Lectures on Multivariable Mathematics: Bases for Vector Spaces

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Table of Contents

Objectives

After this lesson you should be able to:

1. Recall using matrix reductions to solve systems of equations,

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6. Change calculations from one basis to another.

Table of Contents

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$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 9 \\ 0 & 1 & -1 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 0 & 2 & 8 \\ 0 & 1 & -1 & -1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -1 \end{array}\right)$$

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$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -1 \end{array}\right)$$

$x = 4 - z$, $y = z - 1$, & z is a free variable

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$$
\begin{aligned}
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$$

$$
\left(\begin{array}{cc|c} 1 & 0 & 6 \\ 2 & 7 & 0 \\ 0 & 1 & 9 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 7 & -12 \\ 0 & 1 & 9 \end{array}\right)
$$

$$
\rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 0 & -75 \\ 0 & 1 & 9 \end{array}\right)
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. . . .

$$\left(\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 0 & -75 \\ 0 & 1 & 9 \end{array}\right)$$

This is an inconsistent system. The only time there will be a solution is if the values on the right are of the form

$$\langle a_1, 2a_2 + 7a_2, a_2 \rangle.$$

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$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & -4 \\ 0 & 1 & 21 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 17 \\ 0 & 1 & 21 \end{array}\right)$$

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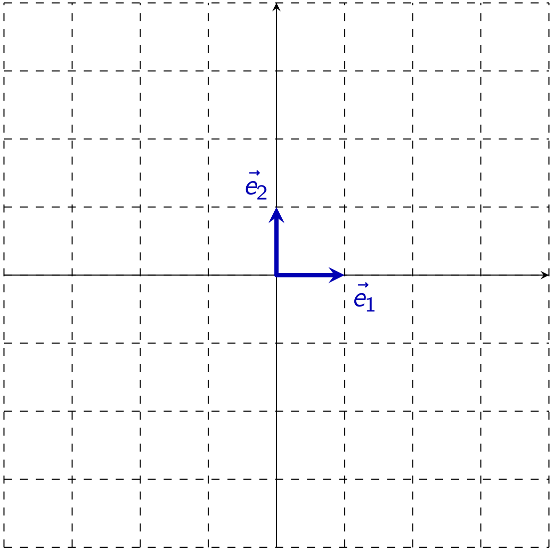
$$x = 17 \ \& \ y = 21$$

Table of Contents

Navigating in two Dimensions



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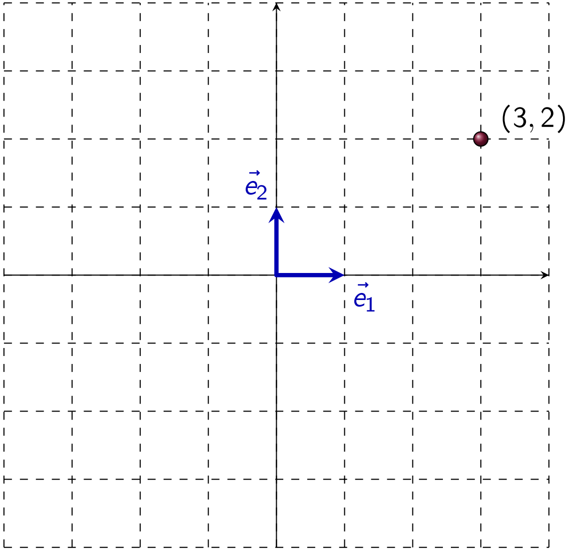

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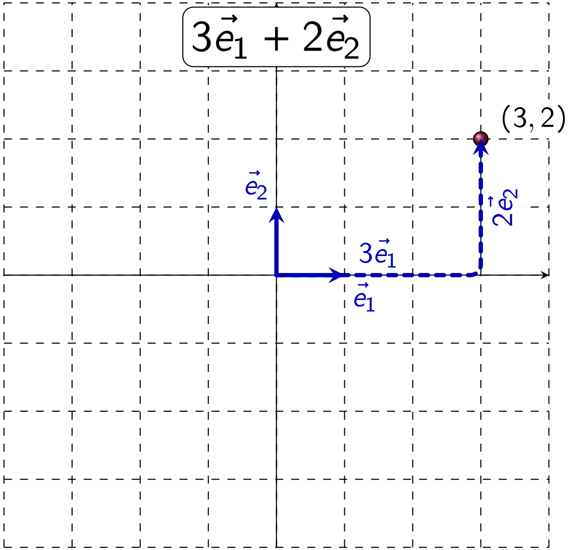

