Discrete Math Review

Dr. Chuck Rocca

roccac@wcsu.edu

http://sites.wcsu.edu/roccac

WESTERN CONNECTICUT STATE UNIVERSITY

MACRICOSTAS SCHOOL OF ARTS & SCIENCES

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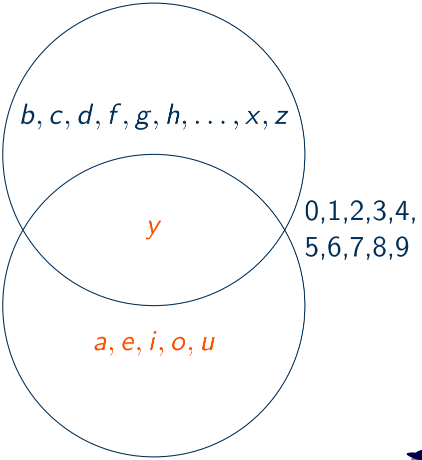
Sets

- Sets:
  $A = \{a, e, i, o, u, y\}$ and
  $B = \{b, c, d, f, g, h, \ldots, z\}$



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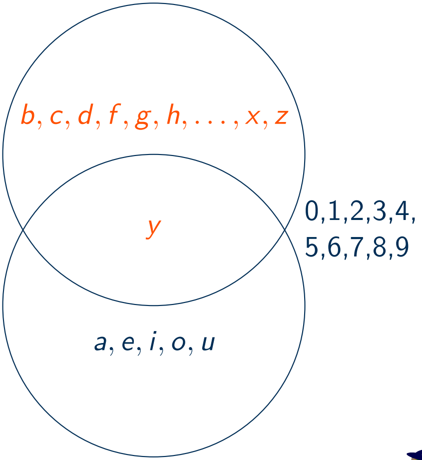

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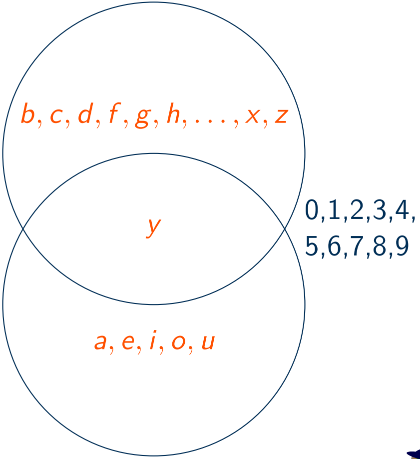
  $A \cup B = \{a, b, c, d, e, \ldots, z\}$



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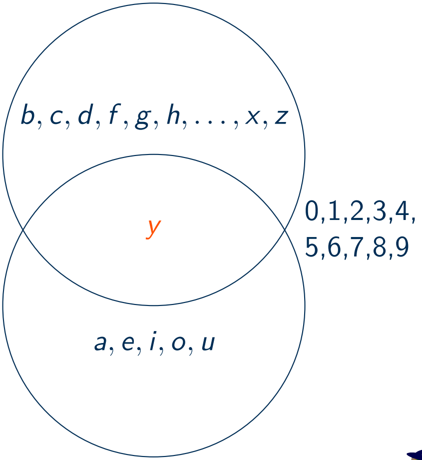

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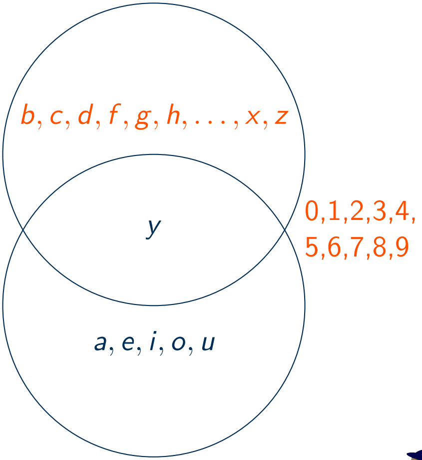

