Lectures on Multivariable Mathematics: Eigenvectors and Eigenvalues

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Objectives

After this lesson you should be able to:

1. Define Eigenvalues and Vectors,

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3. Understand Eigenspaces as Subspaces

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1. Define Eigenvalues and Vectors,
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5. Utilize Diagonalizations.

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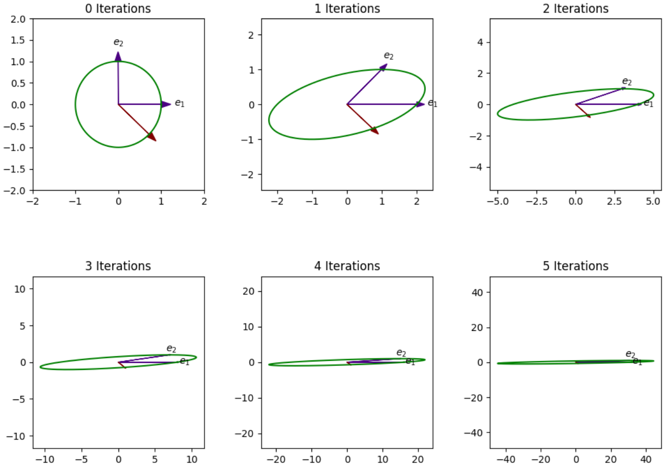

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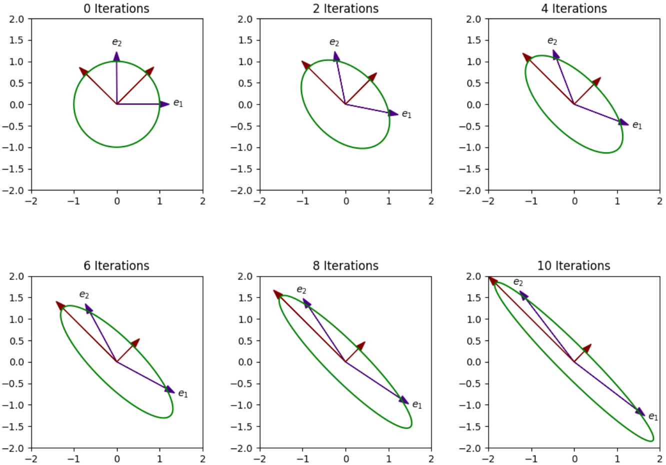

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Eigenvectors and Eigenvalues

Definition (Eigenvectors and Eigenvalues)

Given a matrix A if there exists a vector $\vec{v}$ and constant λ such that

$$A\vec{v} = \lambda\vec{v},$$

then we say that $\vec{v}$ is an **eigenvector** and λ is the corresponding **eigenvalue**.

Finding Eigenvalues and Eigenvectors: General Idea

Given an $n \times n$ matrix A we want to find $\vec{v}$ and λ such that

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Given an $n \times n$ matrix A we want to find $\vec{v}$ and λ such that

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\end{aligned}
$$

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Given an $n \times n$ matrix A we want to find $\vec{v}$ and λ such that

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= \vec{0} \\ A\vec{v} - \lambda I_n\vec{v} &= \vec{0} \end{aligned}$$

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We can think now in terms of finding a non-trivial solution to a homogeneous equation which only exists is $A - \lambda I_n$ is non-invertible, i.e. its determinant is 0.

Finding Eigenvalues: Example 1

For A as in example 1 we want to find λ such that $det(A - \lambda I_2) = 0$:

$$det(A - \lambda I_2) = det\left(\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

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So, we get that the determinant is 0 when $\lambda = 1$ *or* 2.

Finding Eigenvectors: Example 1

With $\lambda_1 = 1$, we look at the equation $(A - I_n)\vec{v}_1 = \vec{0}$, the matrix here is

$$(A - I_n) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and therefore $\vec{v}_1 = \langle t, -t \rangle$ for $t \in \mathbb{R}$.