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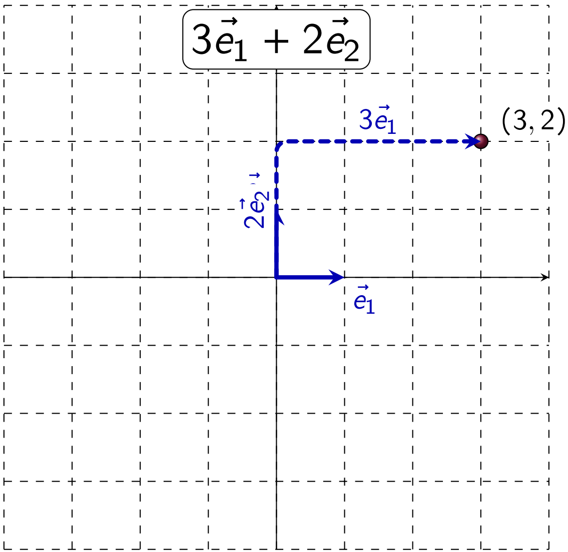

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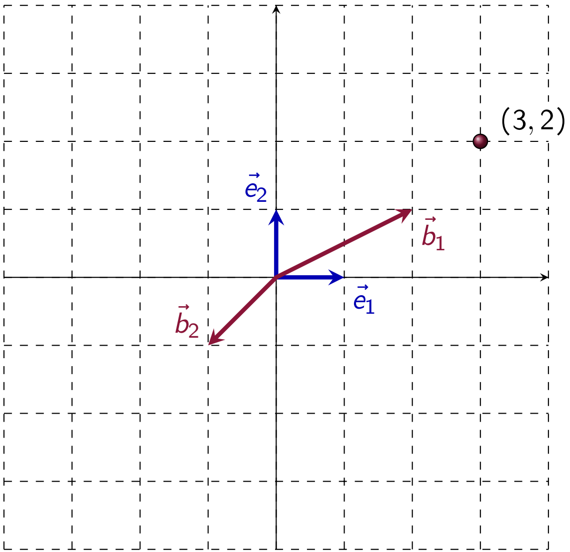

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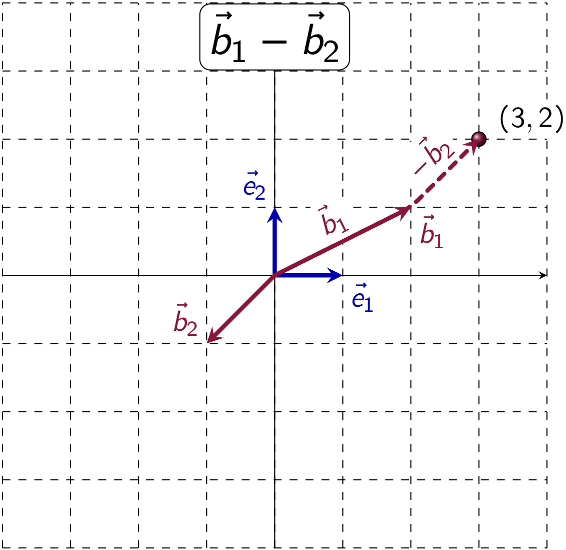

