Limits of Turing Machines

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Table of Contents

Table of Contents

Algorithm

Definition

An *algorithm* is a computational process that is describable in terms of a Turing machine.

Connected Graphs Example

M= "On input $\langle G \rangle$, the string encoding of a graph G:

0. Check that $\langle G \rangle$ is in the correct format
1. Select the first node in G and mark it.
2. Repeat the following until no new nodes are marked:
3. For each node in G, mark it if is is attached by an edge to a node that is already marked.
4. Scan all the nodes of G, if they are all marked, *accept*; otherwise *reject*."

Decidable Unions

Given two *decidable* languages L_1 and L_2 and corresponding Turing machines M_1 and M_2 the union of the languages can be decided by:
M= "On input w:

0. Check that w is in the correct format.
1. Run M_1 on w. If it accepts, *accept*.
2. Run M_2 on w. If it accepts, *accept*.
3. Otherwise *reject*"

Recognizable Unions

Given two *Turing recognizable* languages L_1 and L_2 and corresponding Turing machines M_1 and M_2 the union of the languages can be recognized by:

M= "On input w:

0. Check that w is in the correct format.
1. Run M_1 and M_2 alternately on w step by step. If either accepts, *accept*.
2. If both halt or reject, *reject*."

Decidable vs. Recognizable

Definition

A Language is *Turning-decidable* or simply *decidable* if some Turing machine decides it; the machine always reaches an accept or reject state. Given any word there is a TM that can tell if the word is or is not in the language.

Decidable vs. Recognizable

Definition

A Language is *Turning-decidable* or simply *decidable* if some Turing machine decides it; the machine always reaches an accept or reject state. Given any word there is a TM that can tell if the word is or is not in the language.

Definition

A Language is *Turning-recognizable* if some Turing machine recognizes it; in this case the machine reaches an accept state, reject state, or it may loop (fail to accept). There is a TM that accepts words in the language, but may fail to reach a verdict if a word is not in the language.

Decidable Intersections

Given two *decidable* languages L_1 and L_2 and corresponding Turing machines M_1 and M_2 the intersections of the languages can be decided by:
M="On input w:

0. Check that w is in the correct format.
1. Run M_1 and M_2 on w. If they both accept, *accept*.
2. Otherwise *reject*"

Decidable Intersections

Given two *decidable* languages L_1 and L_2 and corresponding Turing machines M_1 and M_2 the intersections of the languages can be decided by:
M="On input w:

0. Check that w is in the correct format.
1. Run M_1 and M_2 on w. If they both accept, *accept*.
2. Otherwise *reject*"

Why didn't we say "Run M_1 and M_2 alternately on w step by step?"

Decidable Complements

Given a *decidable* language L_1 and corresponding Turing machine M_1 the complement of the languages can be decided by:
M="On input w:

0. Check that w is in the correct format.
1. Run M_1 on w. If it accepts, *reject*.
2. Otherwise *accept*"

Table of Contents

Deterministic Finite Automaton

Theorem

The set

$$A_{DFA} = \{\langle B, w \rangle \,|B \text{ is a DFA that accepts string } w\}$$

is a decidable language.

Deterministic Finite Automaton

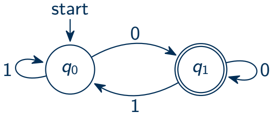
M="On input $\langle B, w\rangle$, where B is a DFA and w is a string:

0. Check the format of the input.
1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*; otherwise, *reject*."

Deterministic Finite Automaton

M="On input $\langle B, w\rangle$, where B is a DFA and w is a string:

0. Check the format of the input.
1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*; otherwise, *reject*."



Deterministic Finite Automaton

M= "On input $\langle B, w \rangle$, where B is a DFA and w is a string:

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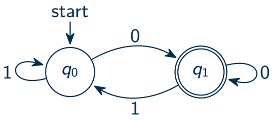

Machine $\langle B \rangle$

#	q_0	q_1	#	0	1	#	q_0	0	q_1	q_0	1	q_0	q_1	0	q_1	q_1	1	q_0	#	q_0	#	q_1	#	␣	···

Word w

0	1	1	0	1	0	1	1	1	0	␣	···

Nondeterministic Finite Automaton

Theorem

The set

$$A_{NFA} = \{\langle B, w\rangle \,|B \textit{ is an NFA that accepts string } w\}$$

is a decidable language.

N="On input $\langle B, w\rangle$, where B is a NFA and w is a string:

1. Convert B to a DFA C.
2. Run M from the previous theorem on input $\langle C, w\rangle$.
3. If the simulation ends in an accept state, *accept*; otherwise, *reject*."