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  $A \cap B = \{y\}$
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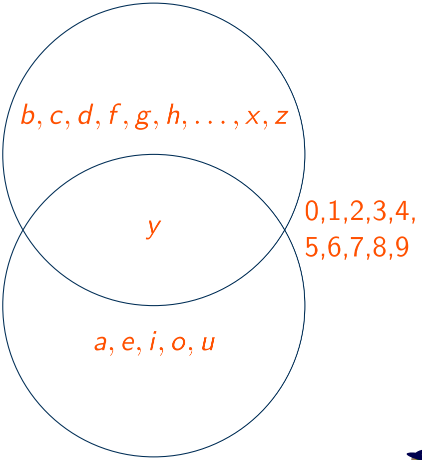
  $A^c = (B \setminus \{y\}) \cup \{0, 1, \ldots, 9\}$
- Universal Set:
  $\mathscr{U} = A \cup B \cup \{0, 1, \ldots, 9\}$



New Sets from Old

- $A = \{a, b, c\}$ and $B = \{0, 1, 2\}$

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- $A = \{a, b, c\}$ and $B = \{0, 1, 2\}$
- Cartesian Product:

$$A \times B = \{(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2), (c, 0), (c, 1), (c, 2)\}$$

New Sets from Old

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- Cartesian Product:

$$A \times B = \{(a,0),(a,1),(a,2),(b,0),(b,1),(b,2),(c,0),(c,1),(c,2)\}$$

- Power Set:

$$\mathscr{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

$$|\mathscr{P}(A)| = 2^{|A|}$$

Cardinality of a Power Set: $|\mathscr{P}(S)| = 2^{|S|}$

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- $A = \{0, 1\}$ and $B = \{0, 1, 2\}$
- $\mathscr{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
- $\mathscr{P}(B) = ?$

$$\begin{aligned}\mathscr{P}(B) &= \mathscr{P}(A) \cup \left(\bigcup_{s \in \mathscr{P}(A)} \{s \cup \{2\}\} \right) \\ &= \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} \cup \{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\} \\ &= \{\emptyset, \{0\}, \{1\}, \{0, 1\}, \{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}\end{aligned}$$

Cardinality of a Power Set: $|\mathscr{P}(S)| = 2^{|S|}$

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$$\begin{aligned} |\mathscr{P}(B)| &= |\mathscr{P}(A)| + \left| \bigcup_{s \in \mathscr{P}(A)} \{s \cup \{2\}\} \right| \\ &= |\mathscr{P}(A)| + \sum_{s \in \mathscr{P}(A)} |\{s \cup \{2\}\}| \\ &= |\mathscr{P}(A)| + |\mathscr{P}(A)| \\ &= 2 \cdot |\mathscr{P}(A)| \end{aligned}$$

Cardinality of a Power Set: $|\mathscr{P}(S)| = 2^{|S|}$

- $A = \{0, 1\}$ and $B = \{0, 1, 2\}$
- $\mathscr{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
- $\mathscr{P}(B) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
- $|\mathscr{P}(B)| = 2 \cdot |\mathscr{P}(A)| = 2 \cdot 2^{|A|} = 2^{|A|+1} = 2^{|B|}$

Relations

Definition (Relation)

A **relation** between two sets is a subset of their Cartesian product.

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A sample relation might be:

$$\mathcal{R} = \{(a,0),(a,1),(a,2),(b,1),(b,2),(c,2)\}$$

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A **relation** between two sets is a subset of their Cartesian product.

Given:

$$A \times A = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$$

A sample relation might be:

$$\mathcal{O} = \{(a,b),(a,c),(b,c)\}$$

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

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A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

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A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{O} = \{(a,b),(a,c),(b,c)\}$$

Since a does not relate to its self ($a \not\sim a$) this is not **reflexive**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{O} = \{(a, b), (a, c), (b, c)\}$$

Since a relates to b ($a \sim b$) but b does not relate to a ($b \nsim a$) this is not **symmetric**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{O} = \{(a, b), (a, c), (b, c)\}$$

Since $a \sim b$ and $b \sim c$ and $a \sim c$ this is **transitive**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{C} = \{(a, a), (a, b), (b, a), (b, b), (c, c)\}$$

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{C} = \{(a,a),(a,b),(b,a),(b,b),(c,c)\}$$

Since $a \sim a$, $b \sim b$, and $c \sim c$ this is **reflexive**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{C} = \{(a,a),(a,b),(b,a),(b,b),(c,c)\}$$

Since $a \sim b$ and $b \sim a$ this is **symmetric**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{C} = \{(a, a), (a, b), (b, a), (b, b), (c, c)\}$$

Since $a \sim b$, $b \sim a$ and $a \sim a$ (also, $b \sim a, a \sim b$, and $b \sim b$) this is **transitive**.

