# Quotients and Homomorphisms

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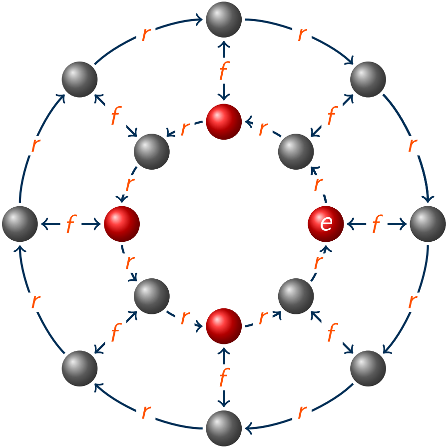
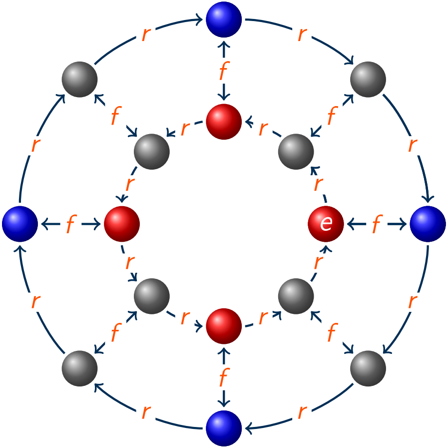
# $D_8 = \left\langle r, f \middle| r^8 = f^2 = e, rf = fr^{-1} \right\rangle$




- $\langle r^2 \rangle = \{r^2, r^4, r^6, e\}$

## $D_8 = \left\langle r, f \middle| r^8 = f^2 = e, rf = fr^{-1} \right\rangle$




- $\langle r^2 \rangle = \{r^2, r^4, r^6, e\}$
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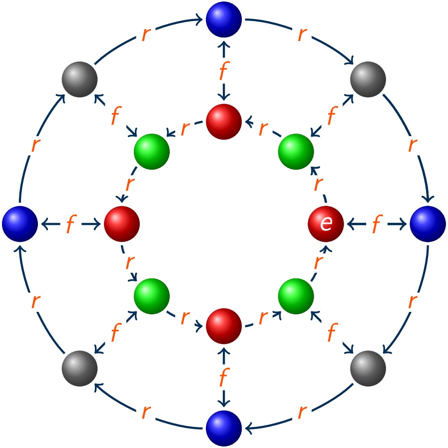


- $\left\langle r^2 \right\rangle = \left\{ r^2, r^4, r^6, e \right\}$
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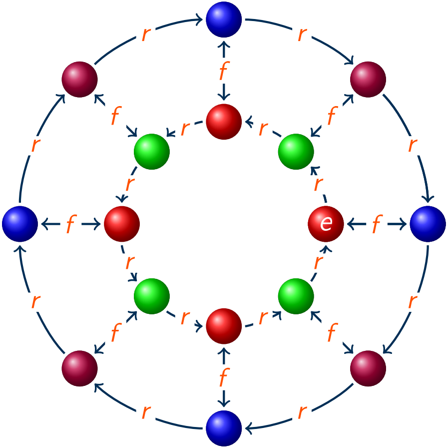


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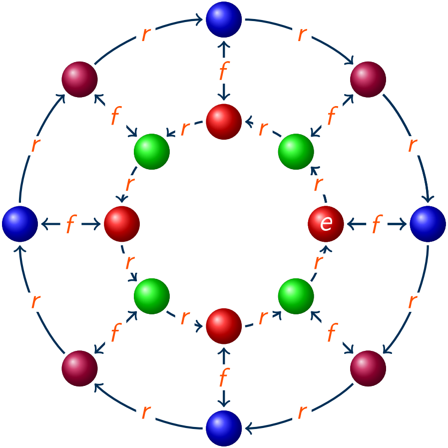


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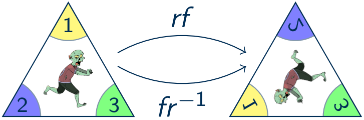
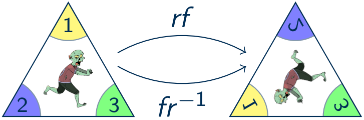
# Calculating in $D_n$

$$D_n = \left\langle r, f \,\middle|\, r^n = f^2 = e, rf = fr^{-1} \right\rangle$$



# Calculating in $D_n$

$$D_n = \langle r, f \,|\, r^n = f^2 = e, rf = fr^{-1} \rangle$$




$r^3 f$

# Calculating in $D_n$

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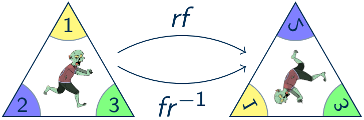
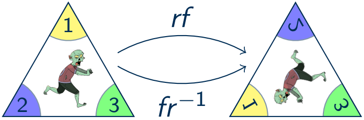



$$r^3 f = r^2(rf)$$

# Calculating in $D_n$

$$D_n = \langle r, f \,|\, r^n = f^2 = e, rf = fr^{-1} \rangle$$




$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \end{aligned}$$

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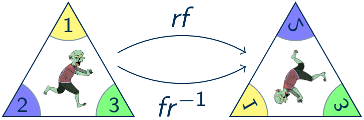