Regular Expressions

Theorem

The set

$$A_{REX} = \{\langle B, w\rangle \,|B \textit{ is a regex that accepts string } w\}$$

is a decidable language.

P="On input $\langle B, w\rangle$, where B is a RegEx and w is a string:

1. Convert B to a NFA C.
2. Run N from the previous theorem on input $\langle C, w\rangle$.
3. If the simulation ends in an accept state, *accept*; otherwise, *reject*."

Empty Languages

Theorem

The set

$$E_{DFA} = \{\langle A \rangle \,|A \text{ is a DFA and } L(A) = \emptyset\}$$

is a decidable language.

T="On input $\langle A \rangle$, the string encoding of DFA A:

1. Select the start state in A and mark it.
2. Repeat the following until no new states are marked:
3. For each state in A, mark it if there is a transition from a marked state.
4. Scan the accept states of A, if any are marked, *reject*; otherwise *accept*."

Equal Languages

Theorem

The set

$$EQ_{DFA} = \{\langle A, B\rangle \,|A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$

is a decidable language.

- Note, DFAs are closed under *unions*, *intersections,* and *compliments*.
- Construct the *symmetric difference* of A and B, $C \equiv A\ XOR\ B$ or

$$L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right).$$

- Check if C is empty using T from the previous theorem.

CFLs

Theorem

The set

$$A_{CFG} = \{\langle G, w\rangle \,|\, G \textit{ is a CFG that generates the string } w\}$$

is a decidable language.

$S =$"On input $\langle G, w\rangle$, where G is a CFG and w is a string:

1. Convert G to Chomsky Normal Form.
2. List all derivations with $2n - 1$ steps, $n = |w|$; except for n=0, then list derivations with one step. (Why $2n - 1$?)
3. If w is generated, *accept*; otherwise *reject*."

CFLs

Theorem

The set

$$E_{CFG} = \{\langle G \rangle \,|\, G \text{ is a CFG and } L(G) = \emptyset\}$$

is a decidable language.

R= "On input $\langle G \rangle$, the string encoding of CFG G:

1. Mark the terminals in G.
2. Repeat the following until no new variables get marked:
3. Mark any variable A where $A \to U_1 U_2 \cdots U_k$ is in G and the U_i are all marked.
4. If the start variable of G is not marked, *accept*; otherwise *reject*."

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle \,|\, G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

is not a decidable language.

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

is not a decidable language. (Proof held until after Chapter 5.)

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle \,|\, G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

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- Recall, CFLs are not necessarily closed under intersections and complementation.

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle | G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

is not a decidable language. (Proof held until after Chapter 5.)

- Recall, CFLs are not necessarily closed under intersections and complementation.
- $L(G) = \{a^m b^n c^n | m, n \geq 0\}$

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle \,|\, G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

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- Recall, CFLs are not necessarily closed under intersections and complementation.
- $L(G) = \{a^m b^n c^n | m, n \geq 0\}$
- $L(H) = \{a^n b^n c^m | m, n \geq 0\}$

CFLs

Theorem

The set

$$EQ_{CFG} = \{\langle G, H\rangle \,|\, G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$

is not a decidable language. (Proof held until after Chapter 5.)

- Recall, CFLs are not necessarily closed under intersections and complementation.
- $L(G) = \{a^m b^n c^n | m, n \geq 0\}$
- $L(H) = \{a^n b^n c^m | m, n \geq 0\}$
- $L(G) \cap L(H) = \{a^n b^n c^n | n \geq 0\}$ is not context-free by the pumping lemma

Hierarchy of Languages

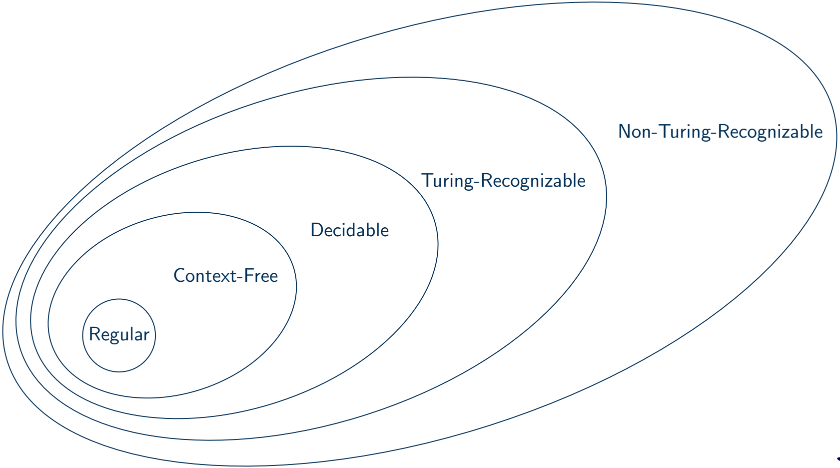


Table of Contents

Cardinality: Naive Definition

Definition

The *cardinality* of a set is the number of elements in the set and two sets have the same cardinality if they have the same number of elements.