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With $\lambda_2 = 2$, we look at the equation $(A - 2I_n)\vec{v}_2 = \vec{0}$, the matrix here is

$$(A - 2I_n) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

and therefore $\vec{v}_2 = \langle t, 0 \rangle$ for $t \in \mathbb{R}$.

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$$(A - 2I_n) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

and therefore $\vec{v}_2 = \langle t, 0 \rangle$ for $t \in \mathbb{R}$.

We choose $\vec{v}_1 = \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle$ and $\vec{v}_2 = \langle 1, 0 \rangle$ to be our representative eigenvectors since they have length 1.

Finding Eigenvalues: Example 2

For A as in example 2 we want to find λ such that $det(A - \lambda I_2) = 0$:

$$det(A - \lambda I_2) = det\left(\begin{pmatrix} 1 & -0.1 \\ -0.1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

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For A as in example 2 we want to find λ such that $det(A - \lambda I_2) = 0$:

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So, we get that the determinant is 0 when $\lambda = 1.1$ *or* 0.9.

Finding Eigenvectors: Example 2

With $\lambda_1 = 0.9$, we look at the equation $(A - I_n)\vec{v}_1 = \vec{0}$, the matrix here is

$$(A - I_n) = \begin{pmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{pmatrix}$$

and therefore $\vec{v}_1 = \langle t, t \rangle$ for $t \in \mathbb{R}$.

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We choose $\vec{v}_1 = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$ and $\vec{v}_2 = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$ to be our representative eigenvectors since they have length 1.

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Diagonal Matrices and Diagonalization

Definition (Diagonal Matrix)

A matrix of the form

$$D = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

in which all the non-diagonal entries are 0 is called a **diagonal matrix**.

Definition (Diagonalizable Matrix)

Given a square matrix A, we say that A is **diagonalizable** if there exists a diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$. That is, A is similar to some diagonal matrix.

When We Can Diagonalize

Theorem (Diagonalization Theorem)

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In particular is the eigenvectors are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and the corresponding eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_n$ then $A = PDP^{-1}$ where the $\vec{v}_i$ are the columns of P and the λ_i are the diagonal entries of D.

Diagonalization Example 1

Using A from example 1 along with its eigenvectors and values we can write

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Diagonalization Example 1

Which can be rewritten as

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

when we multiply on the left by the inverse matrix (which in the case happens to be the same matrix).

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when we multiply on the left by the inverse matrix (which in the case happens to be the same matrix).

And so we have written the matrix as $A = PDP^{-1}$ using the eigenvectors and eigenvalues as described.

Diagonalization Example 2

Using A from example 2 along with its eigenvectors and values we can write

$$\begin{pmatrix} 1 & -0.1 \\ -0.1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0.9 & -1.1 \\ 0.9 & 1.1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0.9 & 0 \\ 0 & 1.1 \end{pmatrix}$$

Diagonalization Example 2

Which can be rewritten as

$$A = \begin{pmatrix} 1 & -0.1 \\ -0.1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0.9 & 0 \\ 0 & 1.1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

when we multiply on the left by the inverse matrix.

Diagonalization Example 2

Which can be rewritten as

$$A = \begin{pmatrix} 1 & -0.1 \\ -0.1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0.9 & 0 \\ 0 & 1.1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

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Fast Exponentiation

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If D is a diagonal matrix with diagonal entries d_{ii}, then D^n is a diagonal matrix whose diagonal entries are d_{ii}^n.

