1 Sets

Set Notation and Common Sets

Definition 1 (Set Notation and Common Sets)**.** Describing a set by listing its elements is called ***set roster*** notation:

$$S = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}\,. \tag{1}$$

Describing a set by setting conditions for membership is called ***set builder*** notation:

$$S = \{2n | n \textit{ is an integer and } 0 < n < 10\}\,. \tag{2}$$

Either way this could be read as

> The set of all integers of the form $2n$ with n from 1 to 9,

or in plainer language

> All the even integers from two to eighteen.

There are six sets that we will frequently use:

1. ***Empty Set***: $\emptyset = \{\}$,
2. ***Natural Numbers***: $\mathbb{N} = \{1, 2, 3, \ldots\}$,
3. ***Integers***: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$,
4. ***Rational Numbers***: $\mathbb{Q} = \{a/b | a, b \in \mathbb{Z} \wedge b \neq 0\}$,
5. ***Real Numbers***: $\mathbb{R} = \mathbb{Q} \cup \{irrationals\}$ or all the numbers used to denote or measure distances on a line, and
6. ***Complex Numbers***: $\mathbb{C} = \{a + bi | a, b \in \mathbb{R} \wedge i^2 = -1\}$.

To read the sets in set builder notation we need to note some new-ish notation:

$$\mathbb{Q} = \underset{\text{“the set containing” or “the set of”}}{\{} a/b \overset{\text{“such that”}}{|} a, b \underset{\text{“element of”}}{\in} \mathbb{Z} \overset{\text{“and”}}{\wedge} b \neq 0\}$$

So we can translate the description of the rationals as

> The rational numbers are ***the set of*** all ratios a/b ***such that*** a and b are ***elements of*** the integers ***and*** b does not equal 0.

Practice 1: Roster to Verbal and Builder

Given the set $A = \{1/2, 2/2, 3/2, 4/2, 5/2, \ldots\}$ write the set as a sentence and then in set builder notation.

Verbal/Written Description:

The set A is the set of

Set Builder Description:

$$A = \left\{ \qquad \middle| \qquad \right\}$$

Practice 2: Builder to Verbal and Roster

Given the set $B = \{n^2/5 | n \in \mathbb{Z}\}$ write the set as a sentence and then in set roster notation.

Verbal/Written Description:

The set B is the set of

Set Roster Description:

$$B = \left\{ \qquad \right\}$$

Practice 3: Verbal to Roster and Builder

The set C is the set of all odd natural numbers less than 30, write the set in set roster notation and then in set builder notation.

Set Roster Description:

$$C = \left\{ \qquad \right\}$$

Set Builder Description:

$$C = \left\{ \qquad \middle| \qquad \right\}$$

New Sets From Old

Definition 2 (Basic Set Combinations)**.** Given sets $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7\}$:

- ***Union of A and B***:

$$A \cup B = \{x | x \in A \ \vee x \in B\} = \{1, 2, 3, 5, 7, 9\} \tag{3}$$

- ***Intersection of A and B***:

$$A \cap B = \{x | x \in A \ \wedge x \in B\} = \{3, 5, 7\} \tag{4}$$

- ***Difference between A and B***:

$$A \setminus B = \{x | x \in A \ \wedge x \notin B\} = \{1, 9\} \tag{5}$$

Practice 4: Finding Combinations of Sets

Given $C = \{n | n \in \mathbb{Z} \wedge -5 \leq n \leq 5\}$ and $D = \{-2, -3, -5, -7, -11, -13, -17, -19\}$ find the following:

- $C \cup D = \Bigg\{ \qquad \Bigg\}$
- $C \cap D = \Bigg\{ \qquad \Bigg\}$
- $C \setminus D = \Bigg\{ \qquad \Bigg\}$
- $D \setminus C = \Bigg\{ \qquad \Bigg\}$

Practice 5: Finding Combinations of Sets

Given $V = \{a, e, i, o, u, y\}$ and $C = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$ find the following:

- $V \cup C = \Bigg\{ \qquad \Bigg\}$
- $V \cap C = \Bigg\{ \qquad \Bigg\}$
- $C \setminus V = \Bigg\{ \qquad \Bigg\}$
- $V \setminus C = \Bigg\{ \qquad \Bigg\}$

Exposition 1: A Note on Universal Sets, Compliments, and Visualization

Again letting $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7\}$ we could let

$$\mathscr{U} = \{n | n \in \mathbb{N} \wedge n \leq 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

be the ***universal set*** from which A and B draw elements so we can now write

$$A = \{x | x \in \mathscr{U} \wedge x \text{ } is \text{ } odd\}$$

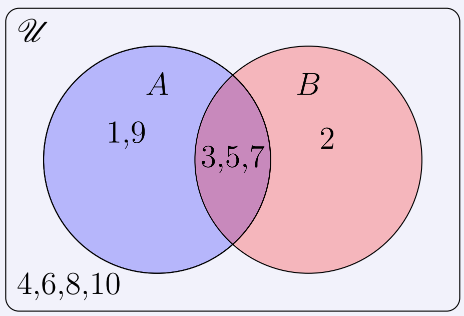
and

$$B = \{x | x \in \mathscr{U} \wedge x \text{ } is \text{ } prime\}.$$

And we say A and B are ***subsets***[a] of $\mathscr{U}$. We can also now define the ***compliment*** of a set A as

$$A^c = \{x | x \in \mathscr{U} \wedge x \notin A\} = \{2, 4, 6, 8, 10\}.^b \tag{6}$$

Finally we can picture all of this with a ***Venn Diagram***:



[a] Or that $\mathscr{U}$ is a superset of A and B.
[b] Some texts us the notation $\overline{A}$ for the compliment of A.