$$
\begin{aligned}
A &= \{1, 2, 3, 4, 5\} \\
B &= \{a, b, c, d, e\} \\
C &= \{!, @, \#, \$, \%, \wedge\} \\
\mathbb{N} &= \{1, 2, 3, 4, ...\} \\
\mathbb{Z} &= \{0, \pm 1, \pm 2, \pm 3, ...\} \\
\mathbb{Q} &= \{a/b | a, b \in \mathbb{Z} \ and \ b \neq 0\}
\end{aligned}
$$

Cardinality: Improved Definition

Definition

The *cardinality* of a finite set is the number of elements in the set. A set is *infinite* if there is a one-to-one correspondence between the set and a proper subset of the set. And, two sets have the same cardinality if there exists a one-to-one correspondence between their elements.

$\mathbb{N}$ to $2\mathbb{N}$

$$\begin{tikzcd} \mathbb{N} & \mathbb{N} \\ 1 \arrow[dr] & 1 \\ & 2 \end{tikzcd}$$

$\mathbb{N}$ to $2\mathbb{N}$

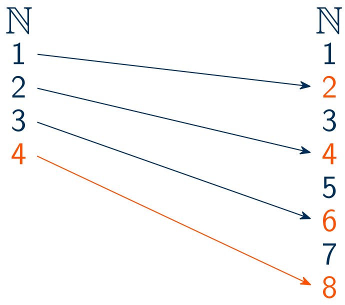
$$
\begin{tikzcd}[row sep=small]
\mathbb{N} & \mathbb{N} \\
1 \arrow[dr] & 1 \\
2 \arrow[ddr] & 2 \\
 & 3 \\
 & 4
\end{tikzcd}
$$

$\mathbb{N}$ to $2\mathbb{N}$

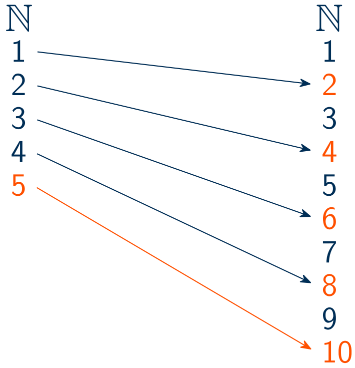
$\mathbb{N}$		$\mathbb{N}$
1	$\to$	2
2	$\to$	4
3	$\to$	6

$\mathbb{N}$: 1, 2, 3, 4, 5, 6

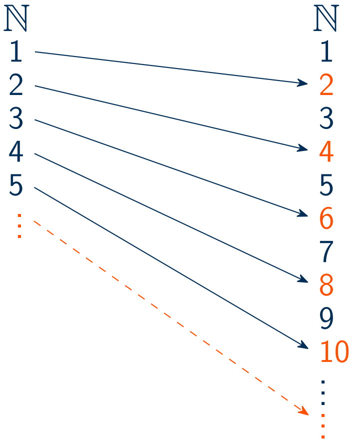
$\mathbb{N}$ to $2\mathbb{N}$



$\mathbb{N}$ to $2\mathbb{N}$



$\mathbb{N}$ to $2\mathbb{N}$



$\mathbb{N}$ to $\mathbb{Z}$

$$\begin{array}{ccc} \mathbb{N} & & \mathbb{Z} \\ 1 & \longrightarrow & 0 \end{array}$$

$\mathbb{N}$ to $\mathbb{Z}$

$$\begin{array}{ccc} \mathbb{N} & & \mathbb{Z} \\ 1 & \longrightarrow & 0 \\ 2 & \longrightarrow & 1 \end{array}$$

$\mathbb{N}$ to $\mathbb{Z}$

$\mathbb{N}$		$\mathbb{Z}$
1	$\longrightarrow$	0
2	$\longrightarrow$	1
3	$\longrightarrow$	-1

$\mathbb{N}$ to $\mathbb{Z}$

$$\begin{array}{ccc} \mathbb{N} & & \mathbb{Z} \\ 1 & \longrightarrow & 0 \\ 2 & \longrightarrow & 1 \\ 3 & \longrightarrow & -1 \\ 4 & \longrightarrow & 2 \end{array}$$