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$$\begin{pmatrix} 0.9 & 0 \\ 0 & 1.1 \end{pmatrix}^n = \begin{pmatrix} 0.9^n & 0 \\ 0 & 1.1^n \end{pmatrix}$$

Fast Exponentiation

Now suppose that A is a diagonalizable matrix with $A = PDP^{-1}$:

$$A^n = (PDP^{-1})^n = \overbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}^{n}$$

Fast Exponentiation

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$$A^n = (PDP^{-1})^n = \overbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}^{n}$$
$$= \overbrace{PD(P^{-1}P)D(P^{-1}\cdots P)DP^{-1}}^{n}$$

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Now suppose that A is a diagonalizable matrix with $A = PDP^{-1}$:

$$\begin{aligned} A^n = (PDP^{-1})^n &= \overbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}^{n} \\ &= \overbrace{PD(P^{-1}P)D(P^{-1}\cdots P)DP^{-1}}^{n} \\ &= P\overbrace{(DD\cdots D)}^{n}P^{-1} \end{aligned}$$

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A^n = (PDP^{-1})^n &= \overbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}^{n} \\
&= \overbrace{PD(P^{-1}P)D(P^{-1}\cdots P)DP^{-1}}^{n} \\
&= P\,\overbrace{(DD\cdots D)}^{n}\,P^{-1} \\
&= PD^nP^{-1}
\end{aligned}$$

Fast Exponentiation

Calculate $F^{12}\vec{x}$ where $\vec{x} = \langle 1, 1 \rangle$ and

$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

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$$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues for F are

$$\lambda_1 = \frac{1 - \sqrt{5}}{2} \text{ and } \lambda_2 = \frac{1 + \sqrt{5}}{2}$$

with corresponding eigenvectors $\vec{v}_i = \langle \lambda_i, 1 \rangle$ for each i.
(Note: $\lambda_i^2 - \lambda_i - 1 = 0$, $\lambda_2 + \lambda_1 = 1$, and $\lambda_2 - \lambda_1 = \sqrt{5}$)

Fast Exponentiation

The eigenvalues for F are

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with corresponding eigenvectors $\vec{v}_i = \langle \lambda_i, 1 \rangle$ for each i.
(Note: $\lambda_i^2 - \lambda_i - 1 = 0$, $\lambda_2 + \lambda_1 = 1$, and $\lambda_2 - \lambda_1 = \sqrt{5}$)

$$F = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{\lambda_2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{\lambda_1}{\sqrt{5}} \end{pmatrix}$$

Fast Exponentiation

Then

$$F^{12}\vec{x} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^{12} & 0 \\ 0 & \lambda_2^{12} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{\lambda_2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{\lambda_1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{\sqrt{5}}\lambda_2^{14} - \frac{1}{\sqrt{5}}\lambda_1^{14} \\ \frac{1}{\sqrt{5}}\lambda_2^{13} - \frac{1}{\sqrt{5}}\lambda_1^{13} \end{pmatrix}$$

Fast Exponentiation

Or, more generally,

$$F^n\vec{x} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{\lambda_2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{\lambda_1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}}\lambda_2^{n+2} - \frac{1}{\sqrt{5}}\lambda_1^{n+2} \\ \frac{1}{\sqrt{5}}\lambda_2^{n+1} - \frac{1}{\sqrt{5}}\lambda_1^{n+1} \end{pmatrix}$$

Fast Exponentiation

Or, more generally,

$$\begin{aligned} F^n \vec{x} &= \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{\lambda_2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{\lambda_1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{5}}\lambda_2^{n+2} - \frac{1}{\sqrt{5}}\lambda_1^{n+2} \\ \frac{1}{\sqrt{5}}\lambda_2^{n+1} - \frac{1}{\sqrt{5}}\lambda_1^{n+1} \end{pmatrix} \\ &= \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} \end{aligned}$$

Where F_n is the n^{th} Fibonacci number.