$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \\ &= r(rf)r^{n-1} \end{aligned}$$

# Calculating in $D_n$

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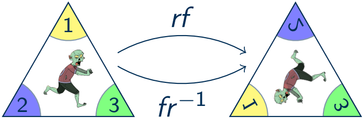


$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \\ &= r(rf)r^{n-1} \\ &= r(fr^{-1})r^{n-1} \end{aligned}$$

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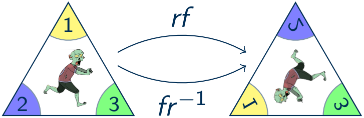
$$D_n = \langle r, f \mid r^n = f^2 = e, rf = fr^{-1} \rangle$$




$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \\ &= r(rf)r^{n-1} \\ &= r(fr^{-1})r^{n-1} \\ &= (rf)r^{n-2} \end{aligned}$$

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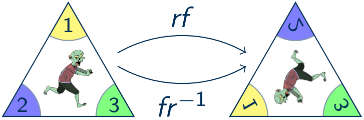


$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \\ &= r(rf)r^{n-1} \\ &= r(fr^{-1})r^{n-1} \\ &= (rf)r^{n-2} \\ &= (fr^{-1})r^{n-2} \end{aligned}$$

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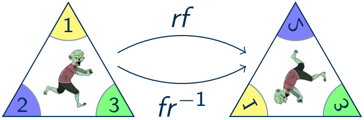


$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \\ &= r(rf)r^{n-1} \\ &= r(fr^{-1})r^{n-1} \\ &= (rf)r^{n-2} \\ &= (fr^{-1})r^{n-2} \\ &= fr^{n-3} \end{aligned}$$

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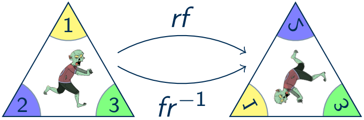


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$$r^k f = fr^{-k} = fr^{n-k}$$

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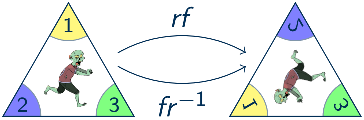


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$$r^k f = fr^{-k} = fr^{n-k}$$

$$fr^k = r^{-k} f = r^{n-k} f$$

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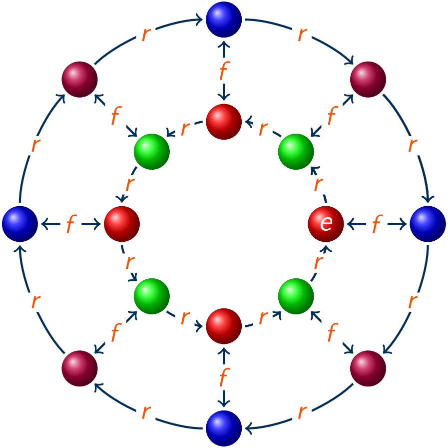


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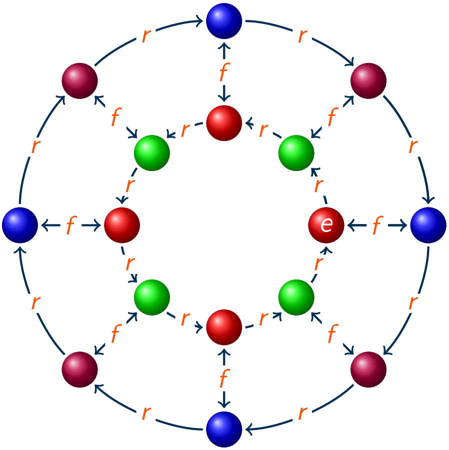


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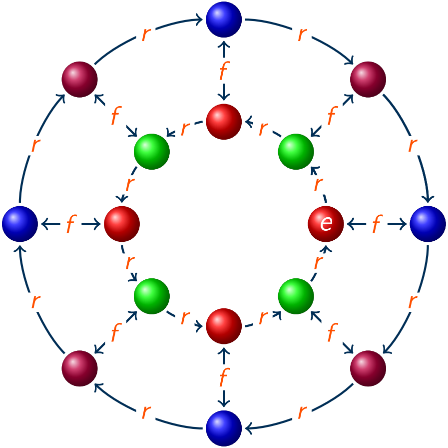


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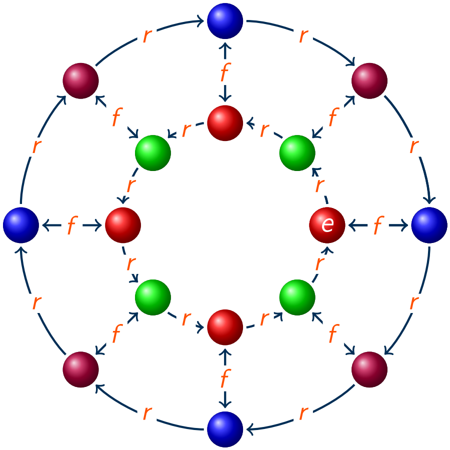


