Algebraic Structures Introduction and Overview v0.2

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Contents

Introduction and Directions

In the following sections you will review calculations in some, hopefully, familiar sets. After each group of calculations you will answer some questions, about arithmetic in those sets, which will help you see the ways in which the sets are similar and the ways in which they are different.

After practicing the arithmetic and completing the observation questions we introduce four basic types of ***algebraic structures***, and a couple variations, which help give names and categories to our observations. You will then assign each of the sets with their operations to one or more structure. Finally, at the very end you are asked to reflect on how you can alter some of the sets in order to get them to fit into other structures.

For this packet you should treat it like a series of worksheets. This means that you will *write* all your answers right on these pages. Your work needs to be neat and legible. To help keep your work neat you should use scrap paper to figure out what you want to write and then write it down neatly on these pages that you will turn in.

1 Integers

In this section consider the set $R = \mathbb{Z}$ with the usual addition and multiplication.

Calculations

Fill in the calculations below.

$+$	-2	-1	0	1	2	3
-2						
-1						
0						
1						
2						
3						

$\times$	-2	-1	0	1	2	3
-2						
-1						
0						
1						
2						
3						

Equations

Try to solve each of these using only numbers in the given set R. Do any of them have more than one solution?

1. $2x = 5$
2. $5x = 3$
3. $x + 3 = 0$
4. $x + 3 = 3$
5. $2x = 1$
6. $5x = 1$
7. $3x = 0$
8. $5x = 0$

Observations

Answer yes or no for each question. If you answer no, give an example.

1. For all a, b in the set R, is $a + b$ in the set?

2. For all a, b in the set R, is $a \times b$ in the set?

3. For all a, b in the set R does $a + b = b + a$?

4. For all a, b in the set R does $a \times b = b \times a$?

5. Does there exists 0_R for all a in the set R such that $a + 0_R = 0_R + a = a$?

6. For all a does there exist b so that $a + b = b + a = 0_R$, with 0_R as above?

7. Is it always true that $a \times b = 0_R$ implies $a = 0_R$ or $b = 0_R$, with 0_R as above?

8. Does there exists 1_R for all a in the set R such that $a \times 1_R = 1_R \times a = a$?

9. For all $a \neq 0_R$ does there exist b so that $a \times b = b \times a = 1_R$, with 1_R as above?

2 Rationals

In this section consider the set $R = \mathbb{Q}$ with the usual addition and multiplication.

Calculations

Fill in the calculations below.

$+$	$-3/5$	$-1/2$	0	1	2	5
-5						
-2						
0						
1						
$1/2$						
$3/5$						

$\times$	$-3/5$	$-1/2$	0	1	2	5
-2						
$-5/3$						
0						
1						
$1/5$						
$1/2$						

Equations

Try to solve each of these using only numbers in the given set R. Do any of them have more than one solution?

1. $2x = 5$
2. $5x = 3$
3. $x + 3 = 0$
4. $x + 3 = 3$
5. $2x = 1$
6. $5x = 1$
7. $3x = 0$
8. $5x = 0$

Observations

Answer yes or no for each question. If you answer no, give an example.

1. For all a, b in the set R, is $a + b$ in the set?

2. For all a, b in the set R, is $a \times b$ in the set?

3. For all a, b in the set R does $a + b = b + a$?

4. For all a, b in the set R does $a \times b = b \times a$?

5. Does there exists 0_R for all a in the set R such that $a + 0_R = 0_R + a = a$?

6. For all a does there exist b so that $a + b = b + a = 0_R$, with 0_R as above?

7. Is it always true that $a \times b = 0_R$ implies $a = 0_R$ or $b = 0_R$, with 0_R as above?

8. Does there exists 1_R for all a in the set R such that $a \times 1_R = 1_R \times a = a$?

9. For all $a \neq 0_R$ does there exist b so that $a \times b = b \times a = 1_R$, with 1_R as above?

3 Complex Numbers

In this section consider the set $R = \mathbb{C}$ with the usual addition and multiplication, i.e.

$$(a+bi)+(c+di) = (a+c)+(b+d)i \text{ and } (a+bi)(c+di) = (ac-bd)+(ad+bc)i.$$

Calculations

Fill in the calculations below.

$+$	3i	1+i	0	1	-3-4i
$-3i$					
$1-i$					
0					
1					
$-3+4i$					

$\times$	3i	1+i	0	1	-3-4i
$-3i$					
$1-i$					
0					
1					
$-3+4i$					

- $(a+bi) \times (a-bi) =$
- $(bi) \times (-bi) =$
- $1/(a+bi) =$
- $1/(bi) =$

Equations

Try to solve each of these using only numbers in the given set R. Do any of them have more than one solution?