Equivalence Relation

Definition (Equivalence Relation)

A relation between a set and its self is an **equivalence relation** if and only if it is **reflexive**, **symmetric**, and **transitive**.

Given the relation on A:

$$\mathcal{C} = \{(a, a), (a, b), (b, a), (b, b), (c, c)\}$$

This relation is am **equivalence relation**.

Function

Definition (Function)

A **function** is a relation between two sets, the first called the **domain** and the second the **co-domain**, such that for all x in the domain there exists a unique y in the co-domain such that (x, y) is in the relation.

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Given:

$$A \times B = \{(a,0),(a,1),(a,2),(b,0),(b,1),(b,2),(c,0),(c,1),(c,2)\}$$

The relation:

$$\mathcal{R} = \{(a,0),(a,1),(a,2),(b,1),(b,2),(c,2)\}$$

is not a function

Function

Definition (Function)

A **function** is a relation between two sets, the first called the **domain** and the second the **co-domain**, such that for all x in the domain there exists a unique y in the co-domain such that (x, y) is in the relation.

Given:

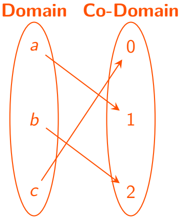
$$A \times B = \{(a,0),(a,1),(a,2),(b,0),(b,1),(b,2),(c,0),(c,1),(c,2)\}$$

But, the relation:

$$\mathcal{S} = \{(a,1),(b,2),(c,0)\}$$

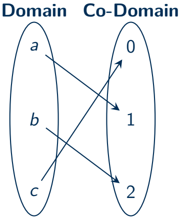
is a function

Visualizing Functions



$1-1$ & *onto*

Visualizing Functions



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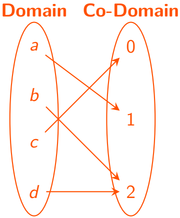

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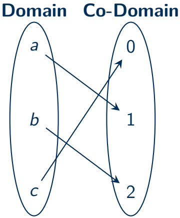

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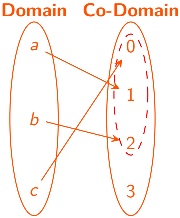

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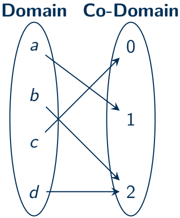

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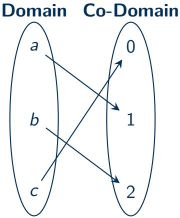
Visualizing Functions



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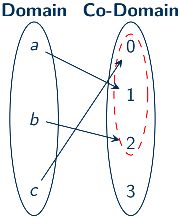

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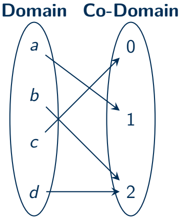

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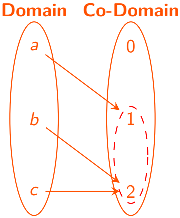

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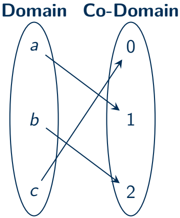

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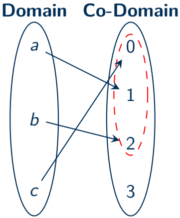

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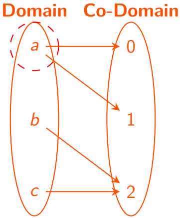
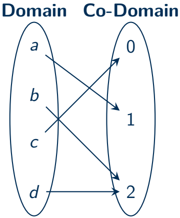


Not a function



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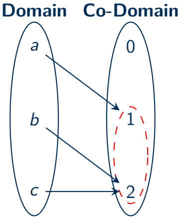

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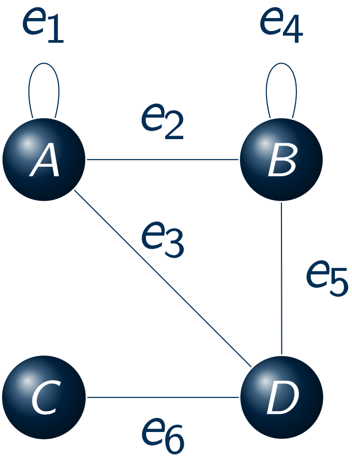
Graphs