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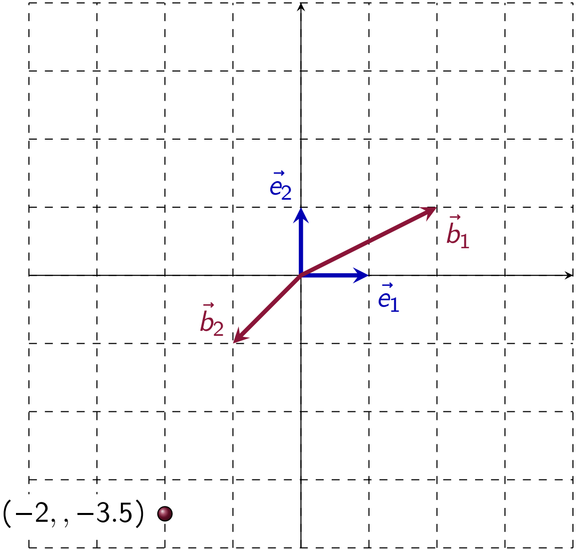

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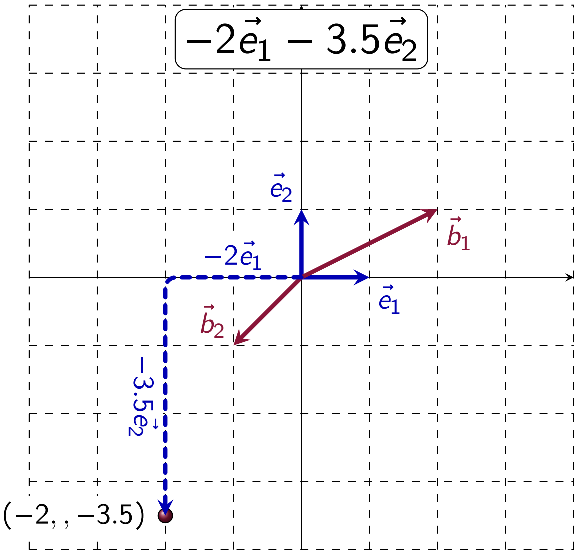

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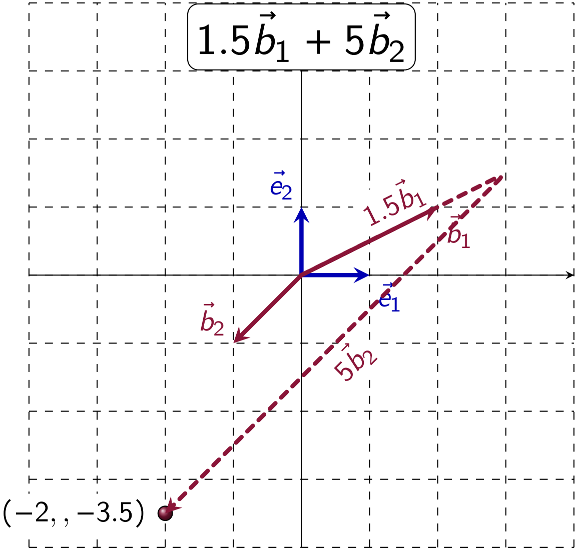

Span of Vectors

Definition (Span)

Given a set of vectors

$$\mathcal{B} = \left\{ \vec{b}_1, \vec{b}_2, \vec{b}_3, \ldots, \vec{b}_k \right\},$$

the span of the set is the set of all possible linear combinations of the vectors

$$\vec{v} = a_1 \vec{b}_1 + a_2 \vec{b}_2 + a_3 \vec{b}_3 + \cdots + a_k \vec{b}_k.$$

Example 1

Let $\mathcal{B} = \{\langle 2, 1\rangle, \langle -1, -1\rangle\}$, then the goal is to see what vectors we can write like so:

$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 = a_1\begin{pmatrix}2\\1\end{pmatrix} + a_2\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}2 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}.$$

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In this case we can use a matrix inverse to get

$$\begin{pmatrix}1 & -1\\1 & -2\end{pmatrix}\vec{v} = \begin{pmatrix}a_1\\a_2\end{pmatrix}.$$

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So if $\vec{v} = \langle 7, -3\rangle$, then

$$\begin{pmatrix}1 & -1\\1 & -2\end{pmatrix}\begin{pmatrix}7\\-3\end{pmatrix} = \begin{pmatrix}10\\13\end{pmatrix}.$$

