Lectures on Multivariable Mathematics: Review of Vector and Matrix Operations (Part 1)

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Objectives

After this lesson you should be able to:

1. Understand vectors graphically and algebraically,

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2. Compute and understand the significance of dot products, and

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3. Multiply matrices with vectors and other matrices.

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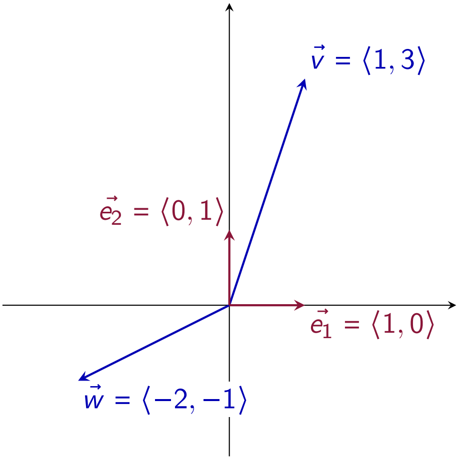
Vectors and Vector Spaces: Basic Examples



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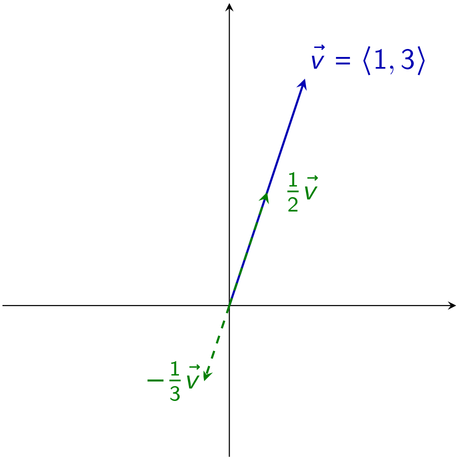

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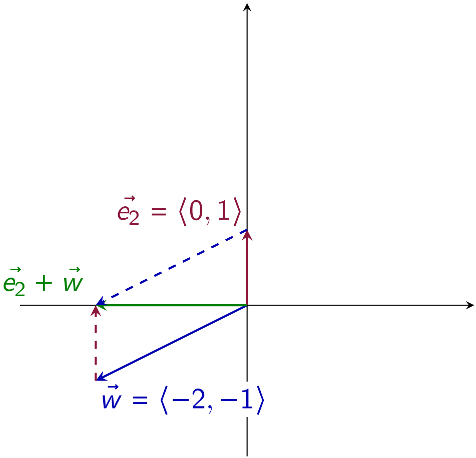

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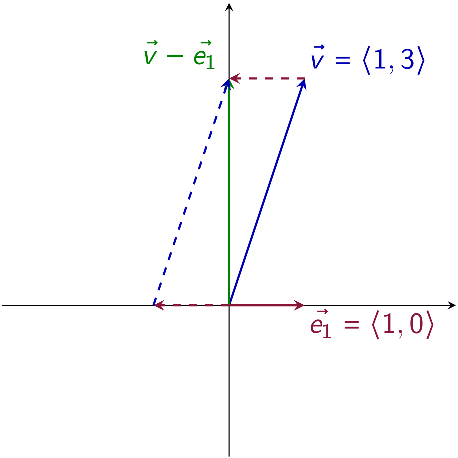

Vectors and Vector Spaces: Definition

Definition (Vector Space)

A *vector space* consist of two sets, a set of vectors V and scalars S, which satisfy the following conditions: for all $\vec{v}$, $\vec{u}$, $\vec{w} \in V$ and c, $d \in S$

1. $\vec{v} + \vec{u} \in V$ (closure of addition)
2. $\vec{v} + \vec{u} = \vec{u} + \vec{v}$ (commutative)
3. $(\vec{v} + \vec{u}) + \vec{w} = \vec{v} + (\vec{u} + \vec{w}$ (associative)
4. $\exists \vec{0} \in V : \vec{v} + \vec{0} = \vec{v}$ (additive identity)
5. $\forall \vec{v} \in V \ \exists (-\vec{v}) \in V : \vec{v} + (-\vec{v}) = \vec{0}$ (additive inverse)

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6. $c\vec{v} \in V$ (closure of scalar multiplication)
7. $c(\vec{v} + \vec{u}) = c\vec{v} + c\vec{u}$ (distributive)
8. $(c + d)\vec{v} = c\vec{v} + d\vec{v}$ (distributive)
9. $(cd)\vec{v} = c(d\vec{v})$ (associative)
10. $\exists 1 \in S : 1\vec{v} = \vec{v}$ (multiplicative identity)

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(V is technically called an *Abelian Group* and S is a *Field*.)