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- $\langle r^2 \rangle r = \{r^3, r^5, r^7, r\}$
- $\langle r^2 \rangle fr = \{r^2 fr, r^4 fr, r^6 fr, fr\}$

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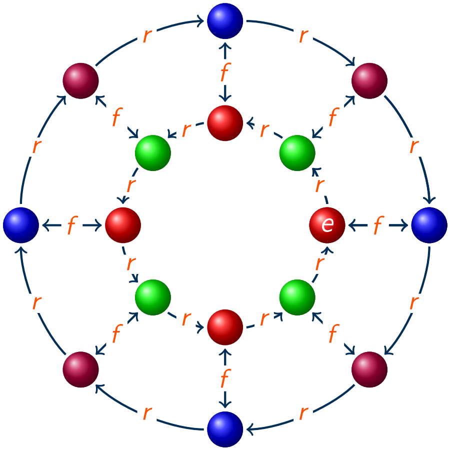


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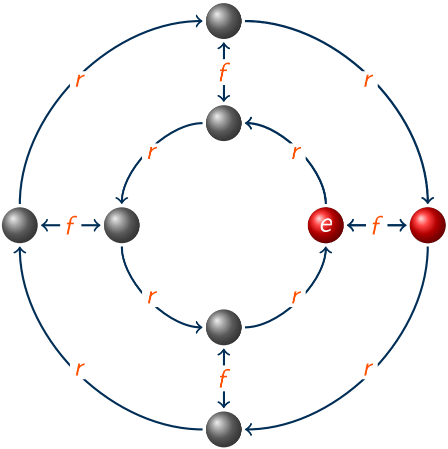
- $f\langle r^2 \rangle = \{fr^2, fr^4, fr^6, f\}$
- $r\langle r^2 \rangle = \{r^3, r^5, r^7, r\}$
- $fr\langle r^2 \rangle = \{fr^3, fr^5, fr^7, fr\}$
- $\langle r^2 \rangle f = \{fr^6, fr^4, fr^2, f\}$
- $\langle r^2 \rangle r = \{r^3, r^5, r^7, r\}$
- $\langle r^2 \rangle fr = \{fr^7, fr^5, fr^3, fr\}$

$$D_8 = \langle r^2 \rangle \cup f\langle r^2 \rangle \cup r\langle r^2 \rangle \cup fr\langle r^2 \rangle$$

# $D_4 = \left\langle r, f \middle| r^4 = f^2 = e, rf = fr^{-1} \right\rangle$




- $\langle f \rangle = \{f, e\}$

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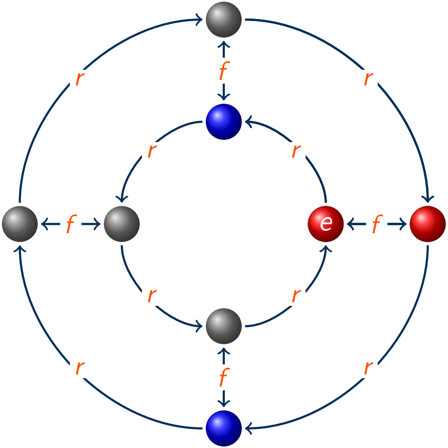


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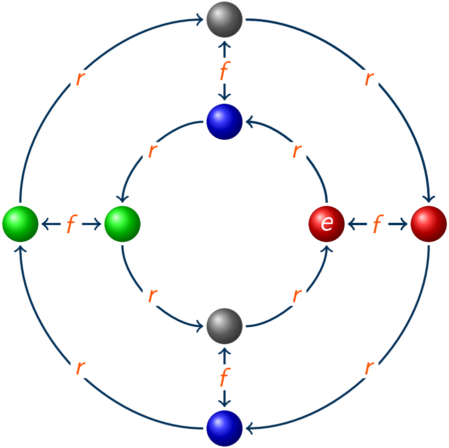


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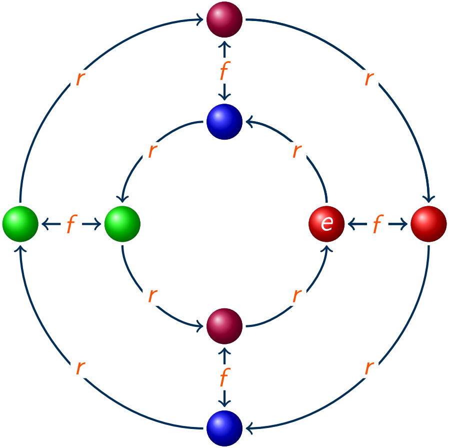


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- $r \langle f \rangle = \{rf, r\}$
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- $r^3 \langle f \rangle = \{r^3 f, r^3\}$

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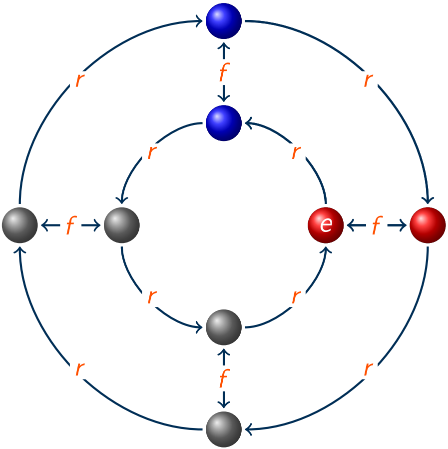


