Lectures on Multivariable Mathematics: Review of Vector and Matrix Operations (Part 2)

Dr. Chuck Rocca
roccac@wcsu.edu

Table of Contents

1. Objectives
2. Equations and Inverse Matrices
3. Linear Transformations
4. Determinants

Table of Contents

1. Objectives
2. Equations and Inverse Matrices
3. Linear Transformations
4. Determinants

Objectives

After this lesson you should recall how to:

1. Solve systems of equations and invert matrices,

Objectives

After this lesson you should recall how to:

1. Solve systems of equations and invert matrices,
2. Understand matrix multiplication from a graphical standpoint,

Objectives

After this lesson you should recall how to:

1. Solve systems of equations and invert matrices,
2. Understand matrix multiplication from a graphical standpoint,
3. Compute determinants, and

Objectives

After this lesson you should recall how to:

1. Solve systems of equations and invert matrices,
2. Understand matrix multiplication from a graphical standpoint,
3. Compute determinants, and
4. Understand the algebraic and graphical significance of determinants.

Table of Contents

1 Objectives

2 Equations and Inverse Matrices

3 Linear Transformations

4 Determinants

Systems of Equations and Matrices

$$
\begin{aligned}
2x + 3y - 7z + 10w &= 5 \\
9y + z - 4w &= -2 \\
2x - 7z &= 0
\end{aligned}
$$

Systems of Equations and Matrices

$$\begin{aligned} 2x + 3y - 7z + 10w &= 5 \\ 9y + z - 4w &= -2 \\ 2x - 7z &= 0 \end{aligned}$$

$$\begin{pmatrix} 2 & 3 & -7 & 10 \\ 0 & 9 & 1 & -4 \\ 2 & 0 & -7 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}$$

Types of Systems of Equations: $A\vec{x} = \vec{b}$

Given an $n \times m$ matrix A and $n \times 1$ vector $\vec{b}$

- If $A\vec{x} = \vec{b}$ has a solution, then it is consistent.

Types of Systems of Equations: $A\vec{x} = \vec{b}$

Given an $n \times m$ matrix A and $n \times 1$ vector $\vec{b}$

- If $A\vec{x} = \vec{b}$ has a solution, then it is consistent.
- If $A\vec{x} = \vec{b}$ has no solution, then it is inconsistent.

Types of Systems of Equations: $A\vec{x} = \vec{b}$

Given an $n \times m$ matrix A and $n \times 1$ vector $\vec{b}$

- If $A\vec{x} = \vec{b}$ has a solution, then it is consistent.
- If $A\vec{x} = \vec{b}$ has no solution, then it is inconsistent.
- If $\vec{b} = \vec{0}$, then $A\vec{x} = \vec{0}$ is a homogeneous equation.

Types of Systems of Equations: $A\vec{x} = \vec{b}$

Given an $n \times m$ matrix A and $n \times 1$ vector $\vec{b}$

- If $A\vec{x} = \vec{b}$ has a solution, then it is consistent.
- If $A\vec{x} = \vec{b}$ has no solution, then it is inconsistent.
- If $\vec{b} = \vec{0}$, then $A\vec{x} = \vec{0}$ is a homogeneous equation.
- If $\vec{b} \neq \vec{0}$, then $A\vec{x} = \vec{b}$ is a non-homogeneous equation.

Types of Systems of Equations: $A\vec{x} = \vec{b}$

Given an $n \times m$ matrix A and $n \times 1$ vector $\vec{b}$

- If $A\vec{x} = \vec{b}$ has a solution, then it is consistent.
- If $A\vec{x} = \vec{b}$ has no solution, then it is inconsistent.
- If $\vec{b} = \vec{0}$, then $A\vec{x} = \vec{0}$ is a homogeneous equation.
- If $\vec{b} \neq \vec{0}$, then $A\vec{x} = \vec{b}$ is a non-homogeneous equation.
- If $n < m$, then the equation is under determined.

Types of Systems of Equations: $A\vec{x} = \vec{b}$

Given an $n \times m$ matrix A and $n \times 1$ vector $\vec{b}$

- If $A\vec{x} = \vec{b}$ has a solution, then it is consistent.
- If $A\vec{x} = \vec{b}$ has no solution, then it is inconsistent.
- If $\vec{b} = \vec{0}$, then $A\vec{x} = \vec{0}$ is a homogeneous equation.
- If $\vec{b} \neq \vec{0}$, then $A\vec{x} = \vec{b}$ is a non-homogeneous equation.
- If $n < m$, then the equation is under determined.
- If $n > m$, then the equation is over determined.