Practice 6: Finding Combinations with a Venn Diagram

Use the Venn diagram to help you fill in the following sets.

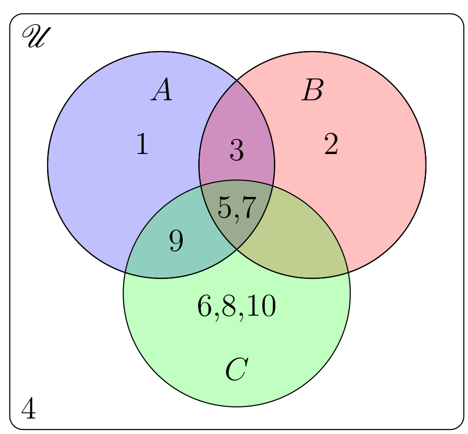
$$C = \left\{ \qquad \qquad \qquad \right\}$$

$$C^c = \left\{ \qquad \qquad \qquad \right\}$$

$$A \cap C^c = \left\{ \qquad \qquad \qquad \right\}$$

$$A^c \cap B \cap C = \left\{ \qquad \qquad \qquad \right\}$$

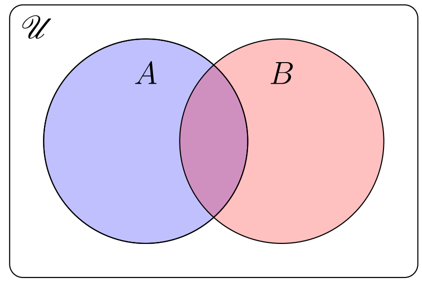
$$A \cup B^c \cup C^c = \left\{ \qquad \qquad \qquad \right\}$$



Practice 7: Understanding Combinations and Venn Diagrams

Use the Venn diagram to help you match up the following sets.

___ (a) A^c	(1) $A^c \cap B$
___ (b) $B \setminus A$	(2) $(A^c \cup B^c)^c$
___ (c) $A \cup B^c$	(3) $\mathscr{U} \setminus A$
___ (d) $A \cap B$	(4) $(A^c \cap B)^c$
___ (e) $(A \cup B) \setminus (A \cap B)$	(5) $(B \setminus A) \cup (A \setminus B)$



Definition 3 (Products and Power Sets)**.** Again letting $A = \{1,3,5,7,9\}$ and $B = \{2,3,5,7\}$ we can define the ***Cartesian Product*** of A and B as

$$\begin{aligned} A \times B &= \{(a,b) | a \in A \wedge b \in B\}\,, \qquad (7)\\ &= \{(1,2),(1,3),(1,5),(1,7),(3,2),(3,3),(3,5),(3,7),(5,2),(5,3),\\ &\quad (5,5),(5,7),(7,2),(7,3),(7,5),(7,7),(9,2),(9,3),(9,5),(9,7)\} \end{aligned}$$

a new set with ***ordered pairs*** of numbers instead of just individual numbers as elements. A set X is a ***subset*** of a set B if every element in X is also in B, and we write

$$X \subset B \text{ if and only if } \forall x \in X : x \in B.$$

We can use this to define the ***power set*** of a set

$$\begin{aligned} \mathscr{P}(B) &= \{\text{All the subsets of } B)\} \qquad (8)\\ &= \{\emptyset, \{2\},\{3\},\{5\},\{7\},\{2,3\},\{2,5\},\{2,7\},\{3,5\},\{3,7\},\{5,7\}\\ &\quad \{2,3,5\},\{2,3,7\},\{2,5,7\},\{3,5,7\},\{2,3,5,7\}\}\,, \end{aligned}$$

which is a new set with sets as elements.

Practice 8: More New Sets

Use $C = \{a,b,c\}$, $D = \{a,b,c,d\}$, and $N = \{1,2,3\}$ to fill in each of the following:

$$C \times N = \{ \qquad\qquad\qquad\qquad \}$$
$$N \times C = \{ \qquad\qquad\qquad\qquad \}$$
$$\mathscr{P}(C) = \{ \qquad\qquad\qquad\qquad \}$$
$$\mathscr{P}(D) = \{ \qquad\qquad\qquad\qquad \}$$

On a side note, what is the distinction between $C = \{a,b,c\}$ and the set $\{\{a\},\{b\},\{c\}\}$?