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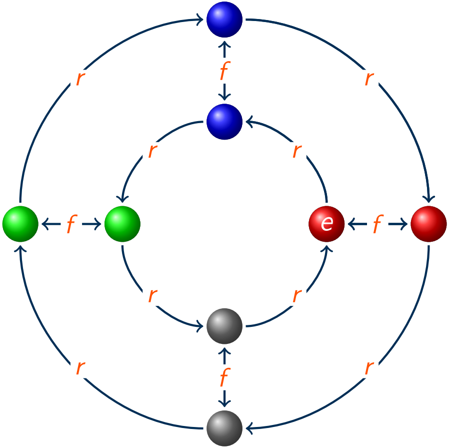


- $\langle f \rangle = \{f, e\}$
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- $r^3 \langle f \rangle = \left\{r^3 f, r^3\right\}$
- $\langle f \rangle r = \{fr, r\} = \left\{r^3 f, r\right\}$
- $\langle f \rangle r^2 = \left\{fr^2, r^2\right\} = \left\{r^2 f, r^2\right\}$

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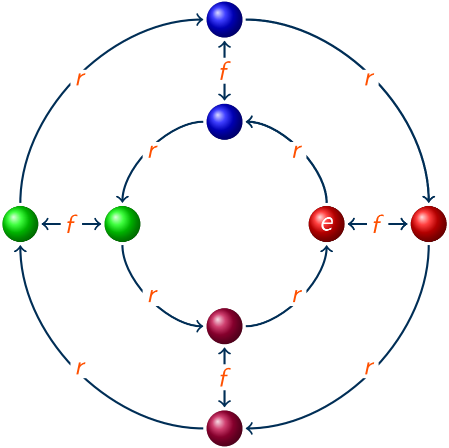


- $\langle f \rangle = \{f, e\}$
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- $r^2\langle f \rangle = \left\{r^2f, r^2\right\}$
- $r^3\langle f \rangle = \left\{r^3f, r^3\right\}$
- $\langle f \rangle r = \{fr, r\} = \left\{r^3f, r\right\}$
- $\langle f \rangle r^2 = \left\{fr^2, r^2\right\} = \left\{r^2f, r^2\right\}$
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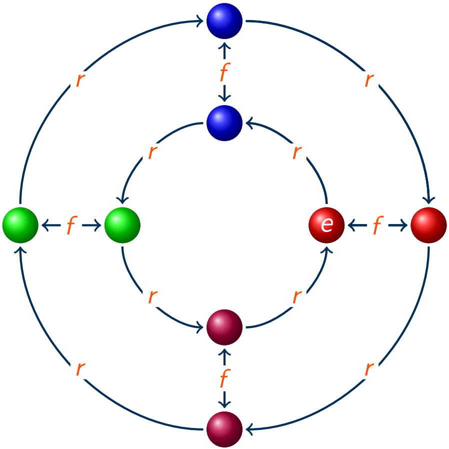


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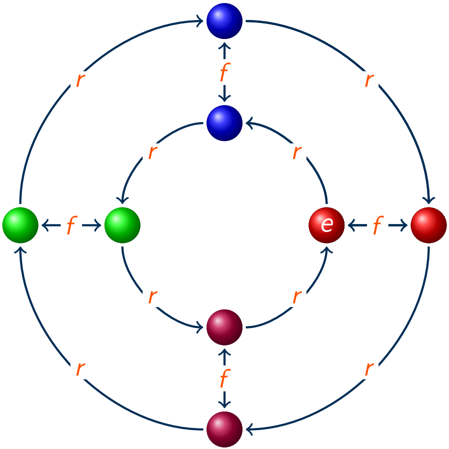
- $r^3 \langle f \rangle = \{r^3 f, r^3\}$
- $\langle f \rangle r = \{fr, r\} = \{r^3 f, r\}$
- $\langle f \rangle r^2 = \{fr^2, r^2\} = \{r^2 f, r^2\}$
- $\langle f \rangle r^3 = \{fr^3, r^3\} = \{rf, r^3\}$
- $\exists g \in D_4 : g \langle f \rangle \neq \langle f \rangle g$

# $D_4 = \langle r, f \mid r^4 = f^2 = e, rf = fr^{-1} \rangle$




- $\langle f \rangle = \{f, e\}$
- $r \langle f \rangle = \{rf, r\}$
- $r^2 \langle f \rangle = \{r^2 f, r^2\}$
- $r^3 \langle f \rangle = \{r^3 f, r^3\}$
- $\langle f \rangle r = \{fr, r\} = \{r^3 f, r\}$
- $\langle f \rangle r^2 = \{fr^2, r^2\} = \{r^2 f, r^2\}$
- $\langle f \rangle r^3 = \{fr^3, r^3\} = \{rf, r^3\}$
- $\exists g \in D_4 : g \langle f \rangle \neq \langle f \rangle g$

$$D_4 = \langle f \rangle \cup r \langle f \rangle \cup r^2 \langle f \rangle \cup r^3 \langle f \rangle$$

# Definition of Cosets

**Definition (Coset)**

Given a group $G$, subgroup $H$, and element $g \in G$,

$$gH = \{gh|h \in H\}$$

is a **left coset of** $H$ and

$$Hg = \{hg|h \in H\}$$

is a **right coset of** $H$.