Types of Systems of Equations: $A\vec{x} = \vec{b}$

Given an $n \times m$ matrix A and $n \times 1$ vector $\vec{b}$

- If $A\vec{x} = \vec{b}$ has a solution, then it is consistent.
- If $A\vec{x} = \vec{b}$ has no solution, then it is inconsistent.
- If $\vec{b} = \vec{0}$, then $A\vec{x} = \vec{0}$ is a homogeneous equation.
- If $\vec{b} \neq \vec{0}$, then $A\vec{x} = \vec{b}$ is a non-homogeneous equation.
- If $n < m$, then the equation is under determined.
- If $n > m$, then the equation is over determined.

Inverse Matrix: As Equations

Given an $n \times n$ matrix A, if the inverse, A^{-1}, exists, with

$$AA^{-1} = A^{-1}A = I_n,$$

we can view finding it as solving n equations of the form

$$A\vec{x} = e_i$$

with $1 \leq i \leq n$. Where e_i is 1 in position i and 0 otherwise. To see this, let's find the inverse of

$$A = \begin{pmatrix} 2 & 3 & -1 \\ -3 & -6 & 4 \\ 1 & 2 & -1 \end{pmatrix}.$$

Inverse Matrix: Using Row Reduction

Start with an augmented matrix and row reduce it:

$$\left(\begin{array}{rrr|rrr} 2 & 3 & -1 & 1 & 0 & 0 \\ -3 & -6 & 4 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right)$$

Inverse Matrix: Using Row Reduction

Start with an augmented matrix and row reduce it:

$$\left(\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ -3 & -6 & 4 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right)$$

Swap Rows

Inverse Matrix: Using Row Reduction

$$\left(\begin{array}{rrr|rrr} 1 & 2 & -1 & 0 & 0 & 1 \\ 2 & 3 & -1 & 1 & 0 & 0 \\ -3 & -6 & 4 & 0 & 1 & 0 \end{array}\right)$$

Inverse Matrix: Using Row Reduction

$$\left(\begin{array}{rrr|rrr} 1 & 2 & -1 & 0 & 0 & 1 \\ 2 & 3 & -1 & 1 & 0 & 0 \\ -3 & -6 & 4 & 0 & 1 & 0 \end{array}\right)$$

(Row 2 - 2 Row 1) and (Row 3 + 3 Row 1)

Inverse Matrix: Using Row Reduction

$$\left(\begin{array}{rrr|rrr} 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{array}\right)$$

Inverse Matrix: Using Row Reduction

$$\left(\begin{array}{rrr|rrr} 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{array}\right)$$

(Row 1 + 2 Row 2) then (- Row 2)

Inverse Matrix: Using Row Reduction

$$\left(\begin{array}{rrr|rrr} 1 & 0 & 1 & 2 & 0 & -3 \\ 0 & 1 & -1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{array}\right)$$

Inverse Matrix: Using Row Reduction

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 0 & -3 \\ 0 & 1 & -1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{array}\right)$$

(Row 1 - Row 3) and (Row 2 + Row 3)

Inverse Matrix: Using Row Reduction

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -6 \\ 0 & 1 & 0 & -1 & 1 & 5 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{array}\right)$$

Inverse Matrix: Using Row Reduction

$$A^{-1} = \begin{pmatrix} 2 & -1 & -6 \\ -1 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix}$$

Elementary Matrices

Identity

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Elementary Matrices

Identity

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Swap

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Elementary Matrices

Identity

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Swap

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Row 2 - 2 Row 1

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Elementary Matrices

Identity

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Swap

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Row 2 - 2 Row 1

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Row 2

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse Matrix: For a 2×2 Matrix

For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the inverse is of the form

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Solving Equations

Notes:

- We can solve equations by row reducing an augmented matrix like we did to find the inverse,

Solving Equations

Notes:

- We can solve equations by row reducing an augmented matrix like we did to find the inverse,
- If we have an equation $A\vec{x} = \vec{b}$ and A has an inverse, then $\vec{x} = A^{-1}\vec{b}$.

