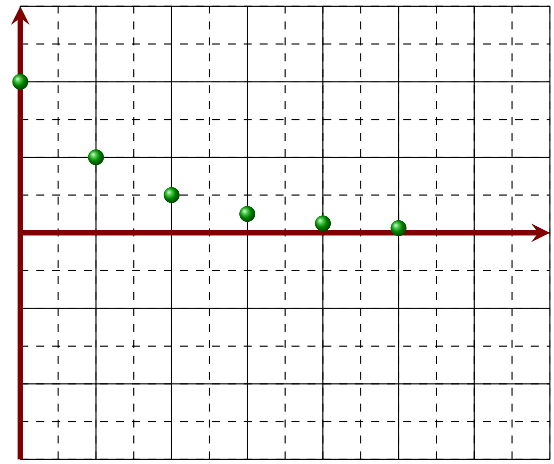
1 Sequences

$a_n = 1/2^n$

- $a_0 = 1/2^0 = 1$
- $a_1 = 1/2^1 = 1/2$
- $a_2 = 1/2^2 = 1/4$
- $a_3 = 1/2^3 = 1/8$
- $a_4 = 1/2^4 = 1/16$
- $a_5 = 1/2^5 = 1/32$

Description: Monotonically Decreasing



$b_n = 2^n$

- $b_0 =$ ________________
- $b_1 =$ ________________
- $b_2 =$ ________________
- $b_3 =$ ________________
- $b_4 =$ ________________
- $b_5 =$ ________________

Description: ________________________________

$c_n = (-2/3)^n$

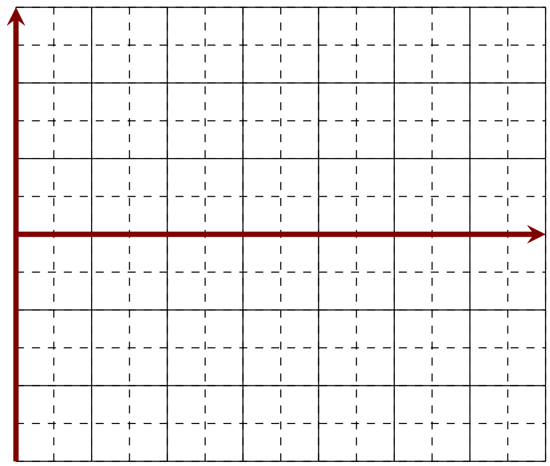
- $c_0 =$ ________________
- $c_1 =$ ________________
- $c_2 =$ ________________
- $c_3 =$ ________________
- $c_4 =$ ________________
- $c_5 =$ ________________

Description: ________________________________

$d_n = 1 - 1/3^n$

- $d_0 =$ ____________
- $d_1 =$ ____________
- $d_2 =$ ____________
- $d_3 =$ ____________
- $d_4 =$ ____________
- $d_5 =$ ____________

Description: ________________________



$f_n = 1 + (-1/2)^n$

- $f_0 =$ ____________
- $f_1 =$ ____________
- $f_2 =$ ____________
- $f_3 =$ ____________
- $f_4 =$ ____________
- $f_5 =$ ____________

Description: ________________________

$h_n = (-1)^n + (-1/2)^n$

- $h_0 =$ ____________
- $h_1 =$ ____________
- $h_2 =$ ____________
- $h_3 =$ ____________
- $h_4 =$ ____________
- $h_5 =$ ____________

Description: ________________________

$j_n = (-1)^n - (-1/2)^n$

- $j_0 =$ __________
- $j_1 =$ __________
- $j_2 =$ __________
- $j_3 =$ __________
- $j_4 =$ __________

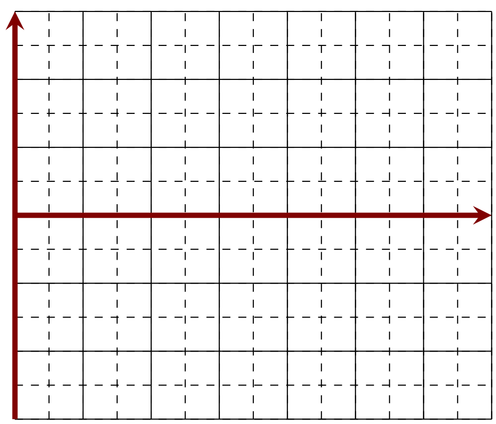
Description: ____________________

Theorem 1. *A bounded sequence has a* **convergent subsequence***. (The converse is not true.)*

