Basic Counting

Definition 1 (Multiplication Principle)**.** If we have k ***independent*** choices to make and n_i options for each choice, then the total number of possible choices is

$$\prod_{i=1}^{k} n_i = n_1 \times n_2 \times \cdots \times n_k.$$

Definition 2 (Addition Principle)**.** If we have k ***disjoint*** options and each option can be chosen in n_i different ways, then the total number of options is

$$\sum_{i=1}^{k} n_i = n_1 + n_2 + \cdots + n_k.$$

Exposition 1: A Trip to Sandwich-Way

As you enter Sandwich-Way you see the following directions for making a sandwich:

1. Choose one of 12 bread options.
2. Choose one of 16 "protien" options.
3. Choose up to 1 of 7 cheese options.
4. Choose up to 3 of 10 vegetable options.
5. Choose up to 3 of 13 condiments.

You then find yourself wondering how many different sandwiches you can make.
What are the number of sandwiches if we use the minimum number of ingredients?

$$\begin{aligned} Bread \times Protien \times Cheese \times Vegitable \times Condiment \\ &= 12 \times 16 \times 1 \times 1 \times 1 \\ &= 192\ Sandwiches \end{aligned}$$

What are the number of sandwiches if we use the maximum number of ingredients (without repetition)?[a]

$$\begin{aligned} Bread \times Protien \times Cheese \times Vegitable \times Condiment \\ &= 12 \times 16 \times 7 \times (10 \cdot 9 \cdot 8) \times (13 \cdot 12 \cdot 11) \\ &= 1,660,538,880\ Sandwiches \end{aligned}$$

How many sandwiches are there if we either minimize or maximize the number of ingredients?

$$192\ Sandwiches + 1,660,538,880\ Sandwiches = 1,660,539,072\ Sandwiches$$

[a] For simplicity we are assuming we care what order cheeses and condiments are added.

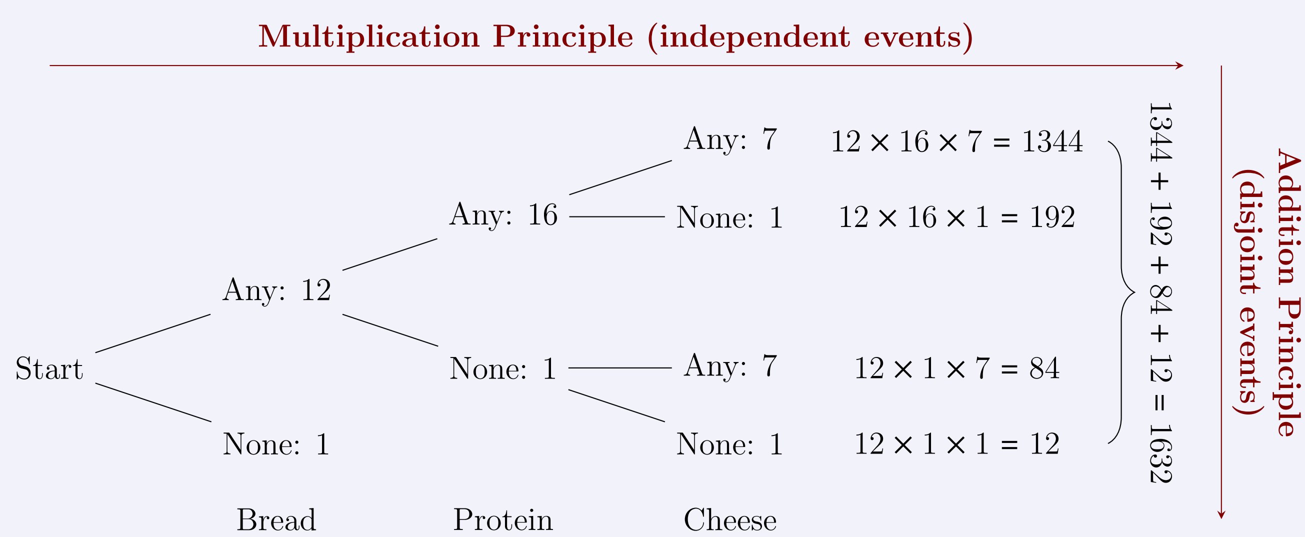
Practice 1: Returning for Another Sandwich

How many different sandwiches are there with no vegetables?