Solving Equations

Notes:

- We can solve equations by row reducing an augmented matrix like we did to find the inverse,
- If we have an equation $A\vec{x} = \vec{b}$ and A has an inverse, then $\vec{x} = A^{-1}\vec{b}$.
- if we have several equations to solve, or need general solutions, it is better to find A^{-1} an use it. Otherwise it may be better to just solve individual equations.

Table of Contents

1 Objectives

2 Equations and Inverse Matrices

3 Linear Transformations

4 Determinants

Linear Transformations: Definition

Definition (Linear Transformation)

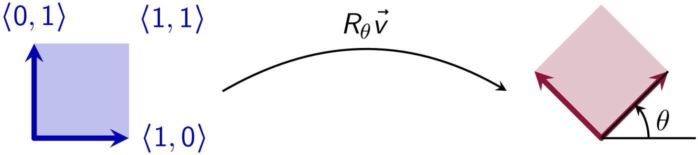
A function from one vector space to another, $\phi : V \to W$, is a linear transformation provided

$$\phi(c\vec{v} + d\vec{w}) = c\phi(\vec{v}) + d\phi(\vec{w})$$

for any scalars c and d and vectors $\vec{v}$ and $\vec{w}$ in V.

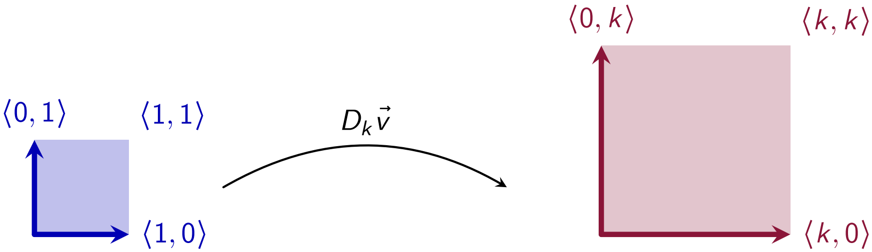
Linear Transformations: Rotation

$$R_\theta \vec{v} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \cos(\theta)v_x - \sin(\theta)v_y \\ \sin(\theta)v_x + \cos(\theta)v_y \end{pmatrix}$$



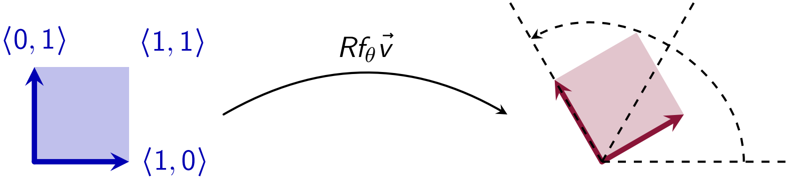
Linear Transformations: Dilation

$$D_k\vec{v} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} kv_x \\ kv_y \end{pmatrix}$$



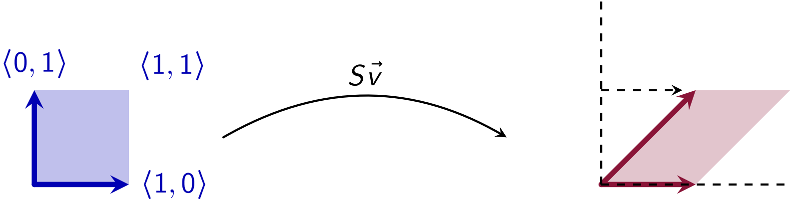
Linear Transformations: Reflection

$$Rf_\theta \vec{v} = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \cos(2\theta)v_x + \sin(2\theta)v_y \\ \sin(2\theta)v_x - \cos(2\theta)v_y \end{pmatrix}$$



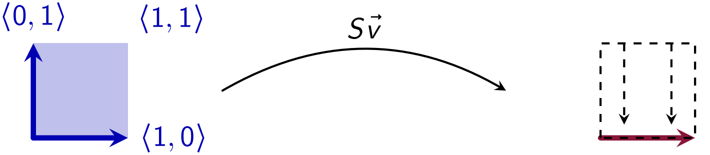
Linear Transformations: Shear

$$S\vec{v} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x + kv_y \\ v_y \end{pmatrix} \qquad S\vec{v} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x \\ kv_k + v_y \end{pmatrix}$$