2 Relations

Basics of Relations

Definition 4 (Relations)**.** A ***relation*** between to sets A and B is a subset of their Cartesian product $A \times B$. For example given $A = \{1,3,5,7,9\}$ and $B = \{2,3,5,7\}$,

$$L = \{(1,2),(1,3),(1,5),(1,7),(2,3),(2,5),(2,7),(3,5),(3,7),(5,7)\}$$

is a relation between A and B. We normally prefer to describe the relation, for example we can describe L by

$$\forall x \in A\, \forall y \in B : x\, L\, b \text{ if and only if } x < y,$$

or equivalently

$$L = \{(x,y) | x \in A \wedge y \in B \wedge (x < y)\}\,.$$

Practice 9: Sample Relation

Let $E = \{2n | n \in \mathbb{N}\}$ and $O = \{2n+1 | n \in \mathbb{N}\}$. Define a relation by

$$\forall x \in E\, \forall y \in O : x\, D\, y \text{ if and only if } x = 2y.$$

List five additional ordered pairs in the relation D:

$$D = \{(2,1),(14,7),(26,13), \qquad\qquad\qquad\qquad\qquad\qquad \}$$

Write the set of ordered pairs for D in set builder notation:

Practice 10: Sample Relation

Define a relation on the set $\mathbb{Z}$ by

$$M_5 = \{(x,y) | x, y \in \mathbb{Z} \wedge \exists q \in \mathbb{Z} : x - y = 5q\} \tag{9}$$

List five additional ordered pairs in the relation M_5:

$$M_5 = \{(17,12),(30,5),(-20,-45), \qquad\qquad\qquad\qquad\qquad\qquad \}$$

Write the a description of M_5 using quantifiers:

Properties of Relations

Definition 5 (Properties of Relations)**.** Of special interest are relations between a set and its self, such as M_5 described in equation (9). In particular when we have a relation R between a set A and its self we look for the following properties:

1. ***Reflexive:*** $\forall a \in A : a\,R\,a$,
2. ***Symmetric:*** $\forall a, b \in A :$ if $a\,R\,b$, then $b\,R\,a$,
3. ***Transitive:*** $\forall a, b, c \in A :$ if $a\,R\,b$ and $b\,R\,c$, then $a\,R\,c$, and
4. ***Antisymmetric:*** $\forall a, b \in A :$ if $a\,R\,b$, then b does not relate to a unless $a = b$.

Recall that in equation (9) a pair of integers (x, y) were in the relation M_5 if and only if their difference, $(x - y)$, was a multiple of 5. For example $(17, 12) \in M_5$ since $17 - 12 = 5$, similarly $(12, -28) \in M_5$ since $12 - (-28) = 40 = 8 \times 5$. But we can note that:

$$\begin{aligned}
(12 - 17) &= -1 \times (17 - 12) \\
&= -1 \times 5, \qquad (10) \\
(-28 - 12) &= -1 \times (12 - (-28)) \\
&= -8 \times 5, \text{ and} \qquad (11) \\
(17 - (-28)) &= (17 - 12) + (12 - (-28)) \\
&= 5 + 8 \times 5 \\
&= 9 \times 5. \qquad (12)
\end{aligned}$$

Therefore we get that $(12, 17), (-28, -12)$, and $(17, -28)$ are in M_5 as well.
Further, the calculations in equations (10) and (11) can be generalized to show that *if* $(a, b) \in M_5$*, then* $(b, a) \in M_5$, i.e. M_5 is a ***symmetric*** relation.
The calculation in equation (12) can likewise be generalized to show that *if* (a, b) *and* (b, c) *are in* M_5*, then* (a, c) *is also in* M_5, i.e. M_5 is a ***transitive*** relation.
Finally, since

$$n - n = 0 = 0 \times 5 \qquad (13)$$

for all integers n, we know that *for all integers* n*,* (n, n) *is in* M_5, i.e. M_5 is ***reflexive***.
Since, M_5 is reflexive, symmetric, and transitive we say that it is an ***equivalence relation***.

Practice 11: Relation M_7

Consider the relation M_7 defined by:

$$\forall x, y \in \mathbb{Z} : x \, M_7 \, y \text{ if and only if } \exists q \in \mathbb{Z} : (x - y) = 7q.$$

Rewrite M_7 in set notation and list five ordered pairs of integers in M_7. Then try to show that M_7 is reflexive, symmetric and transitive, i.e. it is an equivalence relation.

Practice 12: Relation L

Consider the relation L defined by:

$$\forall x, y \in \mathbb{Z} : x \, L \, y \text{ if and only if } x < y$$

Rewrite L in set notation and list five ordered pairs of integers in L. Try to show that L is antisymmetric and transitive, then give examples showing that it is not reflexive or symmetric and so is not an equivalence relation.

Practice 13: Relation L'

Consider the relation L' defined by:

$$\forall x, y \in \mathbb{Z} : x\,L'\,y \text{ if and only if } x \leq y$$

Rewrite L' in set notation and list five ordered pairs of integers in L'. (Try to list some that were not in L.) Try to show that L' is reflexive, antisymmetric and transitive, then give examples showing that it is not symmetric and so is not an equivalence relation. Relations like L' are called ***partial-orderings***.

Practice 14: Relation D

Consider the relation D defined by:

$$\forall x, y \in \mathbb{Z} : x\,D\,y \text{ if and only if } x = 2y$$

Rewrite D in set notation and list five ordered pairs of integers in D. Try to show that D is antisymmetric, then give examples showing that it is not reflexive, symmetric, or transitive and so is not an equivalence relation.