And we have

$$\vec{v} = 10\vec{b}_1 + 13\vec{b}_2 = 10\begin{pmatrix}2\\1\end{pmatrix} + 13\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}20\\10\end{pmatrix} + \begin{pmatrix}-13\\-13\end{pmatrix} = \begin{pmatrix}7\\-3\end{pmatrix}. \checkmark$$

Example 2

Now let $\mathcal{C} = \{\langle 2, 0\rangle, \langle -1, 1\rangle, \langle 3, -1\rangle\}$; what vectors can be written as a linear combination of these like

$$\vec{v} = a_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} ?$$

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For example

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \vec{0}.$$

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Now let $\mathcal{C} = \{\langle 2,0\rangle, \langle -1,1\rangle, \langle 3,-1\rangle\}$; what vectors can be written as a linear combination of these like

$$\vec{v} = a_1\begin{pmatrix}2\\0\end{pmatrix} + a_2\begin{pmatrix}-1\\1\end{pmatrix} + a_3\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}2 & -1 & 3\\0 & 1 & -1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}?$$

But we also have

$$\begin{pmatrix}2 & -1 & 3\\0 & 1 & -1\end{pmatrix}\begin{pmatrix}-2\\2\\2\end{pmatrix} = \vec{0}.$$

Solutions are not unique!

Example 3

Finally if $\mathcal{D} = \{\langle 1, 2, 0\rangle, \langle 0, 7, 1\rangle\}$ then the linear combinations look like

$$\vec{v} = a_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

But, this will only ever give vectors of the form $\langle a_1, 2a_1 + 7a_2, a_2\rangle$, and no combination will give, for example, the vector $\langle 1, 0, 1\rangle$.

Table of Contents

Independence of Vectors

Definition (Linear Independence)

A set of vectors

$$\mathcal{B} = \left\{ \vec{b}_1, \vec{b}_2, \vec{b}_3, \ldots, \vec{b}_k \right\},$$

is linearly independent if and only if none of the vectors $\vec{b}_i$ can be written as a linear combination of the other vectors, or equivalently the only solution to

$$a_1 \vec{b}_1 + a_2 \vec{b}_2 + a_3 \vec{b}_3 + \cdots + a_k \vec{b}_k = \vec{0}$$

is the trivial solution when $a_i = 0$ for all i.

Example 1 Revisited

We saw that linear combinations of the set of vectors $\mathcal{B} = \{\langle 2, 1\rangle, \langle -1, -1\rangle\}$ can be written as a matrix product:

$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 = a_1\begin{pmatrix}2\\1\end{pmatrix} + a_2\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}2 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}.$$

And, that the matrix is inevitable

$$B = \begin{pmatrix}2 & -1\\1 & -1\end{pmatrix} \qquad\qquad B^{-1} = \begin{pmatrix}1 & -1\\1 & -2\end{pmatrix},$$

so that solutions to $B\vec{a} = \vec{v}$ will be unique.

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Alternately when B is row reduced we get the identity matrix

$$B = \begin{pmatrix}2 & -1\\1 & -1\end{pmatrix} \rightsquigarrow I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix},$$

which also gives unique solutions.

Example 2 Revisited

Similarly linear combinations of the vectors in $\mathcal{C} = \{\langle 2,0\rangle, \langle -1,1\rangle, \langle 3,-1\rangle\}$ can be written as a matrix product

$$\vec{v} = a_1\begin{pmatrix}2\\0\end{pmatrix} + a_2\begin{pmatrix}-1\\1\end{pmatrix} + a_3\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}2 & -1 & 3\\0 & 1 & -1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}.$$

However when we row reduce this matrix we get:

$$\begin{pmatrix}2 & -1 & 3\\0 & 1 & -1\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 0 & 1\\0 & 1 & -1\end{pmatrix},$$

with a pivot position in each row and a free variable we will always have infinitely many solutions. In particular we could write $\vec{0}$ multiple ways.

Example 3 Revisited

With $\mathcal{D} = \{\langle 1, 2, 0\rangle, \langle 0, 7, 1\rangle\}$ we had

$$\vec{v} = a_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Row reducing this matrix we get:

$$\begin{pmatrix} 1 & 0 \\ 2 & 7 \\ 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

with a pivot position in each column, but not each row, and no free variables we won't always have a solution, but when we do it will be unique.