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is a **left coset of** $H$ and

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is a **right coset of** $H$.

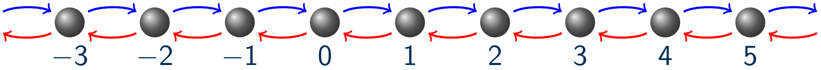
**Definition (Normal Subgroup)**

Given a group $G$ and subgroup $H$, if for all $g \in G$,

$$gH = Hg,$$

then we say that $H$ is a **normal subgroup** of $G$.

# $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$



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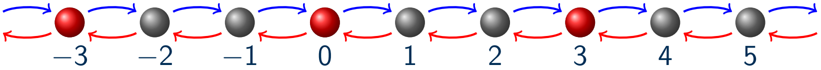
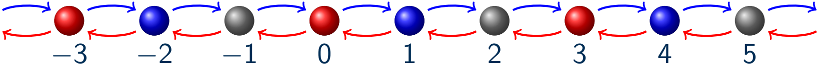


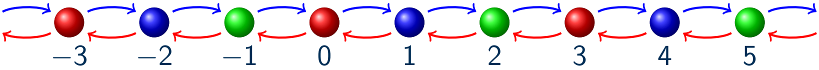

- $3\mathbb{Z} = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$

# $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\}$




- $3\mathbb{Z} = \{0, \pm3, \pm6, \pm9, \ldots\}$
- $1 + 3\mathbb{Z} = \{1, -2, 4, -5, 7, \ldots\}$

# $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$




- $3\mathbb{Z} = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$
- $1 + 3\mathbb{Z} = \{1, -2, 4, -5, 7, \ldots\}$
- $2 + 3\mathbb{Z} = \{2, -1, 5, -4, 8, \ldots\}$

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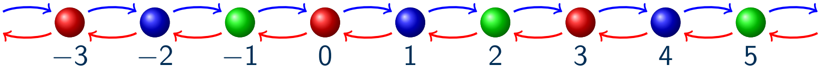


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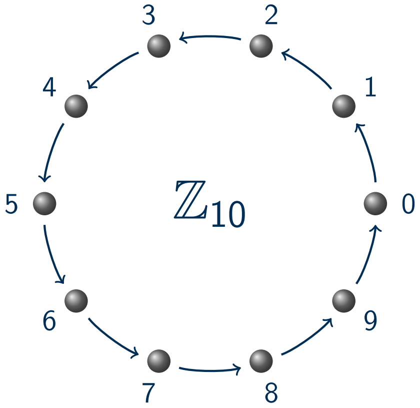
$$\mathbb{Z} = 3\mathbb{Z} \cup (1 + 3\mathbb{Z}) \cup (2 + 3\mathbb{Z})$$

## $\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$



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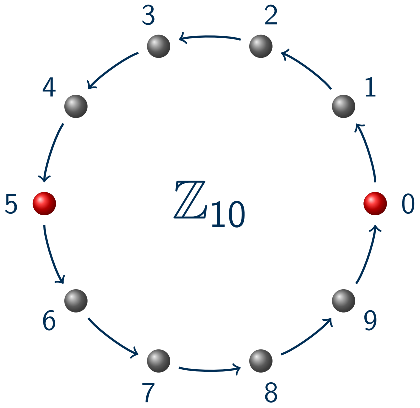
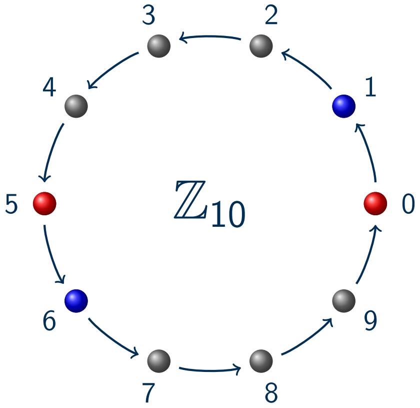


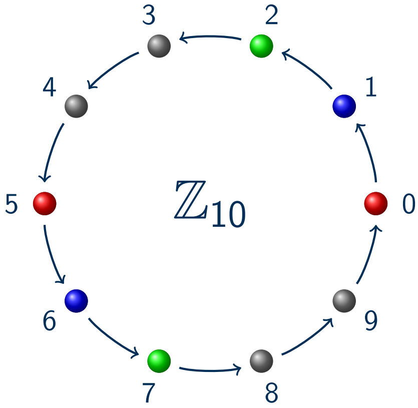

- $\langle 5 \rangle = \{0, 5\}$

# $\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$




- $\langle 5 \rangle = \{0, 5\}$
- $1 + \langle 5 \rangle = \{1, 6\}$

# $\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$




- $\langle 5 \rangle = \{0, 5\}$
- $1 + \langle 5 \rangle = \{1, 6\}$
- $2 + \langle 5 \rangle = \{2, 7\}$