Practice 15: Relation B_ϵ

Consider the relation B_ϵ defined by:

$$\forall x, y \in \mathbb{R} : x \, B_\epsilon \, y \text{ if and only if } |x - y| < \epsilon$$

Rewrite B_ϵ in set notation and list five ordered pairs of real number in B_ϵ in the case when $\epsilon = 0.5$, i.e. $B_{0.5}$. Try to show that $B_{0.5}$ is reflexive and symmetric, then give examples showing that it is not transitive and so is not an equivalence relation.

Practice 16: Relation W

Consider the relation W defined by:

Two words, x and y, are in W if and only if they begin and end with the same letters

For example $(their, tear) \in W$, but $(there, tear) \notin W$. Try to write W in set notation and list five ordered pairs of words in W and five pairs not in W. Try to show that W is reflexive, symmetric, and transitive, i.e. that it is an equivalence relation.

Visualizing Relations

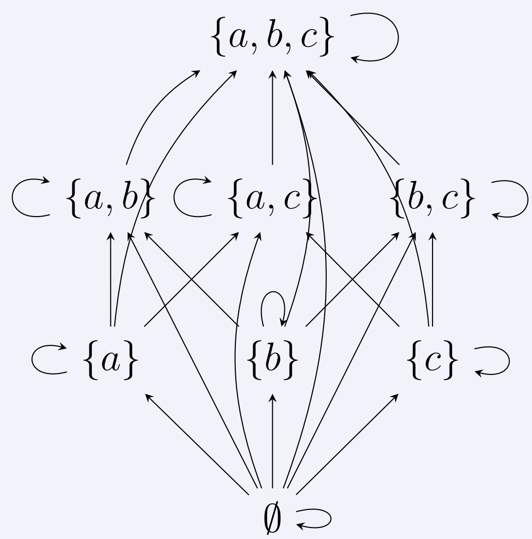
Exposition 2: Subset Relation

Given a set $X = \{a, b, c\}$ we can define a partial-ordering on $\mathscr{P}(X)$ using the subset or equal to relation:

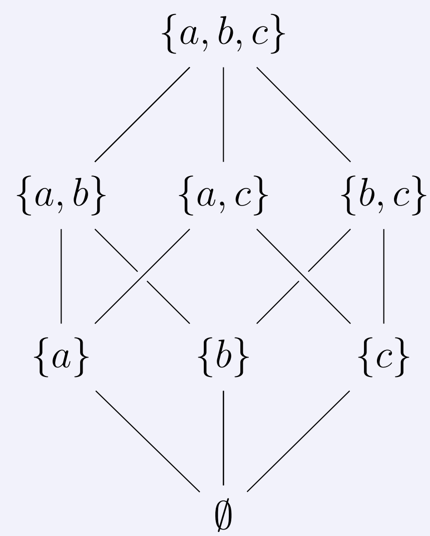
$$A \sim B \text{ if and only if } A \subseteq B.$$

We can then visualize this as a graph where all the subsets are vertices and two vertices are connected if there is a relation between them.

Full Diagram:



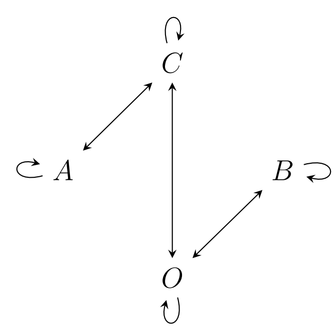
Hasse Diagram:



In the first diagram each time an element x relates to an element y we draw an arrow form x to y. The second diagram is drawn assuming we have a partial-ordering and the physically lowest value in the diagram is the ***minimal value*** in the partial-ordering, the loops are implied, and so are arrows going from lower values to higher.

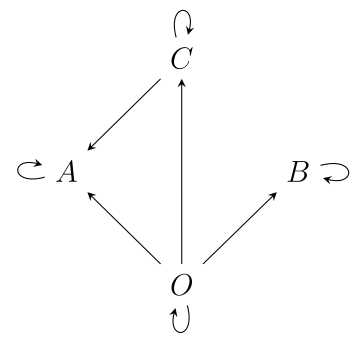
Practice 17: A Relation Defined with a Diagram

1. How do we know this relation is *reflexive*?
2. How do we know this relation is *symmetric*?
3. How do we know this relation is not *transitive*?



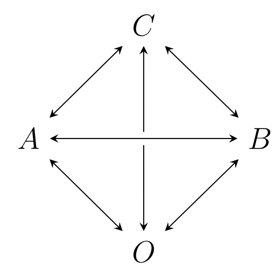
Practice 18: A Relation Defined with a Diagram

1. How do we know this relation is *reflexive*?
2. How do we know this relation is *antisymmetric*?
3. How do we know this relation is *transitive*?



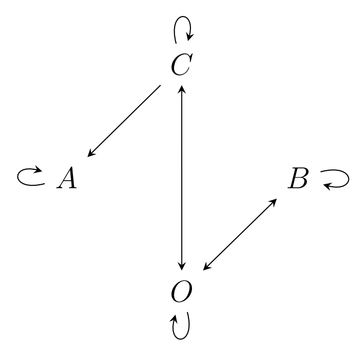
Practice 19: A Relation Defined with a Diagram

1. How do we know this relation is not *reflexive*?
2. How do we know this relation is *symmetric*?
3. How do we know this relation is not *transitive*?