Table of Contents

Basis for a Vector Space

Definition (Basis of a Vector Space)

A set of vectors

$$\mathcal{B} = \left\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \ldots, \vec{b}_k\right\},$$

is basis for a vector space if and only if for every vector $\vec{v}$ the linear combination

$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3 + \cdots + a_k\vec{b}_k$$

has a unique solution. That is $\mathcal{B}$ is a linearly independent set which spans the entire vector space. Note that the coefficients a_i are called the $\mathcal{B}$ – *coordinates* of $\vec{v}$ and we write

$$[\vec{v}]_{\mathcal{B}} = \langle a_1, a_2, \ldots, a_n\rangle.$$

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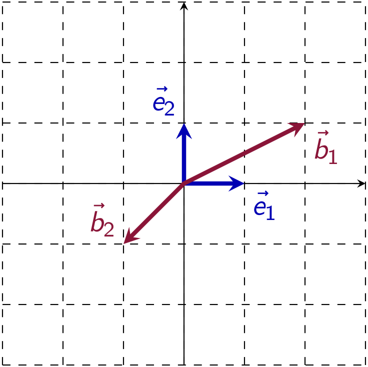
Theorem

A set of vectors is a basis for a vector space if and only if a matrix whose columns are the elements of the set is invertible.

Table of Contents

Calculations in One Basis or Another

Depending on what we want to do sometimes $\mathcal{E} = \{\langle 1, 0\rangle, \langle 0, 1\rangle\}$ is easier to work with and sometimes something like $\mathcal{B} = \{\langle 2, 1\rangle, \langle -1, -1\rangle\}$ is what we might want. So, how can we always change to a more convenient basis for the job?



Change of Basis Matrix

Theorem (Change of Basis Matrix)

Given two bases for the same vector space, $\mathcal{B}$ and $\mathcal{C}$, and matrices B and C whose columns are the vectors in each of the bases, then the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is $C^{-1}B$. That, is if

$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + a_3\vec{b}_3 + \cdots + a_k\vec{b}_k$$

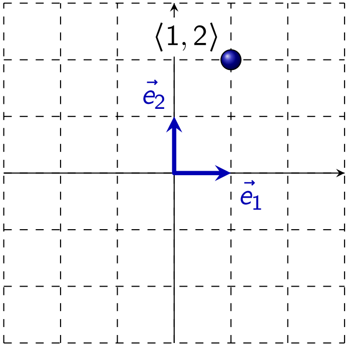
so the $\mathcal{B}-$ coordinates of $\vec{v}$ are

$$[\vec{v}]_{\mathcal{B}} = \langle a_1, a_2, \ldots, a_n \rangle,$$

then the $\mathcal{C}-$ coordinates of $\vec{v}$ will be

$$[\vec{v}]_{\mathcal{C}} = C^{-1}B\,[\vec{v}]_{\mathcal{B}} = C^{-1}B\langle a_1, a_2, \ldots, a_n \rangle.$$

Elementary to $\mathcal{B}$



- $\mathcal{E} = \{\langle 1, 0 \rangle, \langle 0, 1 \rangle\}$

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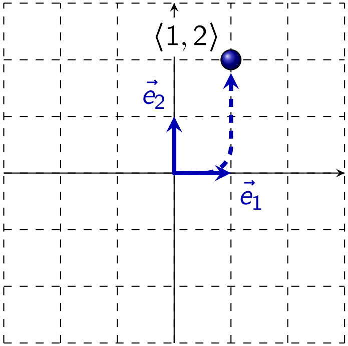

- $\mathcal{E} = \{\langle 1, 0\rangle, \langle 0, 1\rangle\}$
- $\langle 1, 2\rangle = 1 \cdot \vec{e}_1 + 2 \cdot \vec{e}_2$