What if you really like dry sandwiches with no condiments?

Exposition 2: Decision Tree: Picking Bread, Protein, and Cheese

We can map out and compile a sequence of options using a ***decision tree***. For example here is how we can create a sandwich with one or no choice of protein and one or no choice of cheese.



Note that for small examples/simple situations we can actually draw a decision tree, for anything particularly large it is a useful organizational tool but not practical to draw.

Practice 2: License Plates

Suppose that all the license plates in a state must consist of a digit from 1 to 9, followed by four different letters, followed by a different digit from 1-9. How many different license plates could we create? What if you could repeat letters or numbers? What if you could also choose two of four colors?

Practice 3: Shopping for a Party

Preparing for a party you go to the store to get soda, chips, and cookies. For cookies and soda there are regular and sugar free varieties, and for chips there is regular or low sodium.

	Soda		Chips		Cookies	
	Regular	Sugar Free	Regular	Low Salt	Regular	Sugar Free
Varieties	10	7	15	12	14	8

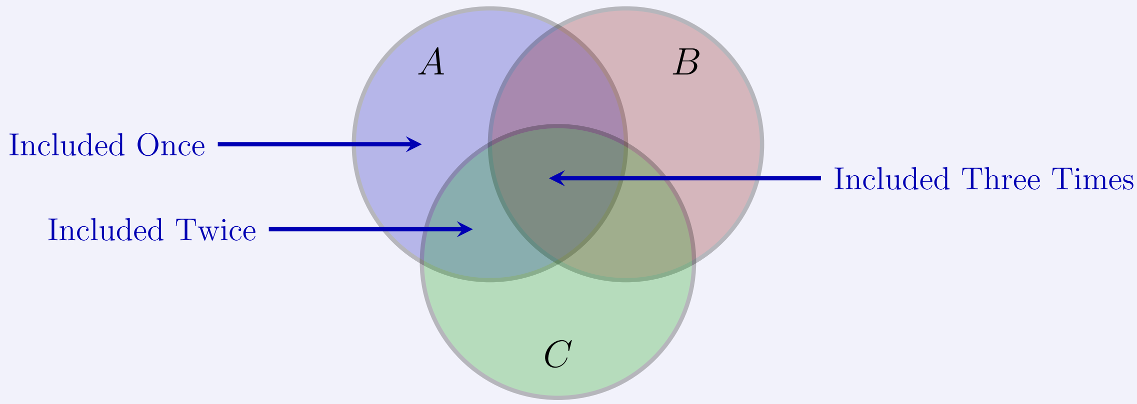
How many buying options do you have if you just get one type of each product? (Use a decision Tree.)

Definition 3 (Inclusion/Exclusion Principle)**.** For any set X let $|X|$ be the ***cardinality***, or size, of X. Given sets A, B, and C,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

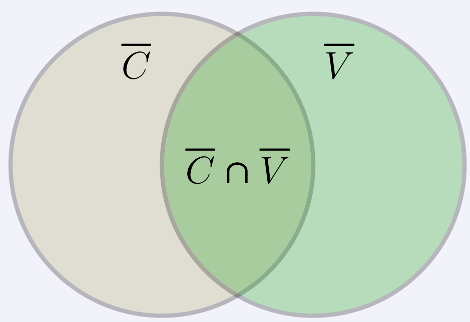
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$



Exposition 3: Inclusion/Exclusion Principle Goes to Lunch

Let $\overline{C}$ be the set of all sandwiches with no cheese and $\overline{V}$ be the set of all sandwiches with no vegetables. How many sandwiches hove no cheese or no vegetables?