$\mathbb{N}$ to $\mathbb{Z}$

$$
\begin{array}{ccc}
\mathbb{N} & & \mathbb{Z} \\
1 & \longrightarrow & 0 \\
2 & \longrightarrow & 1 \\
3 & \longrightarrow & -1 \\
4 & \longrightarrow & 2 \\
5 & \longrightarrow & -2
\end{array}
$$

$\mathbb{N}$ to $\mathbb{Z}$

$$
\begin{array}{ccc}
\mathbb{N} & & \mathbb{Z} \\
1 & \longrightarrow & 0 \\
2 & \longrightarrow & 1 \\
3 & \longrightarrow & -1 \\
4 & \longrightarrow & 2 \\
5 & \longrightarrow & -2 \\
6 & \longrightarrow & 3
\end{array}
$$

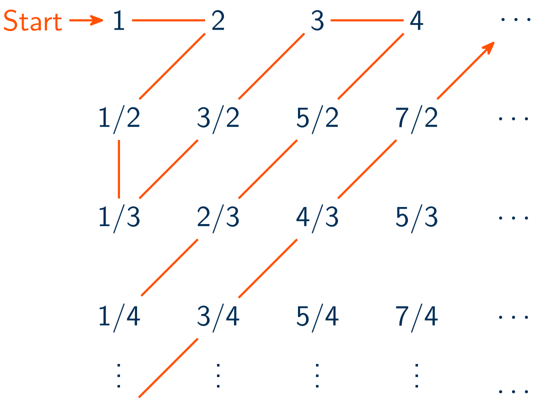
$\mathbb{N}$ to $\mathbb{Z}$

$$
\begin{array}{ccc}
\mathbb{N} & & \mathbb{Z} \\
1 & \longrightarrow & 0 \\
2 & \longrightarrow & 1 \\
3 & \longrightarrow & -1 \\
4 & \longrightarrow & 2 \\
5 & \longrightarrow & -2 \\
6 & \longrightarrow & 3 \\
\vdots & \dashrightarrow & \vdots
\end{array}
$$

$\mathbb{N}$ to $\mathbb{Q}^+$

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & \cdots \\
1/2 & 3/2 & 5/2 & 7/2 & \cdots \\
1/3 & 2/3 & 4/3 & 5/3 & \cdots \\
1/4 & 3/4 & 5/4 & 7/4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ldots
\end{array}
$$

$\mathbb{N}$ to $\mathbb{Q}^+$



Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
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6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
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$\vdots$	$\vdots$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.7$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	0.65395501314···
2	0.73800613014···
3	0.05050813247···
4	0.10810350448···
5	0.04587954758···
6	0.66716666577···
7	0.73243627345···
8	0.27311930829···
9	0.17177211903···
10	0.45518277788···
⋮	⋮

New Number:

$$x = 0.74$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.741$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.7412$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.74128$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.741287$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.7412873$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.74128731$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.741287318$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.7412873187\ldots$$

Cantor Diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	$0.65395501314\cdots$
2	$0.73800613014\cdots$
3	$0.05050813247\cdots$
4	$0.10810350448\cdots$
5	$0.04587954758\cdots$
6	$0.66716666577\cdots$
7	$0.73243627345\cdots$
8	$0.27311930829\cdots$
9	$0.17177211903\cdots$
10	$0.45518277788\cdots$
$\vdots$	$\vdots$

New Number:

$$x = 0.7412873187\ldots$$

Avoiding 0's and 9's when replacing digits since

$$0.19999\ldots = 0.20000\ldots$$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$
$$\mathscr{P}(\mathbb{N}) = \{\emptyset, \{1\}, \{2, 3\}, \{4\}, \{8\}, \{7, 19, 83\}, \{101, 23, 7\}, \ldots\}$$

Power Sets

$$\begin{aligned} \mathbb{N} &= \{1, 2, 3, 4, 5, \ldots\} \\ \mathscr{P}(\mathbb{N}) &= \{s_1, s_2, s_3, s_4, s_5 \ldots\} \end{aligned}$$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$
$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

$$\begin{array}{ccc}
1 & \longrightarrow & s_1 \\
2 & \longrightarrow & s_2 \\
3 & \longrightarrow & s_3 \\
4 & \longrightarrow & s_4 \\
5 & \longrightarrow & s_5 \\
6 & \longrightarrow & s_6 \\
7 & \longrightarrow & s_7 \\
\vdots & \dashrightarrow & \vdots
\end{array}$$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$
$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

$$1 \longrightarrow s_1$$
$$2 \longrightarrow s_2$$
$$3 \longrightarrow s_3$$
$$4 \longrightarrow s_4$$
$$5 \longrightarrow s_5$$
$$6 \longrightarrow s_6$$
$$7 \longrightarrow s_7$$
$$\vdots \dashrightarrow \vdots$$