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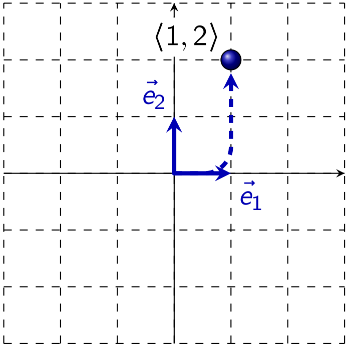

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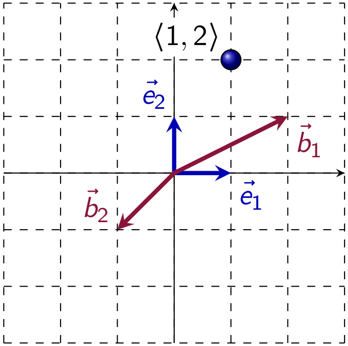

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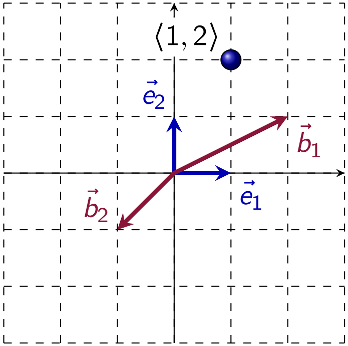

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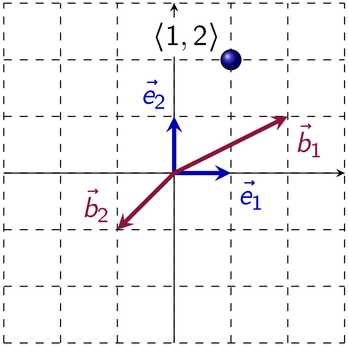
- $[\langle 1,2\rangle]_{\mathcal{E}} = \langle 1,2\rangle$
- $\mathcal{B} = \{\langle 2,1\rangle, \langle -1,-1\rangle\}$
- $[\langle 1,2\rangle]_{\mathcal{B}} = B^{-1} E \, [\langle 1,2\rangle]_{\mathcal{E}}$

Elementary to $\mathcal{B}$



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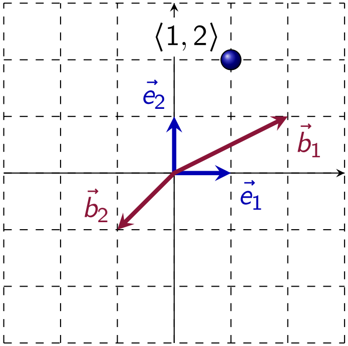
Elementary to $\mathcal{B}$



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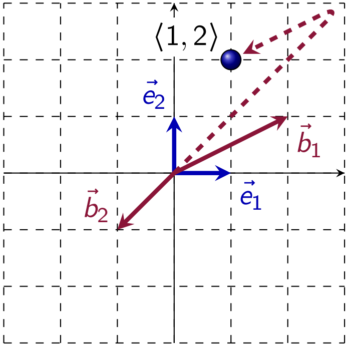

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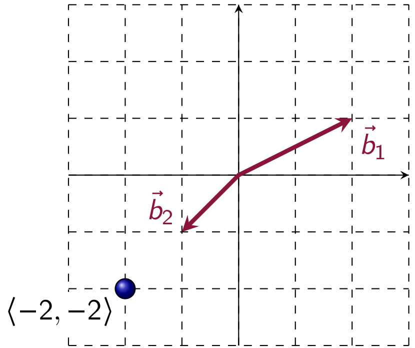

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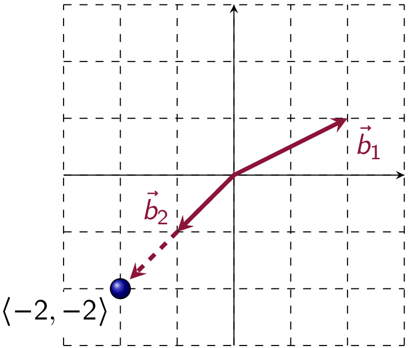
$\mathcal{B}$ to Elementary