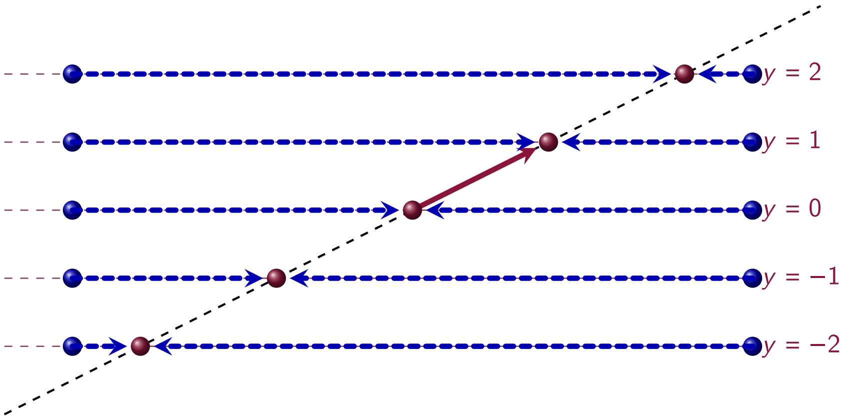
Linear Transformations: Projection

$$P\vec{v} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x \\ 0 \end{pmatrix} \qquad P\vec{v} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ v_y \end{pmatrix}$$



A Combination

$$S P\vec{v} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 2v_y \\ v_y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} v_y$$



Linear Transformation: Kernels

Given a linear transformation defined by an $n \times m$ matrix A, i.e. $T(\vec{x}) = A\vec{x}$, the set of solutions to the homogeneous equation $A\vec{x} = \vec{0}$ is called the kernel:

$$ker(T) = ker(A) = \left\{\vec{x} \in \mathbb{R}^m \middle| A\vec{x} = \vec{0}\right\}.$$

Note that $\vec{x} = \vec{0}$ is always a solution, so the kernel is non-empty. In linear algebra the kernel is also called the Null Space of the matrix A or $Nul(A)$.

Linear Transformations: Inverse

...

Linear Transformations: Matrix of a Transformation

...

Table of Contents

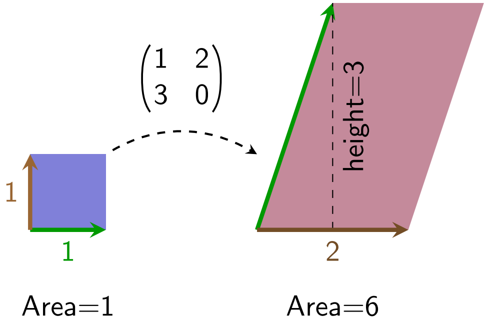
1 Objectives

2 Equations and Inverse Matrices

3 Linear Transformations

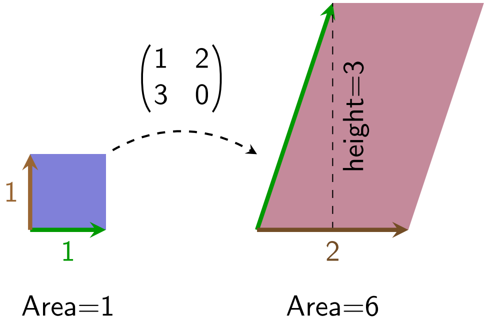
4 Determinants

Determinants from a Geometric Standpoint



How can we look at a matrix and capture the idea that we have a change in area (or volume) as well as a reflection?

Determinants from a Geometric Standpoint



How can we look at a matrix and capture the idea that we have a change in area (or volume) as well as a reflection? Determinants!

