspace spanned by $\vec{u}_i$.

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Quadratic Form: Example and Definition

Quadratics in Linear Algebra:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 7 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + 14xy + y^2.$$

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Then if $x = 1$ and $y = 2$:

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 7 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2(1)^2 + 14(1)(2) + (2)^2 = 34.$$

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Definition (Quadratic Form)

A quadratic form is a function $Q : \mathbb{R}^n \to \mathbb{R}$ with

$$Q(\vec{x}) = \vec{x}^T A \vec{x}$$

where A is a symmetric $n \times n$ matrix.

Change of Variable: Computational View

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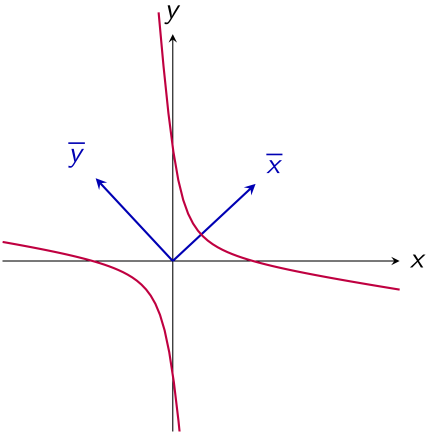
- $\vec{x}^T A\vec{x} = \vec{y}^T D\vec{y}$
- $\vec{y}^T D\vec{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$

Change of Variable: Geometric View



$$2x^2 + 14xy + y^2 = 1 \;\approx\; 8.52\overline{x}^2 - 5.52\overline{y}^2 = 1$$

Quadratic Forms and Principle Axes

Theorem (Principal Axes Theorem)

Let A be an $n \times n$ symmetric matrix, then there is an orthogonal change of variable, $\overrightarrow{x} = P\overrightarrow{y}$, that transforms the quadratic form $\overrightarrow{x}^T A\overrightarrow{x}$ into a quadratic form $\overrightarrow{y}^T D\overrightarrow{y}$ with no cross terms. Note that P and D exist by the Spectral Theorem, theorem 4, and the theorems leading up to it.

Quadratic Forms and Definiteness

Definition (Definiteness)

A quadratic form Q is

1. positive definite if $Q(x) > 0$ for all $x \neq 0$,
2. negative definite if $Q(x) < 0$ for all $x \neq 0$, and
3. indefinite if $Q(x)$ assumes both positive and negative values.

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Theorem (Quadratic Forms and Eigenvalues)

Let A be an $n \times n$ symmetric matrix, then the quadratic form $\vec{x}^T A \vec{x}$ is:

1. *positive definite if and only if the eigenvalues of A are all positive,*
2. *negative definite if and only if the eigenvalues of A are all negative, and*
3. *indefinite if and only if A has positive and negative eigenvalues.*

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Apply the Principal Axes Theorem, theorem 6.

Cholesky Factorization: Definition

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A Cholesky factorization of a symmetric matrix A is a product $A = R^T R$ in which R is upper triangular with positive diagonal entries. Note that such a factorization is possible if and only if A is positive definite.

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$$\begin{pmatrix} 4 & 6 \\ 6 & 10 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

Cholesky Factorization: "Proof"

Assume that A is a symmetric positive definite matrix

$A = PDP^T$ $\qquad$ A is symmetric

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$$
\begin{aligned}
A &= BB \\
&= B^T B \\
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\end{aligned}
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$A = BB$	
$= B^T B$	
$= R^T Q^T QR$	
$= R^T R$	Columns of Q are orthonormal, $Q^T Q = I$

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Basic Example with Algebra

Example

Given $\vec{x} \in \mathbb{R}^3$, find the largest and smallest values of

$$Q(\vec{x}) = 7x_1^2 + 10x_2^2 + 3x_3^2$$

assuming $||\vec{x}|| = 1$.

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$$Q(\vec{x}) = 7x_1^2 + 10x_2^2 + 3x_3^2 \leq 10x_1^2 + 10x_2^2 + 10x_3^2 = 10||\vec{x}||^2 = 10$$

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$$Q(\vec{x}) = 7x_1^2 + 10x_2^2 + 3x_3^2 \geq 3x_1^2 + 3x_2^2 + 3x_3^2 = 3||\vec{x}||^2 = 3$$

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Why do you think we only consider x_i^2 terms and not $x_i x_j$ cross terms?

Basic Example with Matrices

Example

Given $\vec{x} \in \mathbb{R}^3$, find the largest and smallest values of

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

assuming $||\vec{x}|| = 1$.

Basic Example with Matrices

Example

Given $\overrightarrow{x} \in \mathbb{R}^3$, find the largest and smallest values of

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assuming $||\overrightarrow{x}|| = 1$.

Note that

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = Q(\overrightarrow{x})$$

so the answers are the same.

Comment on the Previous Examples

Notice that with

$$Q\left(\vec{x}\right) = 7x_1^2 + 10x_2^2 + 3x_3^2 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The least value corresponds to the vector $\vec{x} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ and the maximum value to $\vec{x} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$. These are eigenvectors corresponding to the eigenvalues 3 and 10. Also, all values of Q, when $||\vec{x}|| = 1$, are between 3 and 10.

Generalized Solutions with Matrices

Given a symmetric $n \times n$ matrix A we can write $A = PDP^T$ and write

$$\vec{x}^{\,T} A \vec{x} = \vec{y}^{\,T} D \vec{y}, \text{ where } \vec{x} = P\vec{y}.$$

Further, if $||\vec{x}|| = 1$, then $||\vec{y}|| = 1$, since P is an orthogonal matrix.

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$$\overrightarrow{y}^T D \overrightarrow{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \cdots + \lambda_n y_n^2$$

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Given a symmetric $n \times n$ matrix A we can write $A = PDP^T$ and write

$$\vec{x}^T A \vec{x} = \vec{y}^T D \vec{y}, \text{ where } \vec{x} = P\vec{y}.$$

Further, if $||\vec{x}|| = 1$, then $||\vec{y}|| = 1$, since P is an orthogonal matrix.

$$\vec{y}^T D \vec{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \cdots + \lambda_n y_n^2$$

$$\max\left(\vec{x}^T A \vec{x}\right) = \max\left(\vec{y}^T D \vec{y}\right) = \max(\lambda_i)\,||\vec{y}||^2 = \max(\lambda_i)$$

Generalized Solutions with Matrices

Given a symmetric $n \times n$ matrix A we can write $A = PDP^T$ and write

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Generalized Solutions with Matrices

Given a symmetric $n \times n$ matrix A we can write $A = PDP^T$ and write

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Arranging D with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \lambda_n$; λ_1 is the maximum value with the corresponding vector equal to the first column of P. And, λ_n is the minimum value with the corresponding vector equal to the last column of P.

Less Basic Example with Matrices

So if we have $A = PDP^T$ like

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 13/2 & 5/2 \\ 0 & 5/2 & 13/2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix},$$

the maximum value is 9 at the vector $\begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$, and the minimum value is 1 at the vector $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$.

Lectures on Multivariable Mathematics: Symmetric Matrices, Quadratic Forms, and Optimization

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