Define a new set X as follows

$$X = \{n | n \notin s_n\}$$

Thus, if

- $1 \in s_1$, then $1 \notin X$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$
$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \dots\}$$

$$\begin{array}{ccc}
1 & \longrightarrow & s_1 \\
2 & \longrightarrow & s_2 \\
3 & \longrightarrow & s_3 \\
4 & \longrightarrow & s_4 \\
5 & \longrightarrow & s_5 \\
6 & \longrightarrow & s_6 \\
7 & \longrightarrow & s_7 \\
\vdots & \dashrightarrow & \vdots
\end{array}$$

Define a new set X as follows

$$X = \{n | n \notin s_n\}$$

Thus, if

- $1 \in s_1$, then $1 \notin X$
- $1 \notin s_1$, then $1 \in X$

Power Sets

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$
$$\mathscr{P}(\mathbb{N}) = \{s_1, s_2, s_3, s_4, s_5 \ldots\}$$

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\end{array}$$

Define a new set X as follows

$$X = \{n | n \notin s_n\}$$

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And therefore $\forall n : X \neq s_n$.

A Theorem on Power Sets

Theorem

Given a set S, the cardinality of $\mathscr{P}(S)$ is always greater than the cardinality of S; there are infinitely many infinities.

Table of Contents

A_{TM} is Undecidable

Theorem

Given the set

$$A_{TM} = \{\langle M, w\rangle \,|\, M \textit{ is a Turing Machine and } M \textit{ accepts } w\},$$

A_{TM} *is undecidable.*

A_{TM} is Undecidable

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A_{TM} is undecidable.

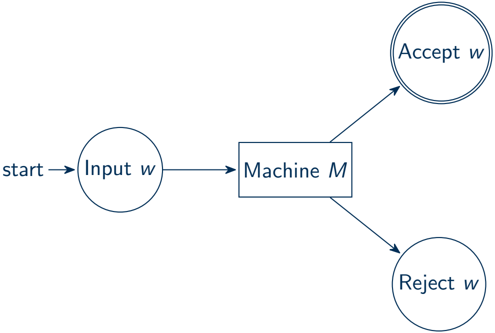
$U =$ " On input $\langle M, w\rangle$, where M is a TM and w is a string:

1. Simulate M on w.
2. If M ever enters its accept state, *accept*; If M ever enters its reject state, *reject*."

This is a *universal Turing machine* and shows that A_{TM} is recognizable.

An Undecidable Language

- $A_{TM} = \{\langle M, w\rangle \,|M \text{ is a Turing Machine and } M \text{ accepts } w\}$



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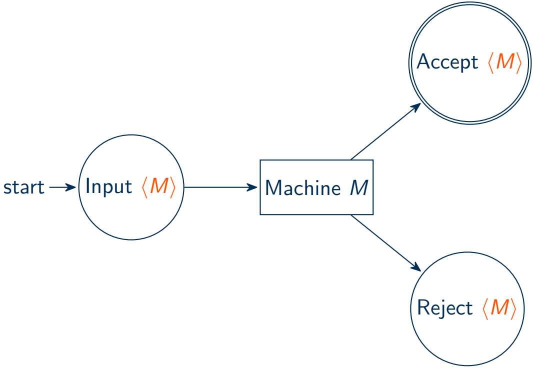

(Think C++ compiler written in C++.)

An Undecidable Language

- $A_{TM} = \{\langle M, w\rangle \mid M \text{ is a Turing Machine and } M \text{ accepts } w\}$
- Suppose there's a *decider* H for A_{TM}

$$H(\langle M, w\rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

An Undecidable Language

- $A_{TM} = \{\langle M, w\rangle \,|\, M$ is a Turing Machine and M accepts $w\}$
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- $\therefore$ Neither D nor H can exist and so A_{TM} is undecidable

Machine D

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$
M_1	*accept*	*reject*	*accept*	*accept*	$\cdots$	*accept*
M_2	*reject*	*accept*	*accept*	*reject*	$\cdots$	*reject*
M_3	*accept*	*accept*	*reject*	*reject*	$\cdots$	*reject*
M_4	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ldots$
D	*reject*	*reject*	*accept*	*reject*	$\cdots$	?

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- $\chi_A = f(A)$ is called the *characteristic sequence* of A

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Corollary

The language $\overline{A_{TM}}$ is non-Turing-recognizable.

Table of Contents

Next Class

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- Specific Undecidable Problem

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- Mapping Reducibility

Limits of Turing Machines

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