Calculating a Determinant: 2×2 Matrix

Given a 2×2 matrix A we can find the determinant like so

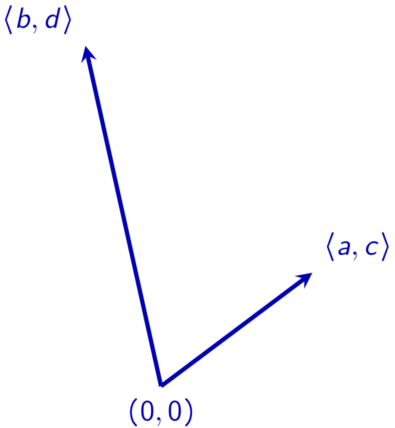
$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For the previous example we get

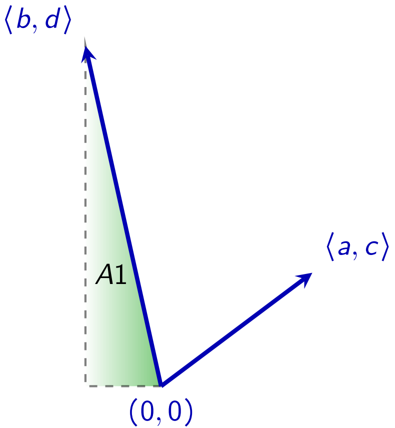
$$det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} = 1 \cdot 0 - 2 \cdot 3 = -6.$$

The negative tells us it changes the orientation of the object and the 6 tells us how it scaled the object.

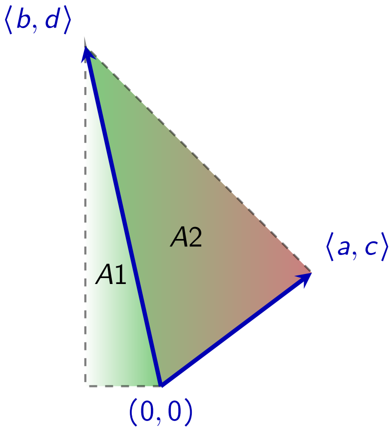
Calculating a Determinant: 2×2 Justification



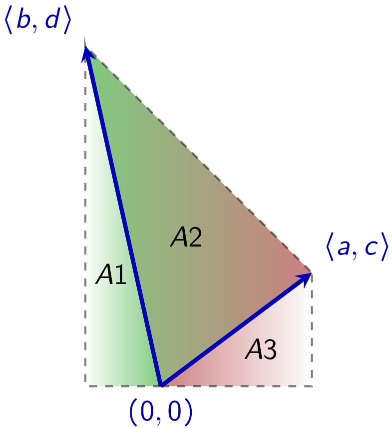
Calculating a Determinant: 2×2 Justification



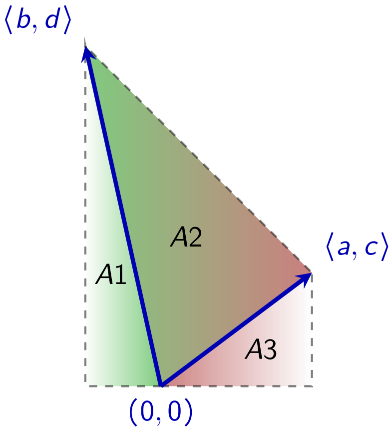
Calculating a Determinant: 2×2 Justification