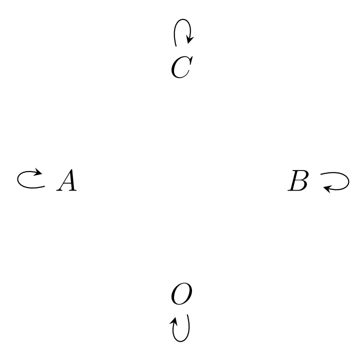
Practice 20: A Relation Defined with a Diagram

1. How do we know this relation is *reflexive*?
2. Why is this relation not *symmetric*, *antisymmetric*, or *transitive*?



Practice 21: A Relation Defined with a Diagram

1. How do we know this relation is *reflexive*?
2. Why is this relation *symmetric*, *antisymmetric*, and *transitive*?



Equivalence Classes

Exposition 3: Return of M_5

Earlier we saw that the relation M_5 (equation (9) p.6) was an equivalence relation, i.e. it is reflexive, symmetric, and transitive. We also saw that the values 17, 12, and -28 are all ***"equivalent"***. If we group all the integers together which are equivalent using M_5 we get five sets:

- $[0] = \{0, \pm 5, \pm 10, \pm 15, \ldots\} = \{5k | k \in \mathbb{Z}\}$
- $[1] = \{1, 6, -4, 11, -9, 16, -14, \ldots\} = \{1 + 5k | k \in \mathbb{Z}\}$
- $[2] = \{2, 7, -3, 12, -8, 17, -13, \ldots\} = \{2 + 5k | k \in \mathbb{Z}\}$
- $[3] = \{3, 8, -2, 13, -7, 18, -12, \ldots\} = \{3 + 5k | k \in \mathbb{Z}\}$
- $[4] = \{4, 9, -1, 14, -6, 19, -11, \ldots\} = \{4 + 5k | k \in \mathbb{Z}\}$

And, because of the symmetry and transitivity of the relation $[n] \cap [m] = \emptyset$ whenever $n \neq m$. We call these sets ***equivalence classes*** and the values 0, 1, 2, 3, and 4 are ***equivalence class representatives***, we let them stand in for all the numbers they are equal to.

Practice 22: Relation M_7 Revisited

Earlier we defined the relation M_7 as:

$$\forall x, y \in \mathbb{Z} : x \, M_7 \, y \text{ if and only if } \exists q \in \mathbb{Z} : (x - y) = 7q.$$

We then *"showed"* that it was an equivalence relation. This relation will have seven equivalence classes with representatives 0, 1, 2, 3, 4, 5, and 6. Try to write down/describe the elements of the equivalence classes:

- $[0] = \left\{ \qquad\qquad\qquad\qquad \right\} = \left\{ \qquad\qquad \middle| \qquad\qquad \right\}$
- $[1] = \left\{ \qquad\qquad\qquad\qquad \right\} = \left\{ \qquad\qquad \middle| \qquad\qquad \right\}$
- $[2] = \left\{ \qquad\qquad\qquad\qquad \right\} = \left\{ \qquad\qquad \middle| \qquad\qquad \right\}$
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- $[5] = \left\{ \qquad\qquad\qquad\qquad \right\} = \left\{ \qquad\qquad \middle| \qquad\qquad \right\}$
- $[6] = \left\{ \qquad\qquad\qquad\qquad \right\} = \left\{ \qquad\qquad \middle| \qquad\qquad \right\}$

Practice 23: Integers Modulo n

The relations M_5 and M_7 are specific examples of ***equivalence modulo*** n:

$$\forall x, y \in \mathbb{Z} : x \equiv y \pmod{n} \text{ if and only if } \exists q \in \mathbb{Z} : (x - y) = qn.$$

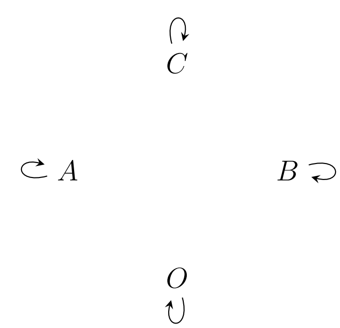
As with M_5 and M_7 this defines and equivalence relation. This relation will have n equivalence classes with representatives $0 \leq k \leq (n-1)$. Try to write down/describe the elements of a general equivalence class:

$$[k] = \Bigg\{ \qquad\qquad\qquad\qquad \Bigg\} = \Bigg\{ \qquad\qquad \Bigg| \qquad\qquad \Bigg\} \tag{14}$$

The idea of equivalence modulo n plays an important role in many areas of Math and Computer Science. Modular calculations are built into most programming languages with commands such as `%, mod, fmod, rem`, and `remainder`.

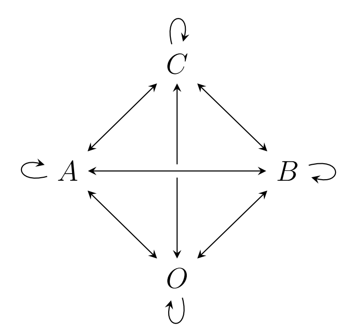
Practice 24: An Equivalence class in a Diagram

The graph represents an, admittedly boring, equivalence relation. What are the four equivalence classes?