- $\overline{C}$ is sandwiches with no cheese.
- $\overline{V}$ is sandwiches with no vegetables.
- $\overline{C} \cup \overline{V}$ is sandwiches with no cheese ***OR*** no vegetables.
- $\overline{C} \cap \overline{V}$ is sandwiches with no cheese ***AND*** no vegetables.



Using the multiplication and addition principles we can get:

$$\begin{aligned} |\overline{C}| &= 12,773,376 \\ |\overline{V}| &= 580,608 \\ |\overline{C} \cap \overline{V}| &= 72,576 \end{aligned}$$

Therefore, the number of sandwiches with no cheese or no vegetables is:

$$\begin{aligned} |\overline{C} \cup \overline{V}| &= |\overline{C}| + |\overline{V}| - |\overline{C} \cap \overline{V}| \\ &= 12773376 + 580608 - 75576 \\ &= 13,278,408 \end{aligned}$$

Practice 4: Marking Cars

Your license plates consisting of a digit from 1 to 9, followed by four different letters, followed by a different digit from 1-9, have been getting complaints because people confuse the B with the 8. To solve this problem you need to exclude one of both of these characters, in how many ways can you do that?

Practice 5: More Party Planning

Suppose you need to get sugar free soda or sugar free cookies, in how many ways could you do that?

	Soda		Chips		Cookies	
	Regular	Sugar Free	Regular	Low Salt	Regular	Sugar Free
Varieties	10	7	15	12	14	8

Permutations and Combinations

Definition 4 (Factorials)**.** The number of ways to arrange n objects is called ***n-factorial*** and is calculated like so:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

Definition 5 (Permutations)**.** ***Permutations*** represent the number of ways to select k object from n options when ***order does matter*** and are calculated like so:

$${}_nP_k = \frac{n!}{(n-k)!} = n \cdot (n-1) \cdot (n-2) \cdots (n-k+1).$$

Definition 6 (Combinations)**.** ***Combinations*** represent the number of ways to select k object from n options when ***order does not matter*** and are calculated like so:

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdots 2 \cdot 1}.$$

This is sometimes also called a ***binomial coefficients*** because we either choose an option or not.

Practice 6: Basic Calculations

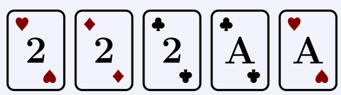
Translate each of the following into a mathematical expression and then complete the calculation.

1. Pick 5 out of 7 distinct objects when order matters:
2. Arrange 6 distinct objects:
3. Pick 3 out of 5 distinct objects when order doesn't matter:
4. Pick 10 out of 15 distinct objects when order matters:
5. Arrange 17 distinct objects:
6. Pick 7 out of 10 distinct objects when order doesn't matter:

Exposition 4: Standard Deck of Cards

Exposition 5: Full House

If we deal 5 cards from a standard deck, there are 3744 full houses like:



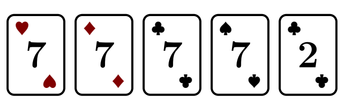
We can use combinations to see why. First ***make a plan*** and second calculate.

1. Pick 1 of 13 ranks for the 3 of a kind
2. Pick 3 of 4 from that rank
3. Pick 1 of 12 ranks for the pair
4. Pick 2 of 4 from that rank

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6$$
$$= 3744 \checkmark$$

Practice 7: Four of a Kind

If we deal 5 cards from a standard deck, there are 624 four of a kinds like:

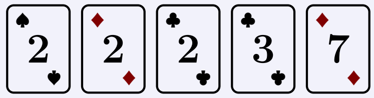


We can use combinations to see why. First ***make a plan*** and second calculate.

1.
2.
3.

Exposition 6: Three of a Kind Done Wrong

Now we want to know how many ways we can make a three of a kind, looking it up the answer is 54912, but why?