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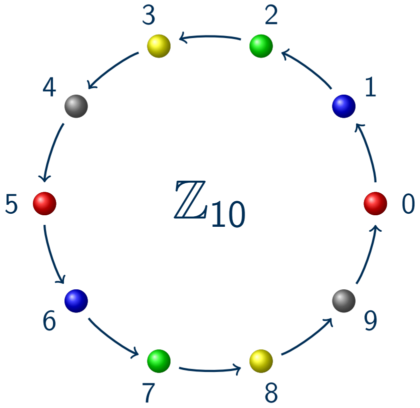


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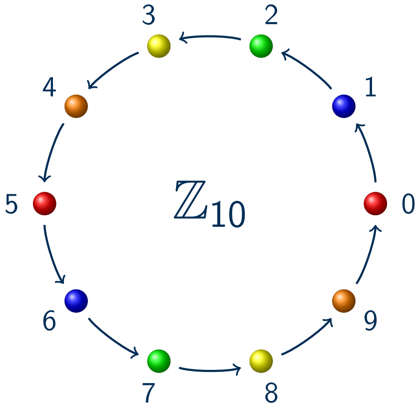


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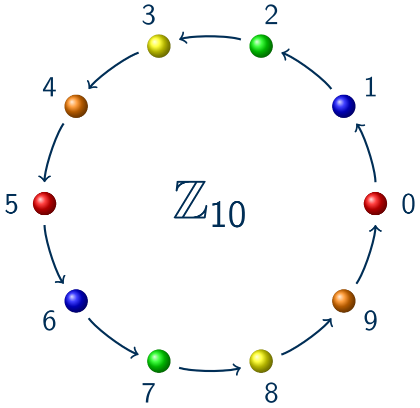


- $\langle 5 \rangle = \{0, 5\}$
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- $4 + \langle 5 \rangle = \{4, 9\}$

$$\mathbb{Z}_{10} = (\langle 5 \rangle) \cup (1 + \langle 5 \rangle) \cup (2 + \langle 5 \rangle) \cup (3 + \langle 5 \rangle) \cup (4 + \langle 5 \rangle)$$

# Table of Contents

# Normal Subgroups

**Definition (Normal Subgroup)**

Given a group $G$ and subgroup $H$, if for all $g \in G$,

$$gH = Hg,$$

then we say that $H$ is a **normal subgroup** of $G$.

Example: $\langle r^2 \rangle \subset D_8$

- $\langle r^2 \rangle = \{r^2, r^4, r^6, e\}$
- $r^k \langle r^2 \rangle = \{r^{2+k}, r^{4+k}, r^{6+k}, r^k\} = \langle r^2 \rangle r^k$
- $f \langle r^2 \rangle = \{fr^2, fr^4, fr^6, f\}$
- $\langle r^2 \rangle f = \{r^2 f, r^4 f, r^6 f, f\} = \{fr^6, fr^4, fr^2, f\}$
- $D_8 = \langle r^2 \rangle \cup r \langle r^2 \rangle \cup f \langle r^2 \rangle \cup rf \langle r^2 \rangle$

# Normal Subgroups

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Given a group $G$ and subgroup $H$, if for all $g \in G$,

$$gH = Hg,$$

then we say that $H$ is a **normal subgroup** of $G$.

Example: $\langle r \rangle \subset D_8$

- $\langle r \rangle = \{r, r^2, r^3, r^4, r^5, r^6, r^7, e\}$
- $f \langle r \rangle = \{fr, fr^2, fr^3, fr^4, fr^5, fr^6, fr^7, f\}$
- $D_8 = \langle r \rangle \cup f \langle r \rangle$

# Normal Subgroups

## Definition (Normal Subgroup)

Given a group $G$ and subgroup $H$, if for all $g \in G$,

$$gH = Hg,$$

then we say that $H$ is a **normal subgroup** of $G$.

Example: $\langle 2 \rangle \subset \mathbb{Z}_8$

- $\langle 2 \rangle = \{2, 4, 6, 0\}$
- $1 + \langle 2 \rangle = \{3, 5, 7, 1\} = \langle 2 \rangle + 1$
- $\mathbb{Z}_8 = \langle 2 \rangle \cup (1 + \langle 2 \rangle)$

# Normal Subgroups

**Definition (Normal Subgroup)**

Given a group $G$ and subgroup $H$, if for all $g \in G$,

$$gH = Hg,$$

then we say that $H$ is a **normal subgroup** of $G$.