Convenient Basis

With F and $\vec{x}$ as before, the product

$$\begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{\lambda_2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{\lambda_1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_2 - 1}{\sqrt{5}} \\ \frac{1-\lambda_1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}}\lambda_1 \\ \frac{1}{\sqrt{5}}\lambda_2 \end{pmatrix}$$

is converting $\vec{x}$ from the elementary basis to a bases with the eigenvectors for F, $\vec{v}_i = \langle \lambda_i, 1 \rangle$.

$$\vec{x} = \frac{\lambda_2}{\sqrt{5}}\vec{v}_2 - \frac{\lambda_1}{\sqrt{5}}\vec{v}_1$$

Convenient Basis

With F and $\vec{x}$ as before, the product

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is converting $\vec{x}$ from the elementary basis to a bases with the eigenvectors for F, $\vec{v_i} = \langle \lambda_i, 1 \rangle$.

$$F^n \vec{x} = F^n \left(\frac{\lambda_2}{\sqrt{5}} \vec{v_2} - \frac{\lambda_1}{\sqrt{5}} \vec{v_1} \right)$$

Convenient Basis

With F and $\vec{x}$ as before, the product

$$\begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{\lambda_2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{\lambda_1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_2 - 1}{\sqrt{5}} \\ \frac{1-\lambda_1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}}\lambda_1 \\ \frac{1}{\sqrt{5}}\lambda_2 \end{pmatrix}$$

is converting $\vec{x}$ from the elementary basis to a bases with the eigenvectors for F, $\vec{v_i} = \langle \lambda_i, 1 \rangle$.

$$F^n \vec{x} = F^n \left(\frac{\lambda_2}{\sqrt{5}} \vec{v_2} - \frac{\lambda_1}{\sqrt{5}} \vec{v_1} \right) = \frac{\lambda_2}{\sqrt{5}} \left(F^n \vec{v_2} \right) - \frac{\lambda_1}{\sqrt{5}} \left(F^n \vec{v_1} \right)$$

Convenient Basis

With F and $\vec{x}$ as before, the product

$$\begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{\lambda_2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{\lambda_1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_2 - 1}{\sqrt{5}} \\ \frac{1-\lambda_1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}}\lambda_1 \\ \frac{1}{\sqrt{5}}\lambda_2 \end{pmatrix}$$

is converting $\vec{x}$ from the elementary basis to a bases with the eigenvectors for F, $\vec{v}_i = \langle \lambda_i, 1 \rangle$.

$$\begin{aligned} F^n\vec{x} = F^n\left(\frac{\lambda_2}{\sqrt{5}}\vec{v}_2 - \frac{\lambda_1}{\sqrt{5}}\vec{v}_1\right) &= \frac{\lambda_2}{\sqrt{5}}\left(F^n\vec{v}_2\right) - \frac{\lambda_1}{\sqrt{5}}\left(F^n\vec{v}_1\right) \\ &= \frac{\lambda_2}{\sqrt{5}}\left(\lambda_2^n\vec{v}_2\right) - \frac{\lambda_1}{\sqrt{5}}\left(\lambda_1^n\vec{v}_1\right) \end{aligned}$$

Convenient Basis

With F and $\vec{x}$ as before, the product

$$\begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{\lambda_2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{\lambda_1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_2 - 1}{\sqrt{5}} \\ \frac{1-\lambda_1}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}}\lambda_1 \\ \frac{1}{\sqrt{5}}\lambda_2 \end{pmatrix}$$

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Linear Recurrence Relations

First Order

$$A_n = kA_{n-1} + l, \ (k \neq 1)$$

$$\begin{pmatrix} A_n \\ 1 \end{pmatrix} = \begin{pmatrix} k & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ 1 \end{pmatrix}$$

$$\lambda^2 - (k+1)\lambda + k = 0$$

$$\lambda_1 = k$$

$$\lambda_2 = 1$$

$$\vec{v}_1 = \langle 1, 0 \rangle$$

$$\vec{v}_2 = \langle l, 1-k \rangle$$

Second Order

$$A_n = kA_{n-1} + lA_{n-2}$$

$$\begin{pmatrix} A_n \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} k & l \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ A_{n-2} \end{pmatrix}$$

$$\lambda^2 - k\lambda - l = 0$$

$$\lambda_1 = \left(k + \sqrt{k^2 + 4l}\right)/2$$

$$\lambda_2 = \left(k - \sqrt{k^2 + 4l}\right)/2$$

$$\vec{v}_1 = \langle \lambda_1, 0 \rangle, \ \vec{v}_2 = \langle \lambda_2, 0 \rangle$$