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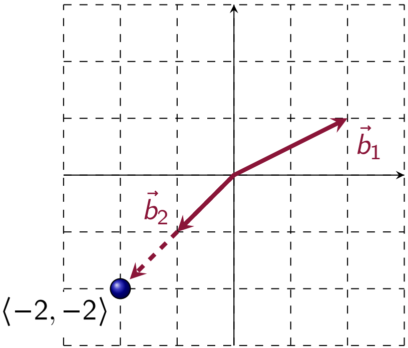

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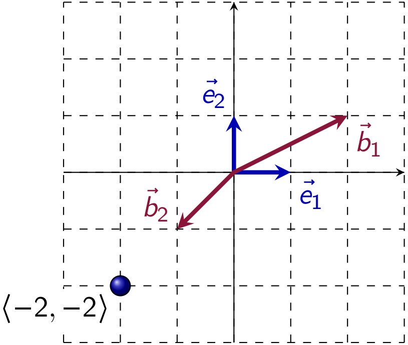

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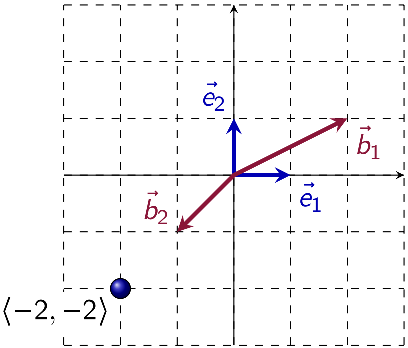

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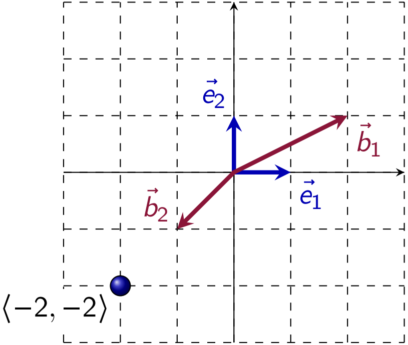

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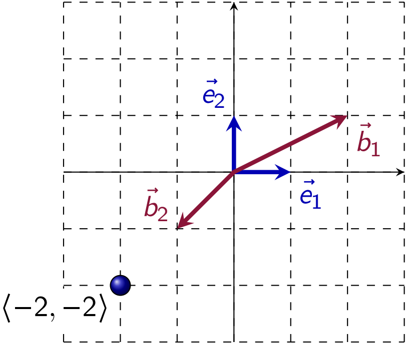

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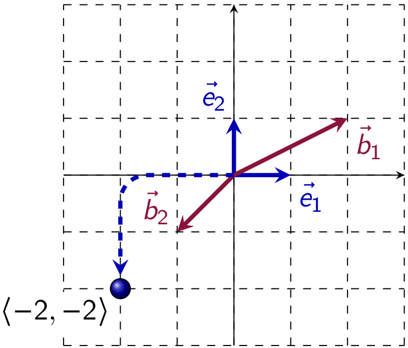

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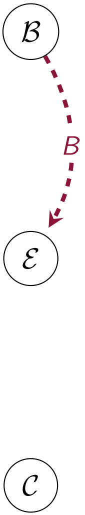
$\mathcal{B}$ to $\mathcal{C}$ to $\mathcal{B}$ (there and back again)

- $\mathcal{E}, \mathcal{B}, \mathcal{C}$ bases

$\mathcal{B}$ to $\mathcal{C}$ to $\mathcal{B}$ (there and back again)



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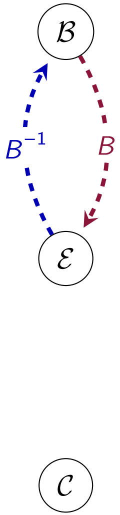

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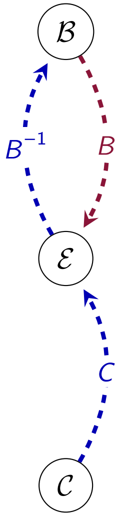

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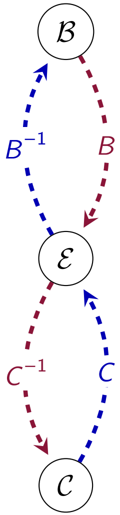