$k_n = 0$ for even n and 2^n for odd n

- $k_0 =$ __________
- $k_1 =$ __________
- $k_2 =$ __________
- $k_3 =$ __________
- $k_4 =$ __________

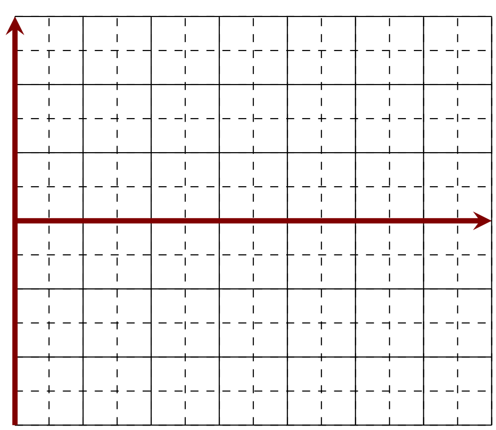
Description: ____________________



$g_n = (-\sqrt{2})^n$

- $g_0 =$ __________
- $g_1 =$ __________
- $g_2 =$ __________
- $g_3 =$ __________
- $g_4 =$ __________

Description: ____________________

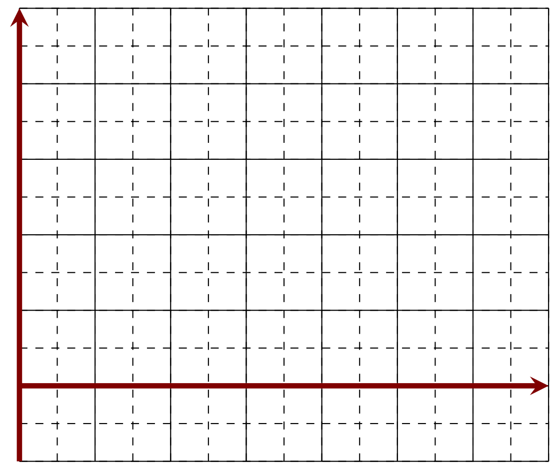


Note that we could also write $g_0 = 1$ and $g_n = -\sqrt{2} \cdot g_{n-1}$ for g_n, this is called a ***recursive definition*** for the sequence.

$r_0 = 3$ and $r_n = r_{n-1}/2$

- $r_0 =$ ____________
- $r_1 =$ ____________
- $r_2 =$ ____________
- $r_3 =$ ____________
- $r_4 =$ ____________
- $r_5 =$ ____________

Description: ________________________



$s_0 = 0$ and $s_n = 2s_{n-1} + 1$

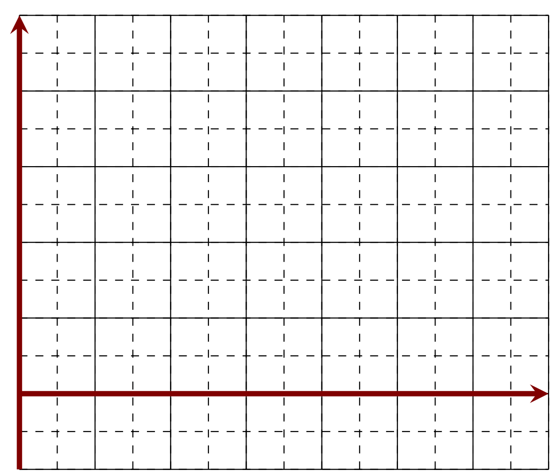
- $s_0 =$ ____________
- $s_1 =$ ____________
- $s_2 =$ ____________
- $s_3 =$ ____________
- $s_4 =$ ____________
- $s_5 =$ ____________

Description: ________________________

$F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$

- $F_0 =$ ____________
- $F_1 =$ ____________
- $F_2 =$ ____________
- $F_3 =$ ____________
- $F_4 =$ ____________
- $F_5 =$ ____________

Description: ________________________



2 Sums

2.1 Arithmetic Sums

$S_n = \sum_{i=1}^{n} i$

$$S_n = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

- $S_1 =$ ____________
- $S_2 =$ ____________
- $S_3 =$ ____________
- $S_4 =$ ____________

For $n = 100$:

$$S_{100} = 1 + 2 + 3 + \cdots + 98 + 99 + 100$$
$$S_{100} = 100 + 99 + 98 + \cdots + 3 + 2 + 1$$