Comment on Time-Space Diagrams

Lorentz Transformations — Special Relativity Ch. 3
(Link to Minute Physics)

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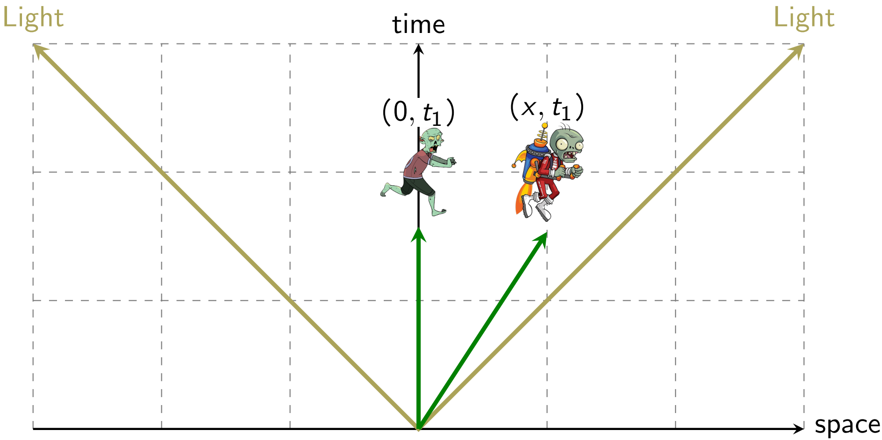

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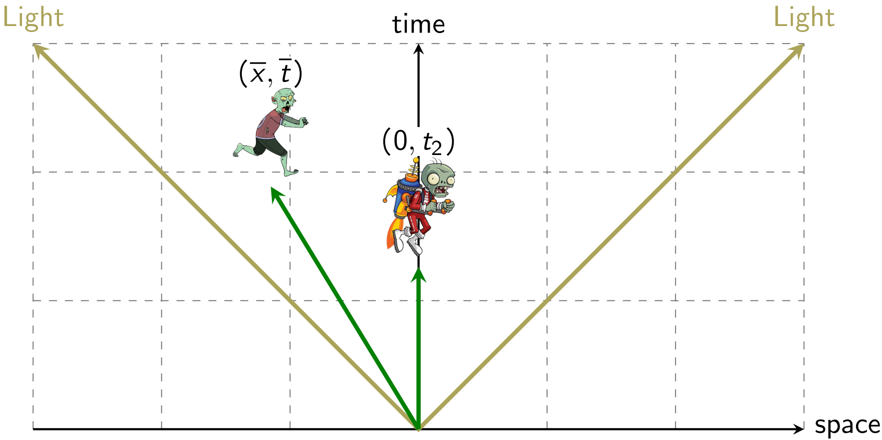

Comment on Time-Space Diagrams

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \left(-\frac{1}{2}\right) \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ t_1 \end{pmatrix} = \begin{pmatrix} 0 \\ t_2 \end{pmatrix}$$

Comment on Time-Space Diagrams

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$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x - t_1 \\ -(x + t_1) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ t_2 \end{pmatrix}$$

Comment on Time-Space Diagrams

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$$\begin{pmatrix} a(t_1 - x) \\ b(t_1 + x) \end{pmatrix} = \begin{pmatrix} t_2 \\ t_2 \end{pmatrix}$$

Comment on Time-Space Diagrams

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$$\begin{pmatrix} a(t_1 - x) \\ b(t_1 + x) \end{pmatrix} = \begin{pmatrix} t_2 \\ t_2 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} t_2/(t_1 - x) \\ t_2/(t_1 + x) \end{pmatrix}$$

Comment on Time-Space Diagrams

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \left(-\frac{1}{2} \right) \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