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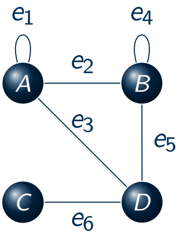
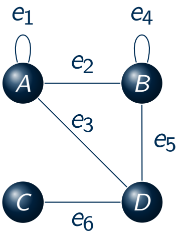


- Vertex Set:

$$V = \{A, B, C, D\}$$

Graphs



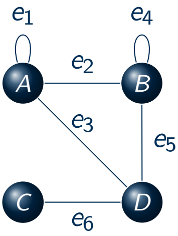
- Vertex Set:

$$V = \{A, B, C, D\}$$

- Edge Set:

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

Graphs



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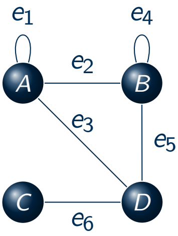
$$E = \{(A, A), (A, B), (A, D), (B, B), (B, D), (C, D)\}$$

Graphs



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- Edge Set:

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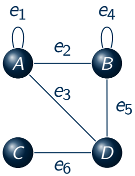
- Graph:

$$\begin{aligned} G &= (V, E) \\ &= (\{A, B, C, D\}, \\ &\qquad \{(A, A), (A, B), (A, D), (B, B), (B, D), (C, D)\}) \end{aligned}$$

Types of Graphs

Graph



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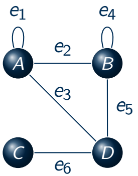
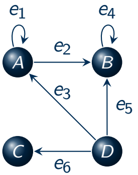


Directed Graph



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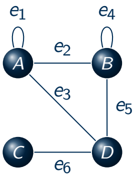
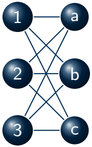


Bipartite Graph



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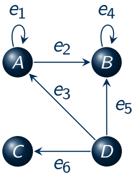

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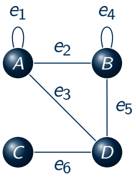

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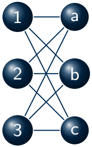

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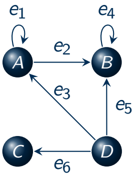
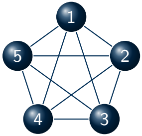


Complete Graph



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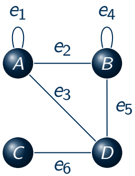
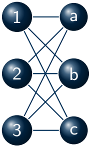
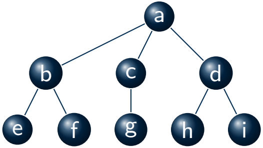


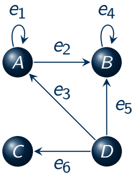
Bipartite Graph



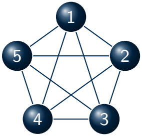
Tree



Directed Graph

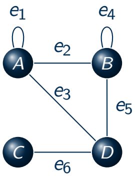


Complete Graph



Types of Graphs

Graph



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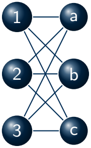

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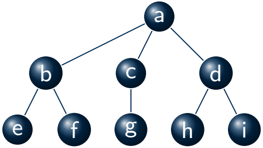

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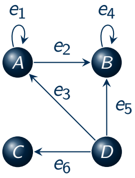
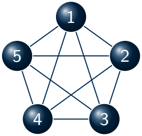
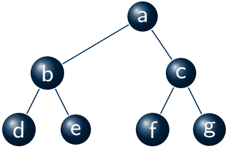