$2 \cdot S_{100} =$ ____________________________________

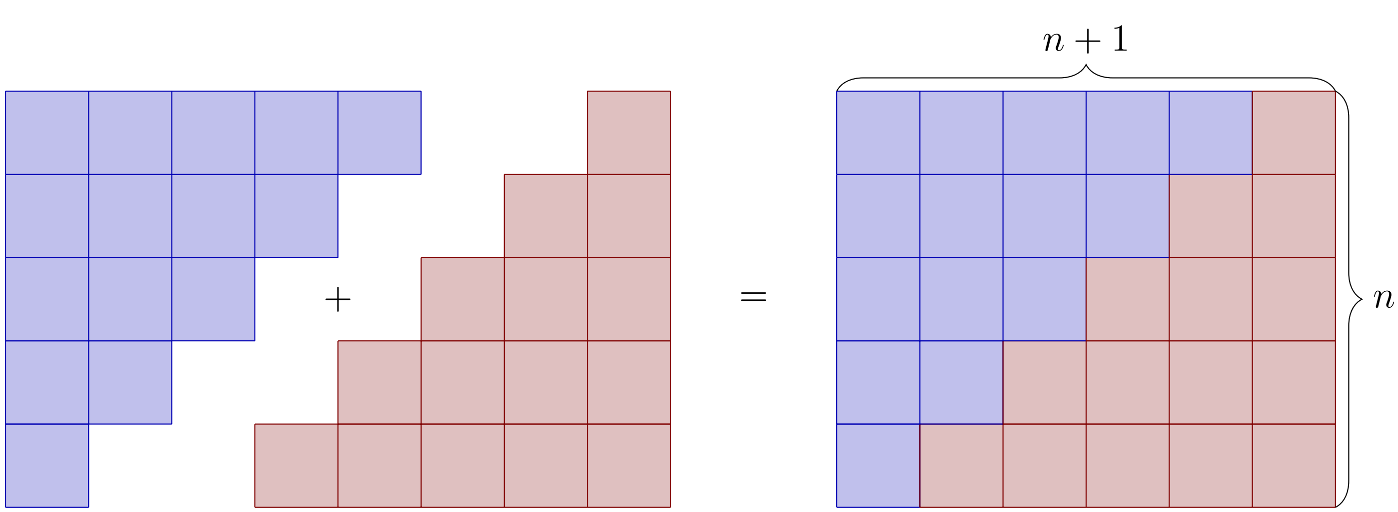
$S_{100} =$ ____________________________________

For $n = k$:

$$S_k = 1 + 2 + 3 + \cdots + (k-2) + (k-1) + k$$
$$S_k = k + (k-1) + (k-2) + \cdots + 3 + 2 + 1$$

$2 \cdot S_k =$ ____________________________________

$S_k =$ ____________________________________



$S_n = \sum_{i=1}^{n} 2 \cdot i$

$$S_n = \sum_{i=1}^{n} 2 \cdot i = 2 + 4 + 6 + \cdots + 2n$$

$S_5 =$ ______________________________

For $n = k$:

$$S_k = 2 + 4 + 6 + \cdots + 2(k-2) + 2(k-1) + 2k$$
$$S_k = 2k + 2(k-1) + 2(k-2) + \cdots + 6 + 4 + 2$$

$2 \cdot S_k =$ ______________________________

$S_k =$ ______________________________

$S_k =$ ______________________________

$S_n = \sum_{i=1}^{n} i + 3$

$$S_n = \sum_{i=1}^{n} i + 3 = 4 + 5 + 6 + \cdots + (n+3) = \sum_{j=4}^{n+3} j$$

$S_5 =$ ______________________________

For $n = k$:

$$S_k = 4 + 5 + 6 + \cdots + (k+1) + 2(k+2) + (k+3)$$
$$S_k = (k+3) + (k+2) + (k+1) + \cdots + 6 + 5 + 4$$

$2 \cdot S_k =$ ______________________________

$S_k =$ ______________________________

$S_k =$ ______________________________

$S_n = \sum_{i=1}^{n} ai + b$

$$S_n = \sum_{i=1}^{n} ai + b = (a+b) + (2a+b) + (3a+b) + \cdots + (n \cdot a + b) = n \cdot b + a \sum_{i=1}^{n} i$$