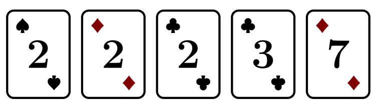
We can use combinations to see why. First ***make a plan*** and second calculate.

1. Pick 1 of 13 ranks for the 3 of a kind
2. Pick 3 of 4 from that rank
3. Pick 2 of 49 other cards

$$\binom{13}{1}\binom{4}{3}\binom{49}{2} = 13 \cdot 4 \cdot \frac{49 \cdot 48}{2}$$
$$= 61152 \text{ ✗}$$

Practice 8: Three of a Kind the Done Right

Now we want to know how many ways we can make a three of a kind, looking it up the answer is 54912, but why?



We can use combinations to see why. First ***make a plan*** and second calculate.

1.

2.

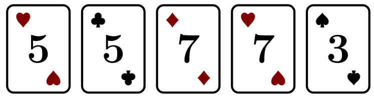
3.

4.

5.

Practice 9: Two Pair

There are 123552 different ways to be dealt two pair.



We can use combinations to see why. First ***make a plan*** and second calculate.

1.

2.

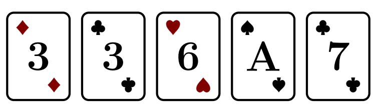
3.

4.

5.

Practice 10: One Pair

There are 2533180 different ways to be dealt a hand with at least one red card.



We can use combinations to see why. First ***make a plan*** and second calculate.

1.

2.

3.

Theorem 1 (Pascal's Formula)**.** *Given positive integers* $0 < k < n$*,*

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}. \tag{1}$$

Exposition 7: Pascal's Triangle

Using Pascal's Formula (equation 1) we can create Pascal's Arithmetic Triangle:

$$\begin{array}{lccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 1 & & 1 & & & & \\
n=2 & & & & 1 & & 2 & & 1 & & & \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & \\
n=4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n=5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

which lists the ***binomial coefficients*** for each value of n.

Theorem 2 (Binomial Theorem)**.** *Given a positive integer* n *and variables* a *and* b*:*

$$\begin{aligned}
(a+b)^n &= (a+b)(a+b)\cdots(a+b) && (2)\\
&= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n && (3)\\
&= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k && (4)
\end{aligned}$$

Practice 11: Sum of Pascal Rows

Show that the sum of the coefficients in row n of Pascal's Triangle is 2^n. (Hint: $(1+1) = 2$)

Practice 12: x^2yz^2

In the product

$$(x+y+z)^5 = (x+y+z)(x+y+z)(x+y+z)(x+y+z)(x+y+z)$$

the coefficient for the term x^2yz^2 is 30. We can use combinations to see why. First ***make a plan*** and second calculate.

1.

2.

3.

Practice 13: Mississippi

There are 34650 distinct ways to rearrange the 11 letters in the word Mississippi. We can use combinations to see why. First ***make a plan*** and second calculate.

1.

2.

3.

4.

Definition 7 (Multinomial Coefficients)**.** There are two ways to think of a ***multinomial coefficient***.
Given k different types of objects with n_i of each type, a ***multinomial coefficient*** where $\sum n_i = n$, represents the number of ways to arrange all the objects.

OR

Given n copies of a set each with the same k distinct objects, a ***multinomial coefficient*** where $\sum n_i = n$, represents the number of ways to select n_i copies of object type i.