Addition, Subtraction, and Multiples of Vectors

$$V = \mathbb{R}^3 \text{ and } S = \mathbb{R}$$

$$\vec{v} = \langle 1, 5, 0 \rangle, \ \vec{u} = \langle -3, 0, 4 \rangle, \ c = 5, \ d = -2$$

- $\vec{v} + \vec{u} = \langle 1, 5, 0 \rangle + \langle -3, 0, 4 \rangle = \langle 1 + -3, 5 + 0, 0 + 4 \rangle = \langle -2, 5, 4 \rangle$

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- $c\vec{v} = 5\langle 1, 5, 0 \rangle = \langle 5 \cdot 1, 5 \cdot 5, 5 \cdot 0 \rangle = \langle 5, 25, 0 \rangle$

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Addition, Subtraction, and Multiples of Vectors

$$V = \mathbb{Q}[x] \text{ and } S = \mathbb{Q}$$

$$\vec{v} = x^4 + 5x^2, \;\; \vec{u} = -3x^4 + 4, \;\; c = 5, \;\; d = -2$$

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- $\vec{0} = 0$

Dot Product of Vectors

- $V = \mathbb{R}^3$:

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$$\langle 7, 8 \rangle \cdot \langle 3, 4 \rangle = 7 \cdot 3 + 8 \cdot 4 = 55$$

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- $V = \mathbb{R}^4$:

$$\langle 1, -2, 0, 5 \rangle \cdot \langle -3, 6, 4, 3 \rangle = -3 + -12 + 0 + 15 = 0$$

Properties of Dot Products

Theorem (Dot Product Properties)

Given a vector space V, vectors $\vec{v}$, $\vec{u}$, $\vec{w} \in V$ and a scalar c:

1. $\vec{v} \cdot \vec{u} = \vec{u} \cdot \vec{v}$
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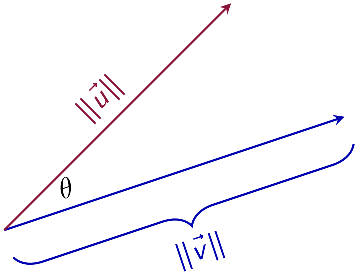
Definition (Norm or Length)

The *norm* or *length* of a vector is a nonnegative scalar, $||\vec{v}||$, defined by

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Graphical Significance of Dot Products



- $\vec{v} \cdot \vec{u} = ||\vec{v}|| \, ||\vec{u}|| \cos(\theta)$

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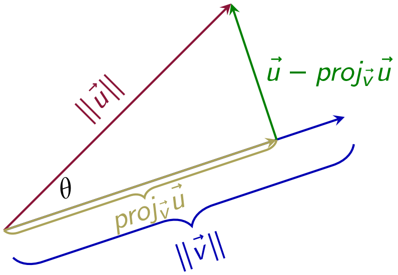

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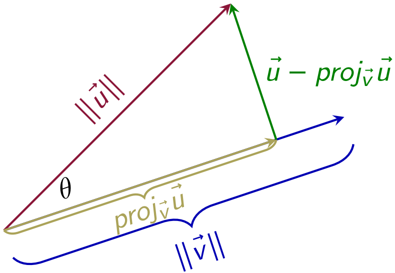

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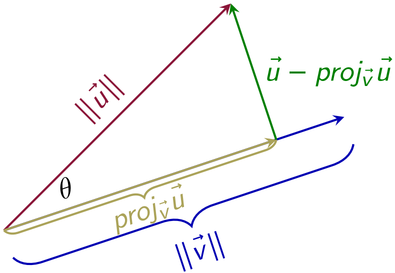

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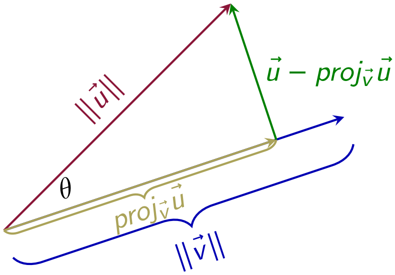

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- $(proj_{\vec{v}}\vec{u}) \cdot (\vec{u} - proj_{\vec{v}}\vec{u}) = 0$