For $n = k$:

$$S_k = (a+b) + (2a+b) + (3a+b) + \cdots + (k \cdot a + b)$$
$$S_k = (k \cdot a + b) + \cdots + (3a+b) + (2a+b) + (a+b)$$

$2 \cdot S_k =$ ______________________________

$S_k =$ ______________________________

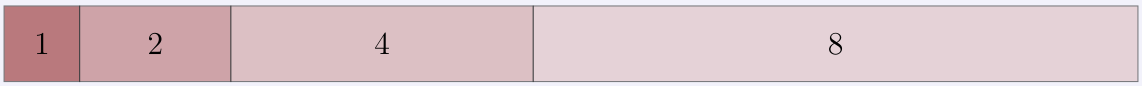
$S_k =$ ______________________________

2.2 Geometric Sums

$S_n = \sum_{i=0}^{n} 2^i$

$$S_n = \sum_{i=0}^{n} 2^i = 1 + 2 + 4 + 8 + \cdots + 2^n$$

$$S_n = \sum_{i=0}^{3} 2^i = \underline{\hspace{6cm}}$$



For $n = 10$:

$$\begin{aligned} S_{10} &= 1 + 2 + 4 + 8 + \cdots + 2^9 + 2^{10} \\ 2 \cdot S_{10} &= 2 + 4 + 8 + 16 + \cdots + 2^{10} + 2^{11} \end{aligned}$$

$$2 \cdot S_{10} - S_{10} = \underline{\hspace{6cm}}$$

$$S_{10} = \underline{\hspace{6cm}}$$

For $n = k$:

$$\begin{aligned} S_k &= 1 + 2 + 4 + \cdots + 2^{k-1} + 2^k \\ 2 \cdot S_k &= 2 + 4 + 8 + \cdots + 2^k + 2^{k+1} \end{aligned}$$

$$2 \cdot S_k - S_k = \underline{\hspace{6cm}}$$

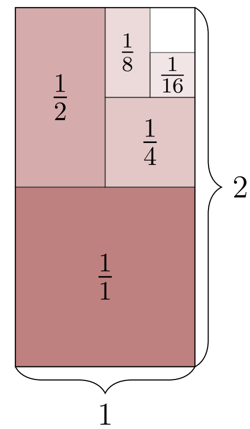
$$S_k = \underline{\hspace{6cm}}$$

$$S_k = \underline{\hspace{6cm}}$$

$S_n = \sum_{i=0}^{n} (1/2)^i$

$$S_n = \sum_{i=0}^{n} (1/2)^i = 1 + 1/2 + 1/4 + 1/8 + \cdots + (1/2)^n$$

$$S_n = \sum_{i=0}^{4} (1/2)^i = \underline{\hspace{6cm}}$$



For $n = k$:

$$S_k = 1/1 + 1/2 + 1/4 + \cdots + 1/2^{k-1} + 1/2^k$$
$$(1/2) \cdot S_k = 1/2 + 1/4 + 1/8 + \cdots + 1/2^k + 1/2^{k+1}$$

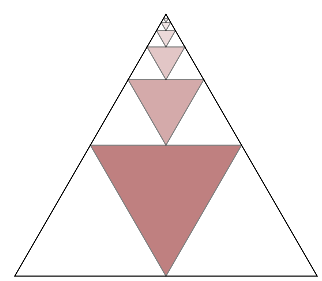
$$(1/2) \cdot S_k - S_k = \underline{\hspace{6cm}}$$

$$S_k = \underline{\hspace{6cm}}$$

$S_n = \sum_{i=0}^{n} (1/4)^i$

$$S_n = \sum_{i=0}^{n} (1/4)^i = 1 + 1/4 + 1/16 + 1/64 + \cdots + (1/4)^n$$

$$S_n = \sum_{i=0}^{4} (1/4)^i = \underline{\hspace{6cm}}$$



For $n = k$:

$$S_k = 1/1 + 1/4 + 1/16 + \cdots + 1/4^{k-1} + 1/4^k$$
$$(1/4) \cdot S_k = 1/4 + 1/16 + 1/64 + \cdots + 1/4^k + 1/4^{k+1}$$