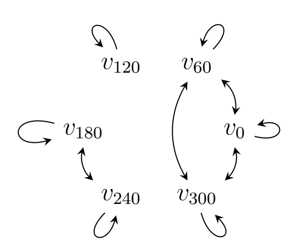
Practice 25: An Equivalence class in a Diagram

The graph represents an equivalence relation. Why is there only one equivalence class?



Practice 26: An Equivalence class in a Diagram

The graph represents an equivalence relation. What are the equivalence classes?



3 Functions

Functions as Relations

Definition 6 (Function). A ***function*** is a relation between two sets, the first called the ***domain*** and the second the ***codomain***, such that for each x in the domain there is *exactly* one y in the codomain associated with x. Further, the set of all y from the codomain which are associated with at least one x in the domain is called the ***range***.

Exposition 4: Function Example

Define a function f from the Domain

$$D = \{a, b, c, d\}$$

to the Codomain

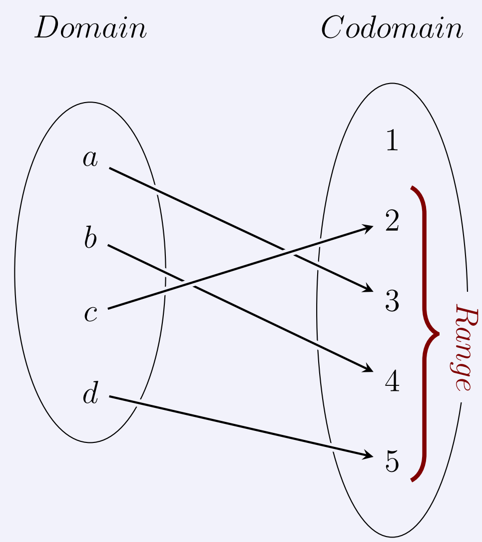
$$C = \{1, 2, 3, 4, 5\}$$

with the following set of pairs:

$$f = \{(a, 3), (b, 4), (c, 2), (d, 5)\}.$$

Since there are no letters associated with the $1 \in C$, the Range is

$$\begin{aligned} R &= \{y \in C | \exists x \in D \wedge f(x) = y\} \\ &= \{2, 3, 4, 5\}. \end{aligned}$$



Practice 27: Set to Diagram

Define a function S from the Primes

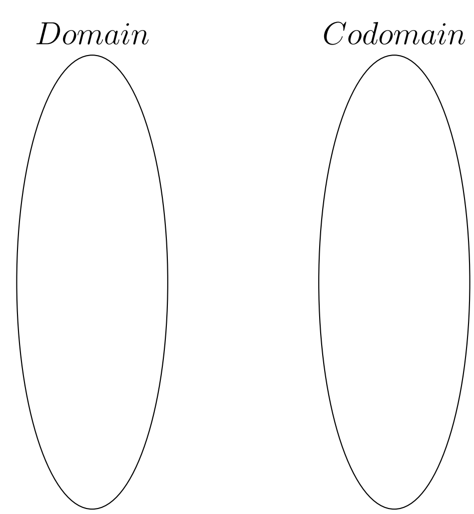
$$P = \{2, 3, 5, 7, 11, \ldots\}$$

to the Alphabet

$$A = \{a, b, c, d, e, \ldots, w, x, y, z\}$$

with the following set of pairs:

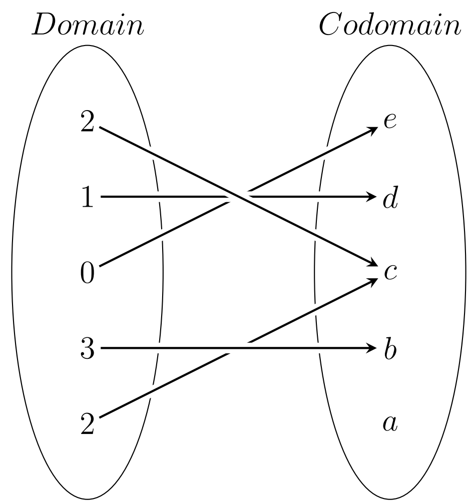
$$S = \{(2, w), (3, h), (5, i), (7, e), (11, l), \ldots\}.$$



Practice 28: Set to Diagram

What ordered pairs are in the function illustrated on the right?

What is the range of the illustrated function?



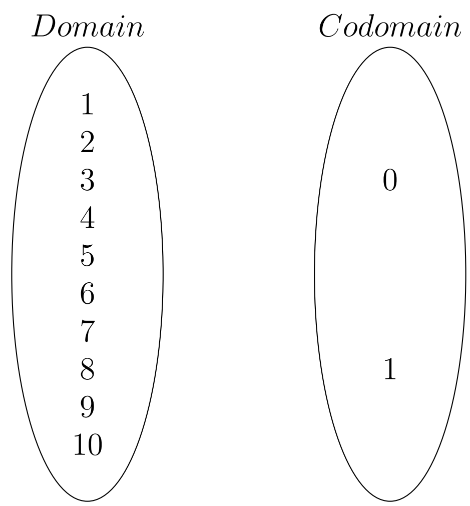
Practice 29: Function Notation

Define

$$P(x) = \begin{cases} 0 & \textit{if } x \textit{ is not prime} \\ 1 & \textit{if } x \textit{ is prime} \end{cases}$$