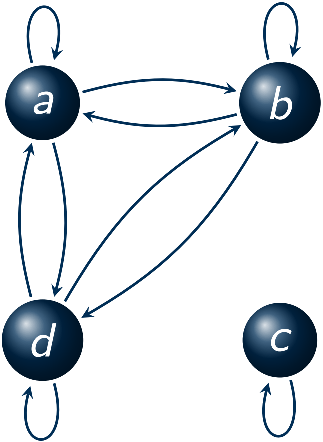
Complete Graph



Binary Tree



Relations as Graphs



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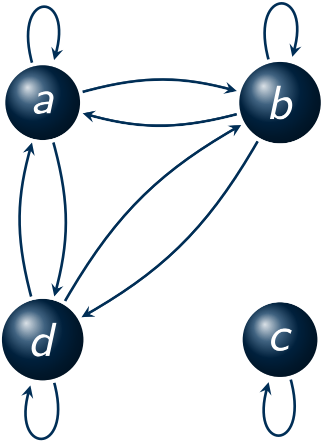
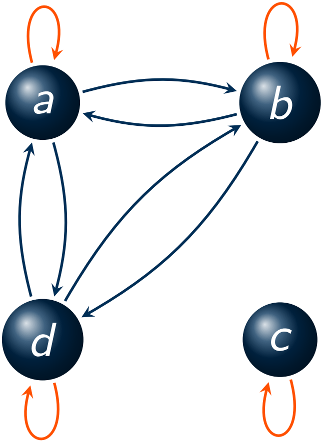


Equivalence Relation?

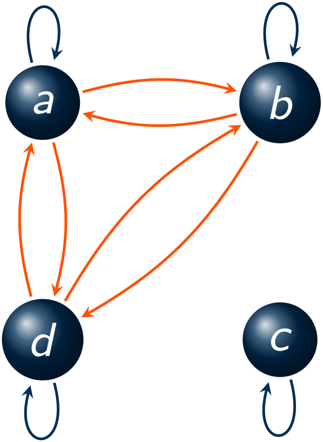
Relations as Graphs



Equivalence Relation?

- Reflexive ✓

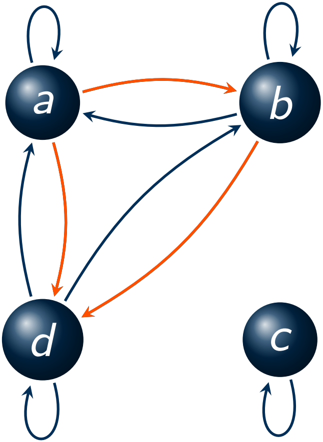
Relations as Graphs



Equivalence Relation?

- Reflexive ✓
- Symmetric ✓

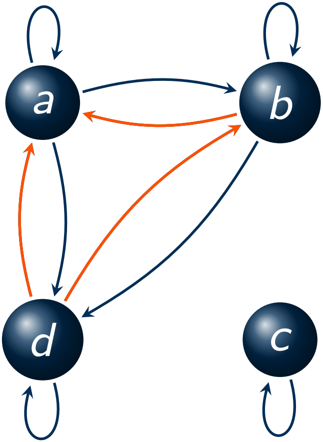
Relations as Graphs



Equivalence Relation?

- Reflexive ✓
- Symmetric ✓
- Transitive ✓

Relations as Graphs



Equivalence Relation?

- Reflexive ✓
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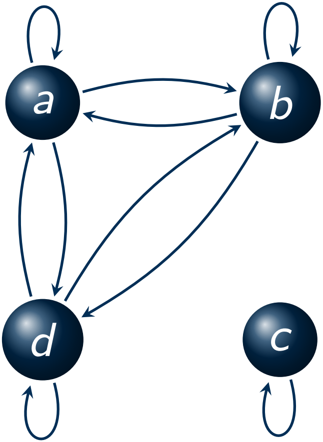
Relations as Graphs



Equivalence Relation✓

- Reflexive ✓
- Symmetric ✓
- Transitive ✓
- Equivalence Classes

$$A = \{a, b, d\} \;\&\; C = \{c\}$$

Table of Contents

Direct Proof

Theorem (De Morgan's Law)

Given two sets A and B, the complement of their union is equal to the intersection of their complements:

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Now suppose $x \in A^c \cap B^c$ or equivalently $x \in A^c$ and $x \in B^c$.

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Now suppose $x \in A^c \cap B^c$ or equivalently $x \in A^c$ and $x \in B^c$. This tells us that $x \notin A$ and $x \notin B$ and thus $x \notin A \cup B$, i.e. $x \in (A \cup B)^c$.

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Now suppose $x \in A^c \cap B^c$ or equivalently $x \in A^c$ and $x \in B^c$. This tells us that $x \notin A$ and $x \notin B$ and thus $x \notin A \cup B$, i.e. $x \in (A \cup B)^c$. Therefore, $A^c \cap B^c \subseteq (A \cup B)^c$ and

$$(A \cup B)^c = A^c \cap B^c$$

as desired.

By Cases

Theorem

Given any integer n, either n^2 or $n^2 - 1$ is divisible by four.