1. $2x = 5$
2. $5x = 3$
3. $x + 3 = 0$
4. $x + 3 = 3$
5. $2x = 1$
6. $5x = 1$
7. $3x = 0$
8. $5x = 0$

Observations

Answer yes or no for each question. If you answer no, give an example.

1. For all a, b in the set R, is $a + b$ in the set?

2. For all a, b in the set R, is $a \times b$ in the set?

3. For all a, b in the set R does $a + b = b + a$?

4. For all a, b in the set R does $a \times b = b \times a$?

5. Does there exists 0_R for all a in the set R such that $a + 0_R = 0_R + a = a$?

6. For all a does there exist b so that $a + b = b + a = 0_R$, with 0_R as above?

7. Is it always true that $a \times b = 0_R$ implies $a = 0_R$ or $b = 0_R$, with 0_R as above?

8. Does there exists 1_R for all a in the set R such that $a \times 1_R = 1_R \times a = a$?

9. For all $a \neq 0_R$ does there exist b so that $a \times b = b \times a = 1_R$, with 1_R as above?

4 Integers Modulo Six

In this section consider the set $R = \mathbb{Z}_6$ with the usual addition and multiplication.

Calculations

Fill in the calculations below.

$+_6$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

$\times_6$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

Equations

Try to solve each of these using only numbers in the given set R. Do any of them have more than one solution?

1. $2x = 5$
2. $5x = 3$
3. $x + 3 = 0$
4. $x + 3 = 3$
5. $2x = 1$
6. $5x = 1$
7. $3x = 0$
8. $5x = 0$

Observations

Answer yes or no for each question. If you answer no, give an example.

1. For all a, b in the set R, is $a + b$ in the set?

2. For all a, b in the set R, is $a \times b$ in the set?

3. For all a, b in the set R does $a + b = b + a$?

4. For all a, b in the set R does $a \times b = b \times a$?

5. Does there exists 0_R for all a in the set R such that $a + 0_R = 0_R + a = a$?

6. For all a does there exist b so that $a + b = b + a = 0_R$, with 0_R as above?

7. Is it always true that $a \times b = 0_R$ implies $a = 0_R$ or $b = 0_R$, with 0_R as above?

8. Does there exists 1_R for all a in the set R such that $a \times 1_R = 1_R \times a = a$?

9. For all $a \neq 0_R$ does there exist b so that $a \times b = b \times a = 1_R$, with 1_R as above?

5 Integers Modulo Seven

In this section consider the set $R = \mathbb{Z}_7$ with the usual addition and multiplication.

Calculations

Fill in the calculations below.

$+_7$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

$\times_7$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

Equations

Try to solve each of these using only numbers in the given set R. Do any of them have more than one solution?

1. $2x = 5$
2. $5x = 3$
3. $x + 3 = 0$
4. $x + 3 = 3$
5. $2x = 1$
6. $5x = 1$
7. $3x = 0$
8. $5x = 0$

Observations

Answer yes or no for each question. If you answer no, give an example.

1. For all a, b in the set R, is $a + b$ in the set?

2. For all a, b in the set R, is $a \times b$ in the set?

3. For all a, b in the set R does $a + b = b + a$?

4. For all a, b in the set R does $a \times b = b \times a$?

5. Does there exists 0_R for all a in the set R such that $a + 0_R = 0_R + a = a$?

6. For all a does there exist b so that $a + b = b + a = 0_R$, with 0_R as above?

7. Is it always true that $a \times b = 0_R$ implies $a = 0_R$ or $b = 0_R$, with 0_R as above?

8. Does there exists 1_R for all a in the set R such that $a \times 1_R = 1_R \times a = a$?

9. For all $a \neq 0_R$ does there exist b so that $a \times b = b \times a = 1_R$, with 1_R as above?

6 Polynomials

In this section consider the set of polynomials with rational coefficients

$$R = \mathbb{Q}[x] = \left\{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \middle| a_i \in \mathbb{Q},\, n \in \mathbb{N}\right\}$$

with the usual addition and multiplication.