Non-Example: $\langle f \rangle \subset D_8$

- $\langle f \rangle = \{f, e\}$
- $r \langle f \rangle = \{rf, r\}$
- $\langle f \rangle r = \{fr, r\} = \{r^7 f, r\} \neq r \langle f \rangle$

# Results on Normal Subgroups

**Theorem**

*If $G$ is an abelian group, then all subgroups are normal.*

# Results on Normal Subgroups

**Theorem**

*If $G$ is an abelian group, then all subgroups are normal.*

**Theorem**

*If $G$ is a finite group and a subgroup $H$ has index 2, then $H$ is normal.*

# Coset Properties

## Theorem

*Given a group $G$, subgroup $H \subseteq G$, and elements $a, b \in G$:*

1. $|H| = |aH|$,
2. $|aH| = |bH|$,
3. $aH = bH$ *or* $aH \cap bH = \emptyset$, *and*
4. $aH = bH$ *if and only if* $b^{-1}a \in H$.

## Theorem

*From the previous theorem, given a group $G$ and subgroup $H \subseteq G$, the cosets of $H$ partition $G$, e.g. for some set of $g_i \in G$*

$$\bigcup_i g_i H = G$$

*and* $g_i H \cap g_j H = \emptyset$ *when* $i \neq j$.

# Index 2 Implies Normal Proof

**Proof.**

- $G$ a group, $H$ a subgroup, and $[G : H] = 2$

□

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Proof.

- $G$ a group, $H$ a subgroup, and $[G : H] = 2$
- $g \in G$ and $g \notin H$

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- $gH = G \setminus H$

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- $gH = G \setminus H$
- $Hg \cap H = \emptyset$ and $G = Hg \cup H$
- $Hg = G \setminus H$
- $\therefore gH = Hg$

□

# Results on Normal Subgroups

**Theorem**

*If $G$ is an abelian group, then all subgroups are normal.*

**Theorem**

*If $G$ is a finite group and a subgroup $H$ has index 2, then $H$ is normal.*

**Theorem**

*Let $G$ be a group with subgroup $N$, then $N$ is normal if and only if for all $g \in G$, $gNg^{-1} = N$.*

# $gNg^{-1} = N$ Proof

**Part 1.**

- $G$ a group, $N$ normal

□

# $gNg^{-1} = N$ Proof

**Part 1.**

- $G$ a group, $N$ normal
- $gN = Ng, \forall g \in G\, \forall n_1 \in N\, \exists n_2 \in N : gn_1 = n_2 g$ (e.g. $fr = r^{-1}f$ in $D_n$)

□

# $gNg^{-1} = N$ Proof

## Part 1.

- $G$ a group, $N$ normal
- $gN = Ng$, $\forall g \in G\, \forall n_1 \in N\, \exists n_2 \in N : gn_1 = n_2 g$ (e.g. $fr = r^{-1}f$ in $D_n$)
- $gNg^{-1} = \{gng^{-1} | n \in N\}$

□

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- $gN = Ng$, $\forall g \in G\, \forall n_1 \in N\, \exists n_2 \in N : gn_1 = n_2 g$ (e.g. $fr = r^{-1}f$ in $D_n$)
- $gNg^{-1} = \left\{gng^{-1} \middle| n \in N\right\}$
- $gn_1g^{-1} = n_2gg^{-1} = n_2 \in N$, so $gNg^{-1} \subseteq N$

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- But, the previous steps are reversible, $n_2 = n_2gg^{-1} = gn_1g^{-1}$, so $N \subseteq gNg^{-1}$

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- $G$ a group, $N$ normal
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- But, the previous steps are reversible, $n_2 = n_2gg^{-1} = gn_1g^{-1}$, so $N \subseteq gNg^{-1}$
- $\therefore gNg^{-1} = N$

$\square$

# $gNg^{-1} = N$ Proof Continued

**Part 2.**

- $G$ a group, $\forall g \in G: \; gNg^{-1} = N$

□

# $gNg^{-1} = N$ Proof Continued

**Part 2.**

- $G$ a group, $\forall g \in G: \ gNg^{-1} = N$
- $\forall g \in G \, \forall n_1 \in N \, \exists n_2 \in N : gn_1g^{-1} = n_2$

□

# $gNg^{-1} = N$ Proof Continued

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- $G$ a group, $\forall g \in G: \; gNg^{-1} = N$
- $\forall g \in G \, \forall n_1 \in N \, \exists n_2 \in N : gn_1g^{-1} = n_2$
- $gn_1 = n_2g$

□

# $gNg^{-1} = N$ Proof Continued

**Part 2.**

- $G$ a group, $\forall g \in G: \; gNg^{-1} = N$
- $\forall g \in G \, \forall n_1 \in N \, \exists n_2 \in N : gn_1g^{-1} = n_2$
- $gn_1 = n_2g$
- $\therefore gN = Ng$ and $N$ is normal

$\square$

# Example and Non-Example in $D_4$

## Example $\langle r^2 \rangle$

$$f \left\langle r^2 \right\rangle f = \left\{ fr^2 f, fef \right\}$$

# Example and Non-Example in $D_4$

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## Non-Example $\langle f \rangle$

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r \langle f \rangle r^3 = \left\{ rfr^3, rer^3 \right\}
$$

# Example and Non-Example in $D_4$

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(But, $\left\{r^2f, e\right\}$, is another subgroup of order 2.)