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Given any integer n, either n^2 or $n^2 - 1$ is divisible by four.

Proof: (Case 1) Let n be an even integer so that we may write $n = 2k$ for some unique k. Then

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and n^2 is divisible by four.

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Given any integer n, either n^2 or $n^2 - 1$ is divisible by four.

Proof: (Case 1) Let n be an even integer so that we may write $n = 2k$ for some unique k. Then

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and n^2 is divisible by four.
(Case2) Now, if n is an odd integer then we write $n = 2k + 1$ for some unique k. Thus,

$$n^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k)$$

and $n^2 - 1$ is divisible by four.

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and $n^2 - 1$ is divisible by four.

Therefore, for any integer n we have shown that either n^2 or $n^2 - 1$ is divisible by four.

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If n^2 is even, then n is even.

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Proof: Suppose that n is odd and is written $n = 2k + 1$ for some unique k. Then we can write

$$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

which is odd. Therefore, if n is odd, then n^2 is odd and so if n^2 is even, then n is even.

Contradiction

Theorem

No integer is both even and odd.

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Proof: Suppose that n is both even and odd so that $n = 2k$ and $n = 2l + 1$ for some unique k and l.

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No integer is both even and odd.

Proof: Suppose that n is both even and odd so that $n = 2k$ and $n = 2l + 1$ for some unique k and l. Then we can write $2k = 2l + 1$ and $1 = 2(k - l)$. If $k - l = 0$, then $1 = 0$ and if $k - l \neq 0$, then 2 divides 1.

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Proof: Suppose that n is both even and odd so that $n = 2k$ and $n = 2l + 1$ for some unique k and l. Then we can write $2k = 2l + 1$ and $1 = 2(k - l)$. If $k - l = 0$, then $1 = 0$ and if $k - l \neq 0$, then 2 divides 1. In either case we derive a contradiction and therefore no integer is both even and odd.

Induction

Theorem

Any tree with n vertices has $n-1$ edges.

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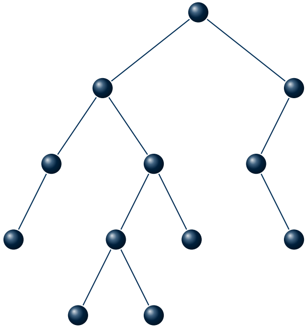
(Base Case) When there is only one vertex there are no edges since trees do not contain loops and there is not a second vertex to connect to.

Induction

Theorem

Any tree with n vertices has $n - 1$ edges.



(Induction Step) Assume that the theorem is true for some $k \geq 2$ and consider a tree with $k + 1 \geq 3$ vertices.

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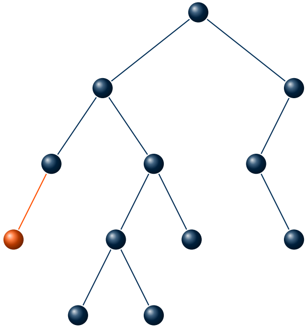

(Induction Step) Assume that the theorem is true for some $k \geq 2$ and consider a tree with $k + 1 \geq 3$ vertices. Since there are at least two vertices the tree must contain at least one leaf.

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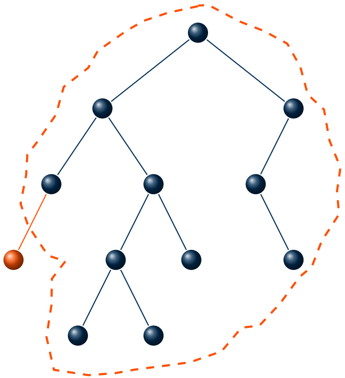

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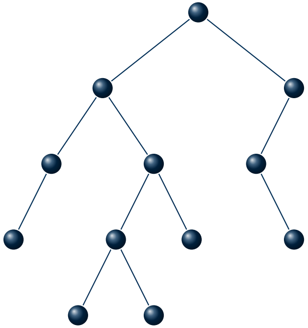

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Table of Contents

Next Class

- Deterministic and Non-Deterministic Finite State Automata

Discrete Math Review

Dr. Chuck Rocca
roccac@wcsu.edu
http://sites.wcsu.edu/roccac

WESTERN CONNECTICUT STATE UNIVERSITY
MACRICOSTAS SCHOOL OF ARTS & SCIENCES