Big-ish Foreshadowing Example and Some Practice

$$\vec{u} = \langle 1, 1, 0 \rangle, \vec{v} = \langle 1, 0, 1 \rangle, \vec{w} = \langle -1, 0, 1 \rangle$$

- $\vec{v} \cdot \vec{w} =$
- $p_v = proj_{\vec{v}} \vec{u} =$
- $p_w = proj_{\vec{w}} \vec{u} =$
- $z = p_v + p_w =$
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Big-ish Foreshadowing Example and Some Practice

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- $p_v + p_w + \hat{z} = z + \hat{z} = \vec{u}$

Table of Contents

Matrices

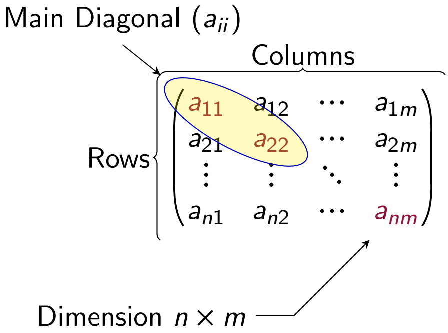
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

Matrices

$$\text{Rows}\left\{\overbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}}^{\text{Columns}}\right.$$

Dimension $n \times m$

Matrices



Main Diagonal (a_{ii})
Columns
Rows
a_{11} a_{12} $\cdots$ a_{1m}
a_{21} a_{22} $\cdots$ a_{2m}
a_{n1} a_{n2} $\cdots$ a_{nm}
Dimension $n \times m$

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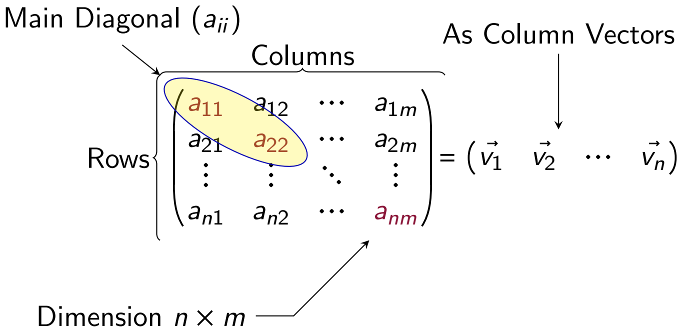
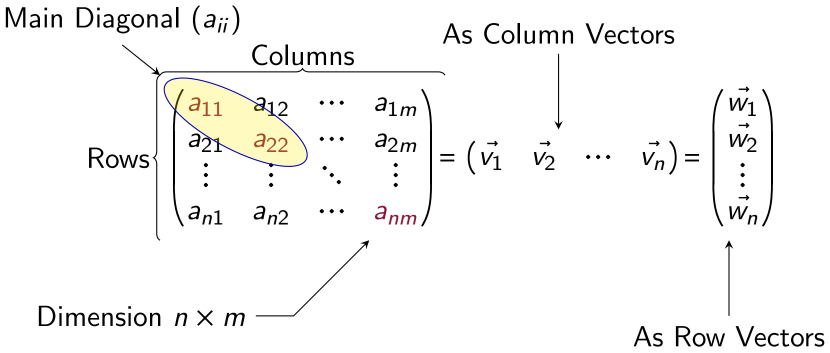

Main Diagonal (a_{ii})
Columns
As Column Vectors
Rows
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{pmatrix}$$
Dimension $n \times m$

Matrices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Dimension $n \times m$
As Row Vectors

Matrix - Matrix Sums

$$\begin{pmatrix} 2 & 0 & 1 \\ 3 & 9 & 5 \\ 0 & 0 & 7 \\ 4 & -3 & 6 \end{pmatrix} + \begin{pmatrix} 5 & 4 & 2 \\ 8 & 7 & -5 \\ 1 & 0 & 0 \\ 0 & 12 & 0 \end{pmatrix}$$

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Note that, in order to add matrices, they must have the same dimension.