Calculating a Determinant: 2×2 Justification

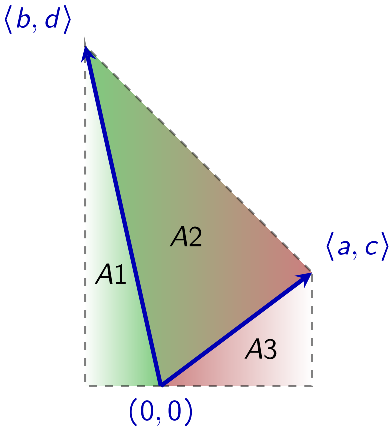


Calculating a Determinant: 2×2 Justification



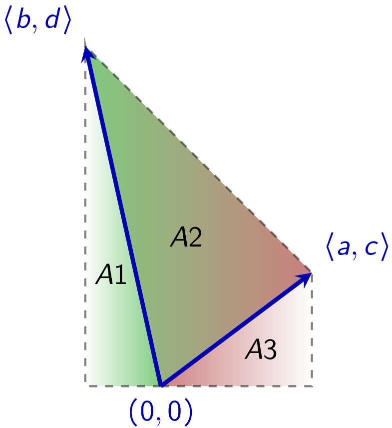
Total Area = $A1 + A2 + A3$

Calculating a Determinant: 2×2 Justification



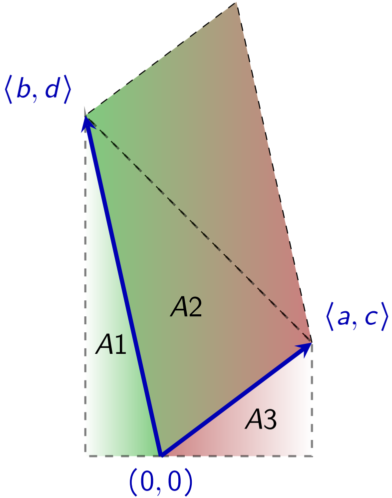
$$\text{Total Area} = A1 + A2 + A3$$
$$\frac{(d+c)(a-b)}{2} = \frac{ac}{2} + A2 - \frac{bd}{2}$$

Calculating a Determinant: 2×2 Justification



$$\begin{aligned} \text{Total Area} &= A1 + A2 + A3 \\ \frac{(d+c)(a-b)}{2} &= \frac{ac}{2} + A2 - \frac{bd}{2} \\ 2A2 &= ad - bc \end{aligned}$$

Calculating a Determinant: 2×2 Justification



$$\begin{aligned} \text{Total Area} &= A1 + A2 + A3 \\ \frac{(d+c)(a-b)}{2} &= \frac{ac}{2} + A2 - \frac{bd}{2} \\ 2A2 &= ad - bc \end{aligned}$$

Calculating a Determinant: 3×3 Matrix (take one)

Given a 3×3 matrix A we can find the determinant by expanding along a row like so

$$det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

For example:

$$det(A) = \begin{vmatrix} 1 & 0 & 7 \\ 1 & -3 & 2 \\ 0 & 5 & 9 \end{vmatrix} = 1(-27-10)-0(9-0)+7(5-0) = -37+35 = -2.$$

So, this A scales by a factor of 2 as well as changing orientation.

Calculating a Determinant: 3×3 Matrix (take two)

Given a 3×3 matrix A we can find the determinant by expanding down a column like so

$$det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - d(bi - ch) + g(bf - ce).$$

For example:

$$det(A) = \begin{vmatrix} 1 & 0 & 7 \\ 1 & -3 & 2 \\ 0 & 5 & 9 \end{vmatrix} = 1(-27-10)-1(0-35)+0(0-21) = -37+35 = -2.$$

So, this A scales by a factor of 2 as well as changing orientation.

Calculating a Determinant: $n \times n$ Matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix}$$

$$\begin{aligned} det(A) = a_{1,1}(-1)^{1+1}det(A_{1,1}) &+ a_{1,2}(-1)^{1+2}det(A_{1,2}) \\ &+ a_{1,3}(-1)^{1+3}det(A_{1,3}) \\ &+ \cdots \\ &+ a_{1,n}(-1)^{1+n}det(A_{1,n}) \end{aligned}$$

Calculating a Determinant: $n \times n$ Matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix}$$

$$\begin{aligned} det(A) = a_{1,1}(-1)^{1+1}det(A_{1,1}) &+ a_{1,2}(-1)^{1+2}det(A_{1,2}) \\ &+ a_{1,3}(-1)^{1+3}det(A_{1,3}) \\ &+ \cdots \\ &+ a_{1,n}(-1)^{1+n}det(A_{1,n}) \end{aligned}$$

Calculating a Determinant: $n \times n$ Matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix}$$

$$\begin{aligned} det(A) = a_{1,1}(-1)^{1+1}det(A_{1,1}) &+ a_{1,2}(-1)^{1+2}det(A_{1,2}) \\ &+ a_{1,3}(-1)^{1+3}det(A_{1,3}) \\ &+ \cdots \\ &+ a_{1,n}(-1)^{1+n}det(A_{1,n}) \end{aligned}$$

Calculating a Determinant: $n \times n$ Matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix}$$

$$\begin{aligned} det(A) = a_{1,1}(-1)^{1+1}det(A_{1,1}) &+ a_{1,2}(-1)^{1+2}det(A_{1,2}) \\ &+ a_{1,3}(-1)^{1+3}det(A_{1,3}) \\ &+ \cdots \\ &+ a_{1,n}(-1)^{1+n}det(A_{1,n}) \end{aligned}$$