Calculations & Equations

1. Simplify $(x^2 + 17x + 1) + (x^3 - x^2 - 17x - 1)$:

2. Simplify $(x - 1)(x^2 + 2)$:

3. Find the quotient and remainder for $(x^6 - 3x^4 - 16) \div (x^4 - 7x^2 + 12)$:

4. Find the quotient and remainder for $(7x^5 + 4x + 9) \div (3x^3 + 2x)$:

5. Is $(5x^4 + 13x + 27) \div (x^2 + 1)$ in $\mathbb{Q}[x]$?

6. Is $(x^4 + 13x^2 - 14) \div (x^2 - 1)$ in $\mathbb{Q}[x]$?

7. Can you find $f(x) \in \mathbb{Q}[x]$ such that $(x^2 + 7)f(x) = (x^2 + 7)$ for all x?

8. Can you find $f(x) \in \mathbb{Q}[x]$ such that $(x^2 + 7) + f(x) = (x^2 + 7)$ for all x?

9. Can you find $f(x) \in \mathbb{Q}[x]$ such that $(x^2 + 7)f(x) = 1$ for all x?

Observations

Answer yes or no for each question. If you answer no, give an example.

1. For all a, b in the set R, is $a + b$ in the set?

2. For all a, b in the set R, is $a \times b$ in the set?

3. For all a, b in the set R does $a + b = b + a$?

4. For all a, b in the set R does $a \times b = b \times a$?

5. Does there exists 0_R for all a in the set R such that $a + 0_R = 0_R + a = a$?

6. For all a does there exist b so that $a + b = b + a = 0_R$, with 0_R as above?

7. Is it always true that $a \times b = 0_R$ implies $a = 0_R$ or $b = 0_R$, with 0_R as above?

8. Does there exists 1_R for all a in the set R such that $a \times 1_R = 1_R \times a = a$?

9. For all $a \neq 0_R$ does there exist b so that $a \times b = b \times a = 1_R$, with 1_R as above?

7 Matrices

In this section consider the set of 2×2 matrices with integer entries

$$R = M_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| a_{ij} \in \mathbb{Z} \right\}$$

with the usual addition and multiplication.

Calculations & Equations

1. Complete the calculation:
$\begin{pmatrix} 3 & 7 \\ 1 & 12 \end{pmatrix} + \begin{pmatrix} 3 & 9 \\ 0 & 2 \end{pmatrix} =$

2. Complete the calculation:
$\begin{pmatrix} 3 & 7 \\ 1 & 12 \end{pmatrix} + \begin{pmatrix} -3 & -7 \\ -1 & -12 \end{pmatrix} =$

3. Complete the calculation:
$\begin{pmatrix} 3 & 7 \\ 1 & 12 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} =$

4. Complete the calculation:
$\begin{pmatrix} 3 & 7 \\ 1 & 12 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} =$

5. Complete the calculation:
$\begin{pmatrix} 3 & 7 \\ 1 & 12 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$

6. Complete the calculation:
$$\begin{pmatrix} 3 & 7 \\ 1 & 12 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} =$$

7. Complete the calculation:
$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 3 & 7 \\ 1 & 12 \end{pmatrix} =$$

8. Complete the calculation:
$$\begin{pmatrix} 3 & 11 \\ 1 & 4 \end{pmatrix} \times \begin{pmatrix} 4 & -11 \\ -1 & 3 \end{pmatrix} =$$

9. Complete the calculation:
$$\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} 4 & -6 \\ -2 & 3 \end{pmatrix} =$$

10. Complete the calculation:
$$\begin{pmatrix} 7 & 3 \\ 5 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & -3 \\ -5 & 7 \end{pmatrix} =$$

Observations

Answer yes or no for each question. If you answer no, give an example.

1. For all a, b in the set R, is $a + b$ in the set?

2. For all a, b in the set R, is $a \times b$ in the set?

3. For all a, b in the set R does $a + b = b + a$?

4. For all a, b in the set R does $a \times b = b \times a$?

5. Does there exists 0_R for all a in the set R such that $a + 0_R = 0_R + a = a$?

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7. Is it always true that $a \times b = 0_R$ implies $a = 0_R$ or $b = 0_R$, with 0_R as above?

8. Does there exists 1_R for all a in the set R such that $a \times 1_R = 1_R \times a = a$?

9. For all $a \neq 0_R$ does there exist b so that $a \times b = b \times a = 1_R$, with 1_R as above?

8 Algebraic Structures

Definitions

An objective of ***Abstract Algebra*** is to recognize when calculations in seemingly very different sets are very much the same. Toward this end we classify sets and operations into different ***algebraic structures***. The key four that we are interested in are ***groups***, ***rings***, ***integral domains***, and ***fields***:

Definition 1 (Groups)**.** A set G together with a binary operation $*$ is a ***group*** if for all $a, b, c \in G$:

1. $a * b \in G$, (closure)
2. $a * (b * c) = (a * b) * c$, (associativity)
3. there exists e_G for all a such that $a * e_G = e_G * a = a$, (identity element) and
4. for all a there exists a' such that $a * a' = a' * a = e_G$ (inverse elements).