Either way it is calculated with:

$$\binom{n}{n_1, n_2, \ldots, n_l} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

Definition 8 (Combinations with Replacement)**.** If we have to make k choices from n options, order does not matter, and we can select the same option more than once, then this is a ***combination with replacement***. the number of ways we can do this is:

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Exposition 8: Robots and Ice Cream

You have a robot that scoops ice cream. It can only follow two commands, scoop to take a scoop of whatever ice cream it is above and move to move to the next flavor. Assuming you get three scoops of ice cream and there are five flavors the commands for the robot look like:

```
def serving([m,m,s,m,s,m,s]):
    move arm
    move arm
    scoop ice cream
    move arm
    scoop ice cream
    move arm
    scoop ice cream
    return ice cream
```

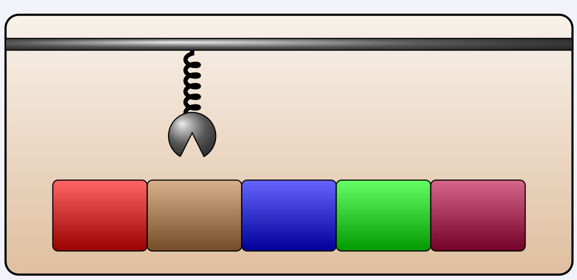
- Why are there only four move commands if there are five flavors?
- If $n = 3$ is the number of scoops and $k = 5$ is the number of flavors, why does it make since to use $n+k-1 = 7$ like in definition 8?

One way of looking at this is that

$$\binom{n+k-1}{n} = \binom{7}{3} = 35$$

is the number of ways we can place the three scoop commands in the seven lines of code, and then put the move commands on the remaining empty lines:

```
def serving([s,s,_,_,s,_,_]):
    scoop ice cream
    scoop ice cream
    _______________
    _______________
    scoop ice cream
    _______________
    _______________
    return ice cream
```



Exposition 9: Zombies and Walls

Another way to think of definition 8 is to ask in how many ways could you divide a row of identical objects into individual groups (we allow some groups to be empty). For example we can split a group of 10 zombies into 4 smaller groups like so

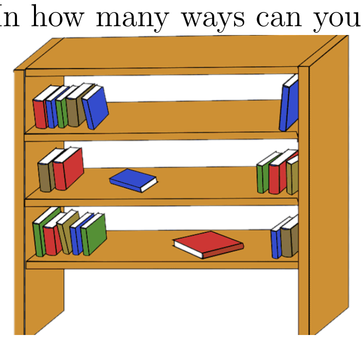
where none of the groups are empty; $1 + 2 + 3 + 4 = 10$. Which we can also think of as 1 zombie before the first wall, 3 total before the second, 6 total before the third, and 10 altogether.
Or, like so

where the first and last groups are empty; $0 + 4 + 6 + 0 = 10$. Which again we can think of as 0 zombies before the first wall, 4 total before the second, and 10 total before the third, and also 10 altogether.
Either way we need to arrange our 10 zombies and 3 walls in a row to get our 4 groupings. Thus the total number of possible choices is

$$\binom{n+k-1}{n} = \binom{10+4-1}{10} = \binom{13}{10} = 286.$$

Practice 14: Book Shelves

In how many ways can you place 23 books onto 3 shelves?



Practice 15: Buying Soda

You go to a store to buy soda for a party, if there are 5 types of soda and you need to get 48 sodas total, in how many ways can you pick out the sodas?

Practice 16: Buying Soda Take 2

You realize that you should really make sure to have at least 6 cans of each type of soda. With this in mind, how many ways can you now pick out sodas?

Practice 17: Get chips Too!!!

You will also need chips. There are 3 types of chips and you want 7 bags. However, there are only 3 bags of one of the types. With this limitation, in how many ways can you select chips?