Calculating a Determinant: $n \times n$ Matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{pmatrix}$$

$$\begin{aligned} det(A) = a_{1,1}(-1)^{1+1}det(A_{1,1}) &+ a_{1,2}(-1)^{1+2}det(A_{1,2}) \\ &+ a_{1,3}(-1)^{1+3}det(A_{1,3}) \\ &+ \cdots \\ &+ a_{1,n}(-1)^{1+n}det(A_{1,n}) \end{aligned}$$

Properties of Determinants

Theorem (Properties of Determinants)

Given an $n \times n$ matrices A and B:

- $det(A^T) = det(A)$

Properties of Determinants

Theorem (Properties of Determinants)

Given an $n \times n$ matrices A and B:

- $det(A^T) = det(A)$
- $det(AB) = det(A)det(B)$

Properties of Determinants

Theorem (Properties of Determinants)

Given an $n \times n$ matrices A and B:

- $det(A^T) = det(A)$
- $det(AB) = det(A)det(B)$
- A^{-1} *exists if and only if* $det(A) \neq 0$

Properties of Determinants

Theorem (Properties of Determinants)

Given an $n \times n$ matrices A and B:

- $det(A^T) = det(A)$
- $det(AB) = det(A)det(B)$
- A^{-1} *exists if and only if* $det(A) \neq 0$
- *if* A^{-1} *exists, then* $det(A^{-1}) = 1/det(A)$

Properties of Determinants

Theorem (Properties of Determinants)

Given an $n \times n$ matrices A and B:

- $det(A^T) = det(A)$
- $det(AB) = det(A)det(B)$
- A^{-1} *exists if and only if* $det(A) \neq 0$
- *if* A^{-1} *exists, then* $det(A^{-1}) = 1/det(A)$
- *interchanging two rows of A switches the sign of the determinant*

Properties of Determinants

Theorem (Properties of Determinants)

Given an $n \times n$ matrices A and B:

- $det(A^T) = det(A)$
- $det(AB) = det(A)det(B)$
- A^{-1} *exists if and only if* $det(A) \neq 0$
- *if* A^{-1} *exists, then* $det(A^{-1}) = 1/det(A)$
- *interchanging two rows of A switches the sign of the determinant*
- *multiplying a row of A by k multiplies the determinant by k*

Properties of Determinants

Theorem (Properties of Determinants)

Given an $n \times n$ matrices A and B:

- $det(A^T) = det(A)$
- $det(AB) = det(A)det(B)$
- A^{-1} *exists if and only if* $det(A) \neq 0$
- *if* A^{-1} *exists, then* $det(A^{-1}) = 1/det(A)$
- *interchanging two rows of A switches the sign of the determinant*
- *multiplying a row of A by k multiplies the determinant by k*
- *adding a multiple of one row of A to a different row doesn't change the determinant*

Determinants and Elementary Matrices

Identity

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1) + 0 + 0 = 1$$

Determinants and Elementary Matrices

Identity

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1) + 0 + 0 = 1$$

A Single Swap

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 + 0 + 1(-1) = -1$$

Determinants and Elementary Matrices

Identity

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1) + 0 + 0 = 1$$

Row 2 - 2 Row 1

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1) + 0 + 0 = 1$$

A Single Swap

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 + 0 + 1(-1) = -1$$

Determinants and Elementary Matrices

Identity

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1) + 0 + 0 = 1$$

Row 2 - 2 Row 1

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(1) + 0 + 0 = 1$$

A Single Swap

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 + 0 + 1(-1) = -1$$

k Row 2

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(k) + 0 + 0 = k$$

Determinants and Diagonal Matrices

Theorem

If A is a upper or lower diagonal matrix, then $\det(A)$ is the product of the entries on the diagonal.

Determinants and Diagonal Matrices

Theorem

If A is a upper or lower diagonal matrix, then $\det(A)$ is the product of the entries on the diagonal.

Expanding down the first column we can see the theorem is true in this case.

$$\begin{vmatrix} 2 & 3 & -1 \\ 0 & -5 & 7 \\ 0 & 0 & 10 \end{vmatrix} = 2(-5 \cdot 10 - 7 \cdot 0) + 0 + 0 = 2 \cdot -5 \times 10 = -100$$

Lectures on Multivariable Mathematics: Review of Vector and Matrix Operations (Part 2)

Dr. Chuck Rocca
roccac@wcsu.edu