Definition 1. Given $a, b \in \mathbb{Z}$, we say that b *divides* a if there exists an unique $q \in \mathbb{Z}$, called the quotient, such that $a = qb$. When b divides a we write $b|a$.

Examples & Non-Examples:

1. $5|60$ since $60 = 12(5)$ ✔
2. $5 \nmid 63$ since $12(5) < 63 < 13(5)$ ✗
3. 4 ___ 7 since ________________
4. 4 ___ 16 since ________________
5. -3 ___ 24 since ________________
6. 5 ___ -17 since ________________
7. 12 ___ 4 since ________________
8. -2 ___ 20 since ________________

Lemma 1. *Given integers a, b, and c with $c \neq 0$, if $c|a$ and $c|b$, then $c|a + b$.*

Examples:

Non-Examples:

Proof. Let $a, b, c \in \mathbb{Z}$, assume $c \neq 0$, $c|a$ and $c|b$ so there exists $q_a, q_b \in \mathbb{Z}$ with $a = q_a c$ and $b = q_b c$. **(1)** Then we can write:

$$a + b = q_a c + q_b c^{\textbf{(2)}} \tag{1}$$

$$= (q_a + q_b)c^{\textbf{(3)}}. \tag{2}$$

So, $(a + b) = qc$, where $q = q_a + q_b$. Now, if $(a + b) = q_1 c = q_2 c$, then $q_1 = q_2$. **(4)** Therefore, we may conclude $c|(a + b)$. **(5)** □

Detail Explanations:

Possible Extensions:

Conjecture (Product Divisibility Conjecture). Given $a, b, c \in \mathbb{Z}$, if $c \neq 0$ and $c|ab$, then $c|a$ or $c|b$.

Definition 2. A ***base b place value number system*** is a method of representing numbers as a string of symbols where the symbols in a given position, or place, can represent the numbers 0 through $(b-1)$ and each place corresponds to a particular power of b. Notationally we will write a number in such a system as

$$(a_k\, a_{k-1}\, a_{k-2}\, \ldots\, a_0)_b$$

where a_i is the symbol or collection of symbols representing the number of copies of b^i which are needed in order to represent the number.

Examples:

1. Base 10: $123 = (1, 2, 3)_{10}$ represents 1 copy of 10^2, 2 copies of 10^1, and 3 copies of 10^0. In base 13 this would be
$$(9, 6)_{13} = 9 \times 13^1 + 6 \times 13^0.$$

2. Base 2: $(1, 0, 0, 1)_2$ represents 1 copy of 2^3, 0 copies of 2^2 and 2^1, and 1 copy of 2^0. In base 10 this is
$$2^3 + 2^0 = 8 + 1 = 9.$$

3. Base 16: $(10, 15, 4, 11)_{16}$ represents 10 copies of 16^3, 15 copies of 16^2, 4 copies of 16^1, and 11 copies of 16^0. In base 10 this is
$$10 \times 16^3 + 15 \times 16^2 + 4 \times 16^1 + 11 \times 16^0 = 44875.$$

4. Base 13: $(7, 3, 10)_{13}$ represents

   In base 10 this is

5. Base 8: $(4, 7, 1)_8$ represents

   In base 10 this is

6. Base 10: 36 represents

   In base 7 this is

7. Base 10: 100 represents

   In base 11 this is

Theorem 2 (Divisibility by $b-1$)**.** *In a base b place value number system, a number is divisible by $(b-1)$ if the sum of its symbols is divisible by $(b-1)$.*

Examples:

Non-Examples:

There are multiple ways to prove theorem 2; here we will use the ***Binomial Theorem***.

Theorem 3 (Binomial Theorem)**.** *Given arbitrary $a, b \in \mathbb{R}$ and exponent n, we may write:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

assuming $(a+b)$ and n are not both 0.

Examples

1. $(x+2)^2 =$

2. $(b+1)^3 =$

3. $(y-1)^4 =$

4. $11^5 = (10+1)^5 =$

5. $6^7 = (7-1)^7 =$

Proof. Let $(a_k\, a_{k-1}\, a_{k-2}\, \ldots\, a_0)_b$ be a number in a base b number system. We may write this number as the sum

$$\sum_{i=0}^{k} a_i\, b^i. \text{ (6)}$$

Applying the *Binomial Theorem*, we can write b^i as

$$b^i = ((b-1)+1)^i = \sum_{l=0}^{i} \binom{i}{l} (b-1)^{i-l} 1^l = Q_i(b-1) + 1, \tag{3}$$

for an appropriate value of Q_i. (7) Using this we may write:

$$\sum_{i=0}^{k} a_i\,(Q_i(b-1)+1) = \sum_{i=0}^{k} a_i\, Q_i(b-1) + a_i \text{(8)} \tag{4}$$

$$= \sum_{i=0}^{k} a_i\, Q_i(b-1) + \sum_{i=0}^{k} a_i \text{(9)} \tag{5}$$

$$= (b-1) \sum_{i=0}^{k} a_i\, Q_i + \sum_{i=0}^{k} a_i. \tag{6}$$

Since the first term in the expression on line (6) is a multiple of $(b-1)$ we may conclude that the entire expression is divisible by $(b-1)$ if and only if $(b-1)$ divides $\sum_{i=0}^{k} a_i$, which is what we wished to prove. (10) □

Detail Explanations:

Possible Extensions:

Conjecture (Reverse Order Conjecture)*.* In a base b place value number system twice the number $(b-1)$ is written with the same symbols as $(b-1)^2$, only the symbols are in the opposite order.