Comment on Time-Space Diagrams

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$$= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \left(-\frac{1}{2} \right) \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \quad a = \frac{t_2}{t_1 - x}, \; b = \frac{t_2}{t_1 + x}$$

Comment on Time-Space Diagrams

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$$= \frac{t_2}{t_1^2 - x^2} \begin{pmatrix} t_1 & -x \\ -x & t_1 \end{pmatrix} \quad t_2 = t_1 \sqrt{1 - \left(\frac{v}{c} \right)^2}$$

$$= \frac{1}{t_2} \begin{pmatrix} t_1 & -x \\ -x & t_1 \end{pmatrix} \quad x = \left(\frac{v}{c} \right) t_1$$

Comment on Time-Space Diagrams

$$\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\left(-\frac{1}{2}\right)\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}$$

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Comment on Time-Space Diagrams

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$$= \frac{1}{t_2} \begin{pmatrix} t_1 & -x \\ -x & t_1 \end{pmatrix} \qquad x = \left(\frac{v}{c} \right) t_1$$

$$= \frac{t_1}{t_2} \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{c^2 - v^2}} \begin{pmatrix} c & -v \\ -v & c \end{pmatrix}$$

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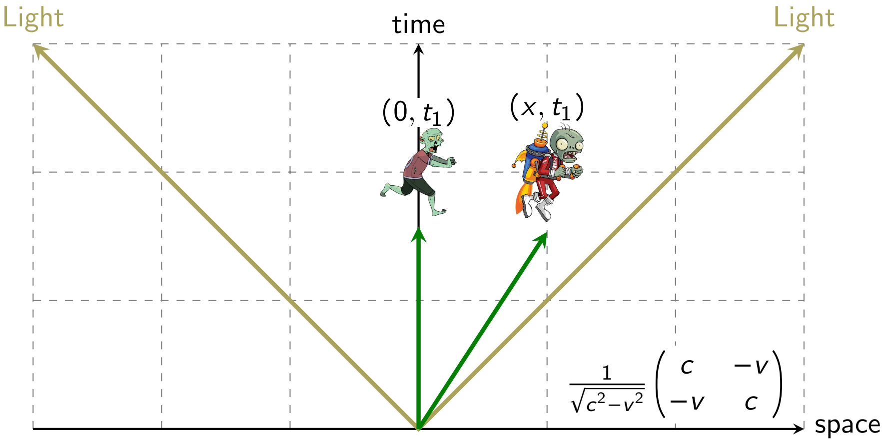

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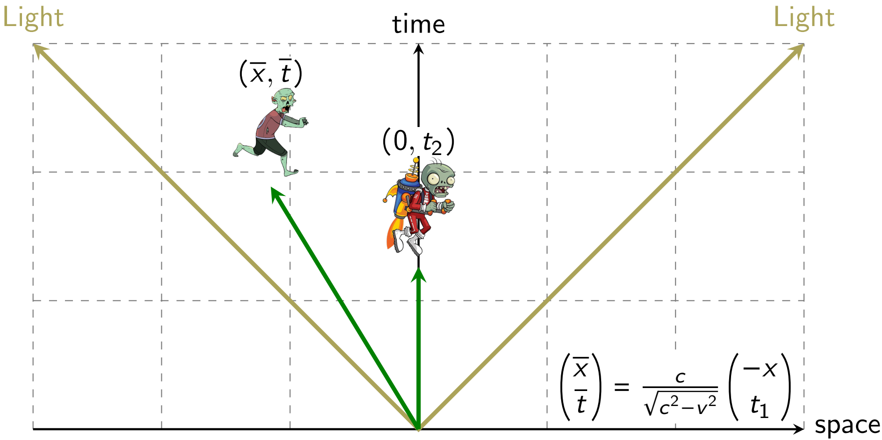

Lectures on Multivariable Mathematics: Eigenvectors and Eigenvalues

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