If in addition $a * b = b * a$ we say the group is an ***abelian group***.

Definition 2 (Rings)**.** A set R together with two binary operations $+$ and $\times$ is a ***ring with unity*** (or ***ring with identity***) if for all $a, b, c \in R$:

1. R with $+$ is an ***abelian group*** with identity 0_R,
2. $a \times b \in R$, (multiplicative closure)
3. $a \times (b \times c) = (a \times b) \times c$, (associativity of multiplication)
4. there exists 1_R for all a such that $a \times 1_R = 1_R \times a = a$, (multiplicative identity element) and
5. $a \times (b + c) = a \times b + a \times c$ and $(b + c) \times a = b \times a + c \times a$, (distributive laws).

If in addition $a \times b = b \times a$ we say the ring is a ***commutative ring***.

Definition 3 (Integral Domains)**.** A set I is an ***integral domain*** if it is a ***commutative ring*** and whenever $a \times b = 0_I$, then either $a = 0$ or $b = 0$. (zero product property)

Definition 4 (Fields)**.** A set F is a ***field*** if it is a ***commutative ring*** and for all $a \in F$ when $a \neq 0_F$ there exists a^{-1} such that $aa^{-1} = a^{-1}a = 1_F$. (multiplicative inverses)

When, in a ring, we have $a \times b = 0$ with $a \neq 0$ and $b \neq 0$ we call a and b ***zero divisors***.

Examples

In section 4 we looked at the integers modulo six and based on that work we can say that

- using just addition, $(\mathbb{Z}_6, +)$, the set is an abelian group,
- using both operations, $(\mathbb{Z}_6, +, \times)$, the set is a commutative ring with unity,
- the set is not an integral domain or field because there are zero divisors ($2 \times 3 \equiv 0 \pmod 6$) and some elements don't have multiplicative inverses ($2x \equiv 1 \pmod 6$ has no solution), and
- the set with just multiplication, $(\mathbb{Z}_6, \times)$, is not a group since there are not multiplicative inverses for all the elements ($2x \equiv 1 \pmod 6$ has no solution).

For each of the other examples, decide what type of algebraic structure the set is or is not when we use one or both of the operations.

	Group	Abelian Group	Ring	Comm. Ring	Int. Domain	Field
$(\mathbb{Z}, +)$						
$(\mathbb{Z}, \times)$						
$(\mathbb{Z}, +, \times)$						
$(\mathbb{Q}, +)$						
$(\mathbb{Q}, \times)$						
$(\mathbb{Q}, +, \times)$						
$(\mathbb{C}, +)$						
$(\mathbb{C}, \times)$						
$(\mathbb{C}, +, \times)$						
$(\mathbb{Z}_6, +)$	✔	✔	✗	✗	✗	✗
$(\mathbb{Z}_6, \times)$	✗	✗	✗	✗	✗	✗
$(\mathbb{Z}_6, +, \times)$	✗	✗	✔	✔	✗	✗
$(\mathbb{Z}_7, +)$						
$(\mathbb{Z}_7, \times)$						
$(\mathbb{Z}_7, +, \times)$						
$(\mathbb{Q}[x], +)$						
$(\mathbb{Q}[x], \times)$						
$(\mathbb{Q}[x], +, \times)$						
$(M_2(\mathbb{Z}), +)$						
$(M_2(\mathbb{Z}), \times)$						
$(M_2(\mathbb{Z}), +, \times)$						

Table 1: Summary of Structure Examples

Give brief justification for your decisions below:

- Section 1: $R = \mathbb{Z}$

- Section 2: $R = \mathbb{Q}$

- Section 3: $R = \mathbb{C}$

- Section 5: $R = \mathbb{Z}_7$

- Section 6: $R = \mathbb{Q}[x]$

- Section 7: $R = M_2(\mathbb{Z})$

You should have noticed that none of the sets were groups using just multiplication. In this last set of questions, reflect on this issue.

1. None of the sets were groups using just multiplication, why?

2. How close to being a group with multiplication were they?

3. Could you make some of them groups with multiplication by taking some elements out of the set?

4. If so, which sets can be modified to be groups with multiplications and how?