Evaluate the function at each point in the domain

$$D = \{1, 2, 3, 5, 6, 7, 8, 9, 10\}.$$



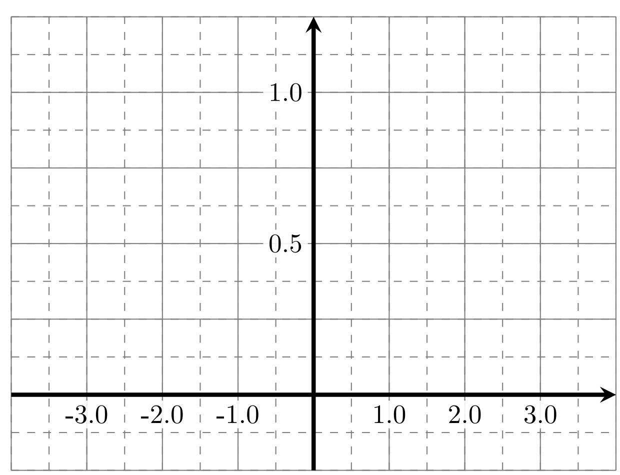
Practice 30: Function Notation

Define

$$S(x) = 1/\sqrt{x^2+1}$$

Evaluate the function at selection of points in the domain

$$D = (-4, 4) \subset \mathbb{R}.$$



Properties of Functions

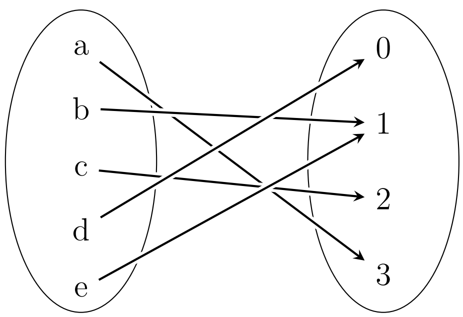
Definition 7 (Properties of Functions)**.** A function $f : A \longrightarrow B$ is said to be ***one-to-one*** $(1-1)$ if for all y in the range there is exactly one x in the domain such that $f(x) = y$. The function is ***onto*** if for all y in the codomain there is at least one x in the domain such that $f(x) = y$. If a function is both one-to-one and onto we say that it is a ***bijection***.

- One-to-One: $\forall x_0, x_1 \in A : f(x_0) = f(x_1) \longrightarrow x_0 = x_1$
- Onto: $\forall y \in B\, \exists x \in A : f(x) = y$

Practice 31: One-to-One, Onto, and Bijections

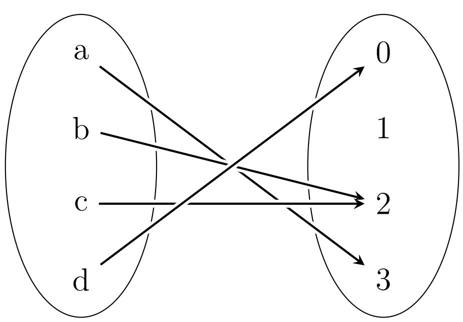
Label each as 1-1, onto, both, or neither.

Example 1:



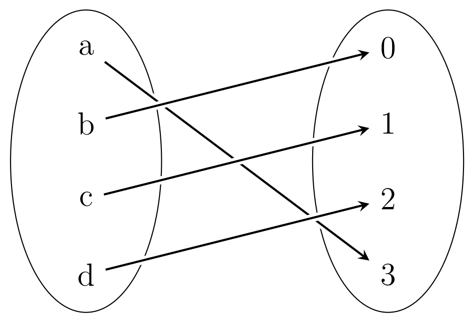
1-1, onto, both, or neither

Example 2:



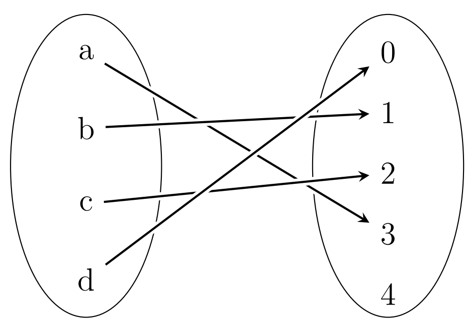
1-1, onto, both, or neither

Example 3:



1-1, onto, both, or neither

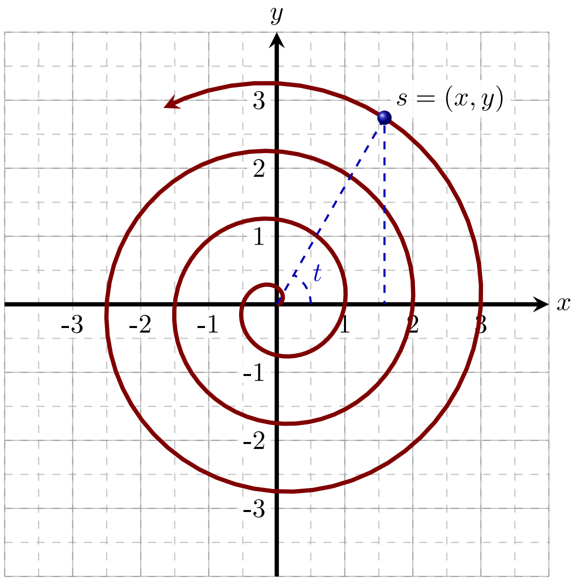
Example 4:



1-1, onto, both, or neither

Practice 32: A Subtle Example

Is the below function 1-1, onto, both, or neither? The answer depends on the *domain* and *codomain*. For each description, let S be the set of points on the spiral.