Practice 18: Home On the Range

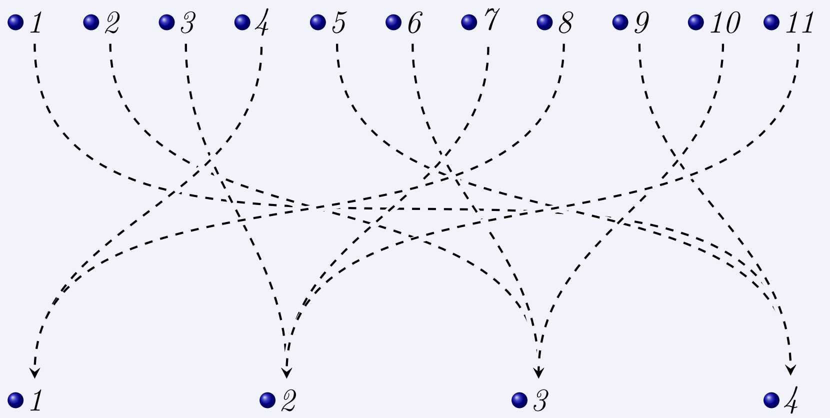
Noting that $0 \leq k \leq j \leq i \leq 9$, how many times does this code print "Discrete Rules!!!" ?

```
count=1
for i in range(10):
    for j in range(i+1):
        for k in range(j+1):
            print("(%d) Discrete Rules!!!"%count)
            count+=1
```

Pigeon Hole Principle

Theorem 3 (The Pigeonhole Principle)**.** *Given finite sets X and Y with $|X| > |Y|$, a function from $f : X \longrightarrow Y$ can not be one-to-one. Moreover, if $k < |X|/|Y|$, then there exists some $y \in Y$ such that $|f^{-1}(y)| \geq k+1$.*

For example given $X = \{1, \ldots, 11\}$ and $Y = \{1, \ldots, 4\}$, $|X| = 11 > |Y| = 4$ and $11/4 > 2$; for any function there will be some y such that $|f^{-1}(y)| \geq 2 + 1 = 3$.



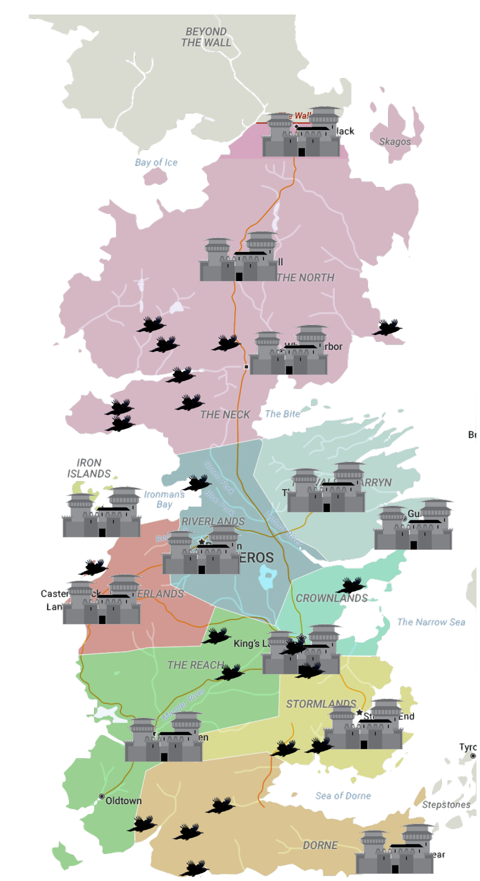
$|f^{-1}(4)| = |\{1, 5, 9\}| = 3$

Practice 19: Pigeonhole Principle: Actual Birds in Actual Holes

If the Citadel in Old Town sends out 21 ravens to 12 castles around Westeros announcing that

 Winter is Coming!!! 

why is there at least one castle that receives 2 ravens?



(Ravens and Castles not to scale.)

Practice 20: Pigeonhole Principle: Picking Things Out

Given the set of integers $X = \{1, ..., 9\}$ any set of six numbers selected at random will have at least two which sum to 10.

Practice 21: Pigeonhole Principle: Remainders

If an integer n is divided by 7, and n is not a multiple of 7, then the number of repeating digits is at most 6. (Try $37 \div 7$.)

Practice 22: Generalized Pigeonhole Principle: Sharing Tables

Given 10 lunch tables and 57 students, show that there must be at least one table with six students.

Theorem 4. *Given finite sets X and Y, there exists a one-to-one and onto function from X to Y if and only if $|X| = |Y|$.*