$$(1/4) \cdot S_k - S_k = \underline{\hspace{6cm}}$$

$$S_k = \underline{\hspace{6cm}}$$

$S_n = \sum_{i=0}^{n} a \cdot r^i$

$$S_n = \sum_{i=0}^{n} a \cdot r^i = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots + a \cdot r^n$$

$$S_n = \sum_{i=0}^{4} a \cdot r^i = \underline{\hspace{10cm}}$$

For $n = k$:

$$S_k = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots + a \cdot r^n$$
$$r \cdot S_k = a \cdot r + a \cdot r^2 + a \cdot r^3 + a \cdot r^4 + \cdots + a \cdot r^{n+1}$$

$$r \cdot S_k - S_k = \underline{\hspace{10cm}}$$

$$S_k = \underline{\hspace{10cm}}$$

$$S_k = \underline{\hspace{10cm}}$$

3 Using and Finding Formulas

Sum Formulas

Arithmetic Sum

$$\sum_{i=1}^{n} a \cdot i + b = n \cdot b + a\frac{n(n+1)}{2}$$

Geometric Sum

$$\sum_{i=0}^{n} a \cdot r^i = a\left(\frac{r^{n+1}-1}{r-1}\right)$$

Using Formulas, an Arithmetic Example

Find the sum of:

$$\sum_{i=5}^{31} i = 5+6+7+\cdots+30+31$$

Solution 1 (Re-Indexing):

$$\begin{aligned}
\sum_{i=5}^{31} i &= 5+6+7+\cdots+30+31 \\
&= (1+4)+(2+4)+(3+4)+\cdots+(26+4)+(27+4) \\
&= \sum_{j=1}^{27}(j+4) && \text{let } j = i-4 \\
&= 27\cdot 4 + \frac{27\cdot 28}{2} \\
&= 108+378 \\
&= 486
\end{aligned}$$

Solution 2 (Adding Zero):

$$\begin{aligned}
\sum_{i=5}^{31} i &= 5+6+7+\cdots+30+31 \\
&= (1+2+3+4+5+6+7+\cdots+30+31)-(1+2+3+4) \\
&= \left(\sum_{i=1}^{31} i\right) - \left(\sum_{i=1}^{4} i\right) \\
&= \left(\frac{31\cdot 32}{2}\right) - \left(\frac{4\cdot 5}{2}\right) \\
&= 496-10 \\
&= 486
\end{aligned}$$

Using Formulas, a Geometric Example

Find the sum of:

$$\sum_{i=10}^{106}\left(\frac{3}{7}\right)^i = \left(\frac{3}{7}\right)^{10} + \left(\frac{3}{7}\right)^{11} + \cdots + \left(\frac{3}{7}\right)^{106}$$

Solution 1 (Re-Indexing):

$$\sum_{i=10}^{106}\left(\frac{3}{7}\right)^i = \left(\frac{3}{7}\right)^{10} + \left(\frac{3}{7}\right)^{11} + \cdots + \left(\frac{3}{7}\right)^{106}$$

$$=$$

$$=$$

$$=$$

$$=$$

Solution 2 (Adding Zero):

$$\sum_{i=10}^{106}\left(\frac{3}{7}\right)^i = \left(\frac{3}{7}\right)^{10} + \left(\frac{3}{7}\right)^{11} + \cdots + \left(\frac{3}{7}\right)^{106}$$

$$=$$

$$=$$

$$=$$

$$=$$

Iteration Technique with $a_0 = 4$ and $a_n = 6 \cdot a_{n-1} + 4$

$$
\begin{aligned}
a_4 &= (6 \cdot a_3 + 4) && a_n = 6 \cdot a_{n-1} + 4 \\
&= 6 \cdot (6 \cdot a_2 + 4) + 4 && a_n = 6 \cdot a_{n-1} + 4 \\
&= 6^2 \cdot a_2 + 6 \cdot 4 + 4 \\
&= 6^2 \cdot (6 \cdot a_1 + 4) + 6 \cdot 4 + 4 && a_n = 6 \cdot a_{n-1} + 4 \\
&= 6^3 \cdot a_1 + 6^2 \cdot 4 + 6 \cdot 4 + 4 \\
&= 6^3 \cdot (6 \cdot a_0 + 4) + 6^2 \cdot 4 + 6 \cdot 4 + 4 && a_n = 6 \cdot a_{n-1} + 4 \\
&= 6^4 \cdot a_0 + 6^3 \cdot 4 + 6^2 \cdot 4 + 6 \cdot 4 + 4 \\
&= 6^4 \cdot 4 + 6^3 \cdot 4 + 6^2 \cdot 4 + 6 \cdot 4 + 4 && a_0 = 4 \\
&= \sum_{i=0}^{4} 4 \cdot 6^i = 4\left(\frac{6^5 - 1}{6 - 1}\right) \\
a_k &= \sum_{i=0}^{k} 4 \cdot 6^i = 4\left(\frac{6^{k+1} - 1}{6 - 1}\right)
\end{aligned}
$$