1. *Domain:* $\mathbb{R}$, *Codomain* $\mathbb{R}$:
$$f(x) = y, \text{ if } (x, y) \in S.$$
2. *Domain:* $\mathbb{R}$, *Codomain* $\mathscr{P}(\mathbb{R})$:
$$f(x) = \{y | (x, y) \in S\}$$
3. *Domain:* $\mathbb{R}^+$, *Codomain* $\mathbb{R} \times \mathbb{R}$:
$$f(t) = \left\langle \frac{t}{2\pi} \cdot \cos(t), \frac{t}{2\pi} \cdot \sin(t) \right\rangle$$
4. *Domain:* S, *Codomain* $\mathbb{R}^+$: for $s \in S$
$$f(s) = \text{length of the spiral up to } s$$

Definition 8 (Inverses)**.** Given a relation R define the ***inverse*** of R to be the set

$$R^{-1} = \{(y, x) | (x, y) \in R\}.$$

If R and R^{-1} are also functions, then we say R is ***inevitable***.

Practice 33: Invertible?

For each function in practice block 31, write the function and its "inverse" as sets of ordered pairs R and R^{-1}. Which functions are invertible? What other property do all those functions have?

Example 1:

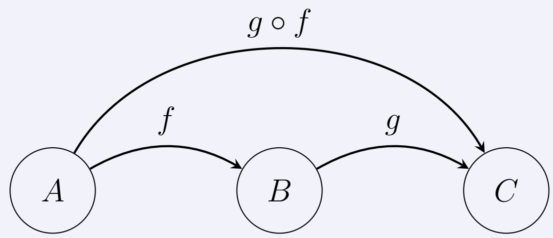
Example 2:

Example 3:

Example 4:

Theorem 1 (Invertible Functions)**.** *A function will be invertible if and only if it is . . .*

Definition 9. Given a function $f : A \to B$ and $g : B \to C$, the ***composition*** of f and g is a function from the domain of f to the codomain of g defined by $g \circ f(x) = g(f(x))$ and we read $g \circ f$ as "g compose f."

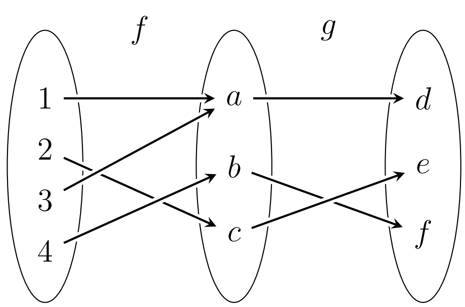


Note, this is well defined as long as the range of f is a subset of the domain of g.

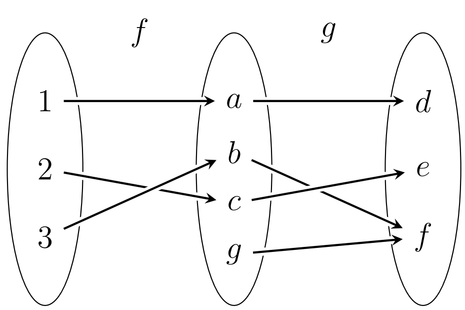
Practice 34: Composing Functions

Is $g \circ f(x)$ 1-1, onto, both, or neither? How does this compare to f and g individually?

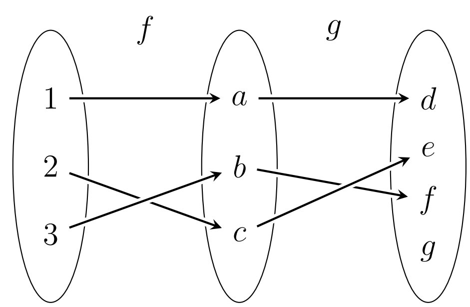
Example 1:



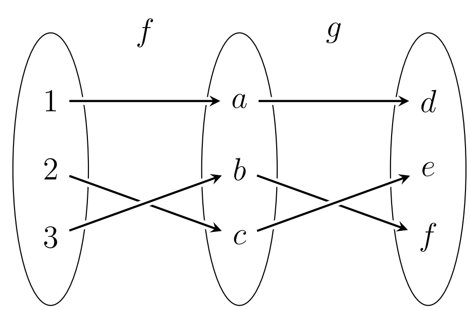
Example 2:



Example 3:



Example 4:



Practice 35: More Compositions

Using $f(x) = x^2$, $g(x) = \sqrt{x}$, and $h(x) = x + 1$, find each of the following:

1. $f \circ h(x) =$

2. $h \circ f(x) =$

3. $f \circ g(x) =$

4. $g \circ f(x) =$