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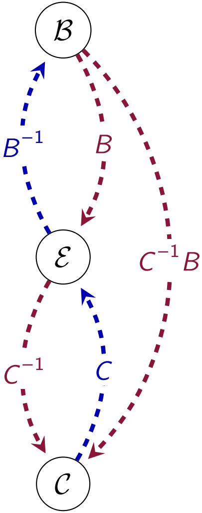

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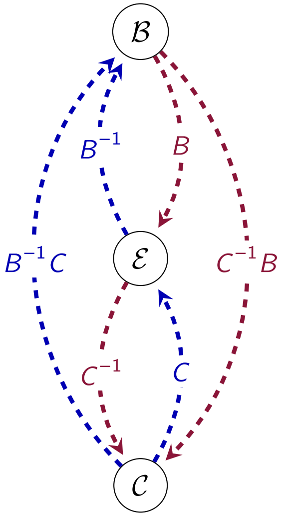

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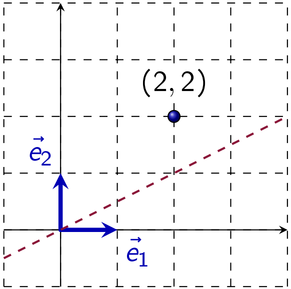
- $B^{-1}C : \mathcal{C} \longrightarrow \mathcal{B}$

Making Calculations Easier: Reflection Problem

Reflect the point $P = (2, 3)$ in the line $y = \frac{1}{2}x$.



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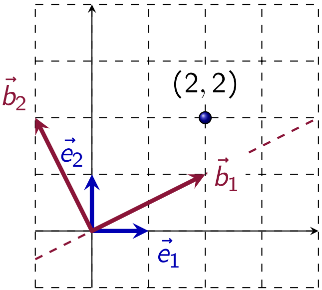

- Create an **orthogonal basis** with a vector parallel to the line: $\mathcal{B} = \{\langle 2, 1\rangle, \langle -1, 2\rangle\}$

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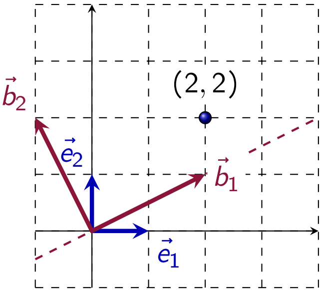

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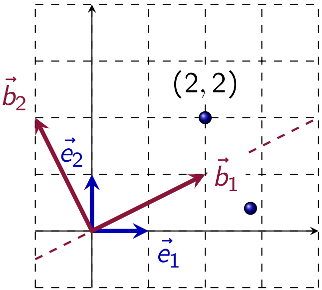

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- Reflect in the second coordinate with

$$Rf_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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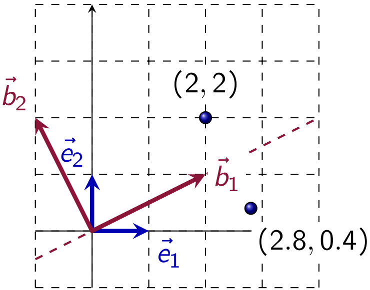

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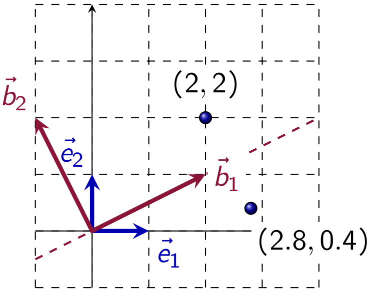

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- Reflect in the second coordinate with

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- Change from $\mathcal{B}$ back to $\mathcal{E}$ with B
- The complete calculation:

$$BRf_2B^{-1}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.8 \\ 0.4 \end{pmatrix}$$

Conjugation

Definition (Conjugation)

Given a transformation P with inverse P^{-1} and another transformation A, we say that

$$B = P \circ A \circ P^{-1}$$

is a **conjugate** of A. If A, B, and P are matrices then we say A and B are **similar matrices**.

Lectures on Multivariable Mathematics: Bases for Vector Spaces

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