Vector - Vector Products

$$\vec{v} = \langle 1, 0, 2 \rangle \;\; \& \;\; \vec{u} = \langle 3, -1, 5 \rangle$$

Inner Product

Outer Product

Vector - Vector Products

$$\vec{v} = \langle 1, 0, 2 \rangle \ \& \ \vec{u} = \langle 3, -1, 5 \rangle$$

Inner Product

$$\vec{v}\vec{u}^T = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$

Outer Product

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$$\vec{v} = \langle 1, 0, 2 \rangle \ \& \ \vec{u} = \langle 3, -1, 5 \rangle$$

Inner Product

$$\begin{aligned} \vec{v}\vec{u}^T &= \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \\ &= (1 \cdot 3 + 0 \cdot -1 + 2 \cdot 5) \\ &= (13) \end{aligned}$$

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$$\vec{v}\vec{u}^T = \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$
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$$= \begin{pmatrix} 1 \cdot 3 + 0 \cdot -1 + 2 \cdot 5 \end{pmatrix}$$
$$= \begin{pmatrix} 13 \end{pmatrix}$$

Outer Product

$$\vec{v}^T\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & -1 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \cdot 3 & 1 \cdot -1 & 1 \cdot 5 \\ 0 \cdot 3 & 0 \cdot -1 & 0 \cdot 5 \\ 2 \cdot 3 & 2 \cdot -1 & 2 \cdot 5 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & -1 & 5 \\ 0 & 0 & 0 \\ 6 & -2 & 10 \end{pmatrix}$$

Matrix - Vector Products

How we carry out the computation (a series of inner products):

$$\begin{pmatrix} 1 & 3 & 8 \\ 0 & 2 & 0 \\ 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} =$$

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How we should think of it:

$$A\vec{v}^{\,T} = \begin{pmatrix}\vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{pmatrix}\vec{v}^{\,T} = \begin{pmatrix}\vec{a}_1\vec{v}^{\,T}\\ \vec{a}_2\vec{v}^{\,T}\\ \vec{a}_3\vec{v}^{\,T}\end{pmatrix}$$

Where each $\vec{a}_i$ is a row vector.

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$$\vec{v}A = \vec{v}\begin{pmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{pmatrix} = \begin{pmatrix}\vec{v}\cdot\vec{a}_1 & \vec{v}\cdot\vec{a}_2 & \vec{v}\cdot\vec{a}_3\end{pmatrix}$$

Where each $\vec{a}_i$ is a column vector.

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How we want to think about it:

$$AB = A \begin{pmatrix} \vec{b}_1 & \vec{b}_2 \end{pmatrix} = \begin{pmatrix} A\vec{b}_1 & A\vec{b}_2 \end{pmatrix}$$

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$$AB = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \end{pmatrix} B = \begin{pmatrix} \vec{a}_1 B \\ \vec{a}_2 B \end{pmatrix}$$

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$$AB = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \end{pmatrix}\begin{pmatrix} \vec{b}_1 & \vec{b}_2 \end{pmatrix} = \begin{pmatrix} \vec{a}_1\cdot\vec{b}_1 & \vec{a}_1\cdot\vec{b}_2 \\ \vec{a}_2\cdot\vec{b}_1 & \vec{a}_2\cdot\vec{b}_2 \end{pmatrix}$$

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    1. A times each column in B,
    2. Each row in A times B,
    3. The rows of A "times" the columns of B, or
    4. All the inner products of rows in A and columns in B.

Particular Matrices

Zero Matrix
(Additive Identity)

$$0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Identity Matrix
(Multiplicative Identity)

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Diagonal Matrix
(Reduced Echelon Form)

$$D = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

Upper Diagonal
(Echelon Form)

$$U = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

Practice Products

$$\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 6 & 0 \\ 1 & 0 & 3 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 4 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 6 & 0 \\ 1 & 0 & 3 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 \\ 7 & 0 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 6 & 0 \\ 1 & 0 & 3 \end{pmatrix} =$$

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Practice Products

$$\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 6 & 0 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 18 & 0 \\ 4 & 0 & 12 \end{pmatrix}$$

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The Transpose

Definition (Transpose)

Given a matrix A, the transpose of A, written A^T, is a matrix in which the $(i,j)^{th}$ entry is the $(j,i)^{th}$ entry in A, i.e. we swap the rows and columns.

$$\vec{v} = \begin{pmatrix} 1 & 7 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 4 & 6 & 1 \\ 3 & 7 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 0 & 2 \\ 8 & 7 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$B^T = \begin{pmatrix} 3 & 8 & 0 \\ 0 & 7 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

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Theorem

Given matrices A and B with appropriate dimensions and a scalar r:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(rA)^T = rA^T$

Lectures on Multivariable Mathematics: Review of Vector and Matrix Operations (Part 1)

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