Iteration Technique with $b_0 = 2$ and $b_n = -2 \cdot b_{n-1} + 2$

Use iteration to show that

$$b_n = 2\left(\frac{(-2)^{n+1} - 1}{-2 - 1}\right).$$

remember to always replace b_n with $-2 \cdot b_{n-1} + 2$ if $n \neq 0$ and 2 if it does.

$$
\begin{aligned}
b_4 &= \\
&= \\
&= \\
&= \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

Trying the Iteration Technique with $F_1 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$

$$
\begin{aligned}
F_5 &= F_4 + F_3 && F_n = F_{n-1} + F_{n-2} \\
&= (F_3 + F_2) + (F_2 + F_1) && F_n = F_{n-1} + F_{n-2} \\
&= F_3 + 2 \cdot F_2 + F_1 && \\
&= (F_2 + F_1) + 2 \cdot (F_1 + F_0) + F_1 && F_n = F_{n-1} + F_{n-2} \\
&= F_2 + 4 \cdot F_1 + 2 \cdot F_0 && \\
&= (F_1 + F_0) + 4 \cdot F_1 + 2 \cdot F_0 && F_n = F_{n-1} + F_{n-2} \\
&= 5 \cdot F_1 + 3 \cdot F_0 &&
\end{aligned}
$$

But $F_4 = 5$ and $F_3 = 3$ so this doesn't really tell us anything.

Theorem 2 (Distinct Roots Version)**.** *Given r_0, r_1, and $r_n = A \cdot r_{n-1} + B \cdot r_{n-2}$, if the roots of*

$$x^2 - Ax - B = 0,$$

are distinct values s_0 and s_1, then $r_n = C \cdot s_0^n + D \cdot s_1^n$ for appropriate values of C and D.

Using Characteristic Polynomials with $F_0 = 1$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$

We have $F_n = F_{n-1} + F_{n-2}$ so that $A = B = 1$ and we need the roots of $x^2 - x - 1$ which are

$$x = \frac{1 \pm \sqrt{5}}{2}$$

and by theorem 2

$$F_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Then

$$
\begin{aligned}
F_0 &= C + D = 1 \text{ and} \\
F_1 &= C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right) = 1.
\end{aligned}
$$

Solving for C and D we get

$$C = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) \text{ and } D = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right).$$

Therefore the closed formula for the Fibonacci sequence is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

Using Characteristic Polynomials with $G_0 = 2$, $G_1 = 3$ and $G_n = 4 \cdot G_{n-1} + 5 \cdot G_{n-2}$

Using theorem 2 as before, we have $G_n = 4 \cdot G_{n-1} + 5 \cdot G_{n-2}$ so that $A = 4$, $B = 5$ and we find the roots of $x^2 - 4x - 5$ which are $x = 5$ and $x = -1$. Then

Theorem 3 (Single Root Version). *Given r_0, r_1, and $r_n = A \cdot r_{n-1} + B \cdot r_{n-2}$, if the only root of*

$$x^2 - Ax - B = 0,$$

is the value s_0, then $r_n = C \cdot s_0^n + D \cdot n \cdot s_0^n$ for appropriate values of C and D.

Using Characteristic Polynomials with $H_0 = 5$, $H_1 = 4$ and $H_n = 4 \cdot H_{n-1} - 4 \cdot H_{n-2}$

Using theorem 3, we have $H_n = 4 \cdot H_{n-1} - 4 \cdot H_{n-2}$ so that $A = 4$, $B = -4$ and we need the roots of $x^2 - 4x + 4$ which are $x = 2$ and $x = 2$. Then