# Kernels are Normal Subgroups

**Theorem**

*Given a homomorphism* $\phi : G \to \overline{G}$, *the kernel of* $\phi$ *is a normal subgroup.*

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- $\phi(kk') = \phi(k)\phi(k') = e_{\overline{G}}e_{\overline{G}} = e_{\overline{G}}$

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- $\therefore$ $ker\,\phi$ is a subgroup

□

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- $\therefore ker\, \phi$ is a subgroup
- $\phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g)^{-1} = \phi(g)e_{\overline{G}}\phi(g)^{-1} = e_{\overline{G}}$

□

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- $\therefore g(ker\, \phi)g^{-1} = ker\, \phi$ and $ker\, \phi$ is normal

□

# Table of Contents

# Quotients and Normal Subgroups

**Theorem**

*If $G$ is a group and $N$ is a normal subgroup, then*

$$G/N = \{gN|g \in G\}$$

*is a group with arithmetic defined by $(gN)(hN) = (ghN)$.*

# Coset Properties

**Theorem**

*Given a group $G$, subgroup $H \subseteq G$, and elements $a, b \in G$:*

1. $|H| = |aH|$,
2. $|aH| = |bH|$,
3. $aH = bH$ *or* $aH \cap bH = \emptyset$, *and*
4. $aH = bH$ *if and only if* $b^{-1}a \in H$.

# Quotients and Normal Subgroups Proof

**Part 1: Closure and Associativity.**

- $g, h \in G$ and $n_1, n_2 \in N$ and $N$ is normal

□

# Quotients and Normal Subgroups Proof

**Part 1: Closure and Associativity.**

- $g, h \in G$ and $n_1, n_2 \in N$ and $N$ is normal
- $gn_1 \in gN$ and $hn_2 \in hN$

□

# Quotients and Normal Subgroups Proof

**Part 1: Closure and Associativity.**

- $g, h \in G$ and $n_1, n_2 \in N$ and $N$ is normal
- $gn_1 \in gN$ and $hn_2 \in hN$
- $gn_1hn_2 = ghn_3n_2 \in ghN$ for some $n_3 \in N$

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- $\therefore$ We get closure

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- $\therefore$ We get closure
- Associativity is "inhereted" from $G$

□

# Quotients and Normal Subgroups Proof

**Part 2: Well Defined.**

- $g, h, g', h' \in G$ with $gN = g'N$ and $hN = h'N$

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- $g, h, g', h' \in G$ with $gN = g'N$ and $hN = h'N$
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- $g, h, g', h' \in G$ with $gN = g'N$ and $hN = h'N$
- $g^{-1}g' = n_1 \in N$ and $h^{-1}h' = n_2 \in N$
- $(gh)^{-1}(g'h') = h^{-1}g^{-1}g'h' = h^{-1}n_1h' = h^{-1}h'n_3 = n_2n_3 \in N$

$\square$

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- $ghN = g'h'N$

□

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- $g, h, g', h' \in G$ with $gN = g'N$ and $hN = h'N$
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- $(gh)^{-1}(g'h') = h^{-1}g^{-1}g'h' = h^{-1}n_1h' = h^{-1}h'n_3 = n_2n_3 \in N$
- $ghN = g'h'N$
- $\therefore (gN)(hN) = ghN$ is well defined

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**Part 3: Identity and Inverses.**

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- $\therefore G/N$ is a group

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# Quotient Group Example: $N = \langle r^2 \rangle \subset D_8$

From before we have:

1. $N = \langle r^2 \rangle = \{r^2, r^4, r^6, e\}$
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$$D_8/N \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

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- We will show, eventually, that this is why $D_8/\langle r^2 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

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4. $\therefore N = ker\,\phi$

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# Table of Contents

# First Isomorphism Theorem

**Theorem**

*If $\phi : G \to \overline{G}$ is a surjective homomorphism with kernel $K = \ker \phi$, then $G/K \cong \overline{G}$.*

# Examples

**$\phi : D_8 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$**

Define $\phi(f^l r^k) = (l, k) \pmod 2$, then from before $ker\,\phi = \langle r^2 \rangle$ and $D_8/ker\,\phi \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This agrees with the conclusion of the **First Isomorphism Theorem**.

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**$\phi : \mathbb{Z} \to \mathbb{Z}_n$**

If we define $\phi(z) = z \pmod n$, then the kernel will be $ker\,\phi = n\mathbb{Z}$ since those are precisely the numbers equal to zero modulo $n$. The **First Isomorphism Theorem** tells us then that $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

# Null Space

**$T : \mathbb{Z}^2 \to \mathbb{Z}^2$**

- $T(\vec{v}) = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 3y \\ 2x + 6y \end{pmatrix}$

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- $Col\ T = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle \cong \mathbb{Z}^2/ker\ T = \left\{ \begin{pmatrix} z \\ 0 \end{pmatrix} + ker\ T \middle| z \in \mathbb{Z} \right\}$

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- Why didn't we have to show $\overline{\phi}$ is surjective?

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- $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z} = \{a + n\mathbb{Z} | a \in \mathbb{Z}\}$

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- $S/K \cong \phi(S) \subseteq \overline{S}$
- There is a class of substructures of $S$ that are all possible kernels

# Quotients and Homomorphisms

Dr. Chuck Rocca