Cosets and Lagrange

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MACRICOSTAS SCHOOL OF ARTS & SCIENCES

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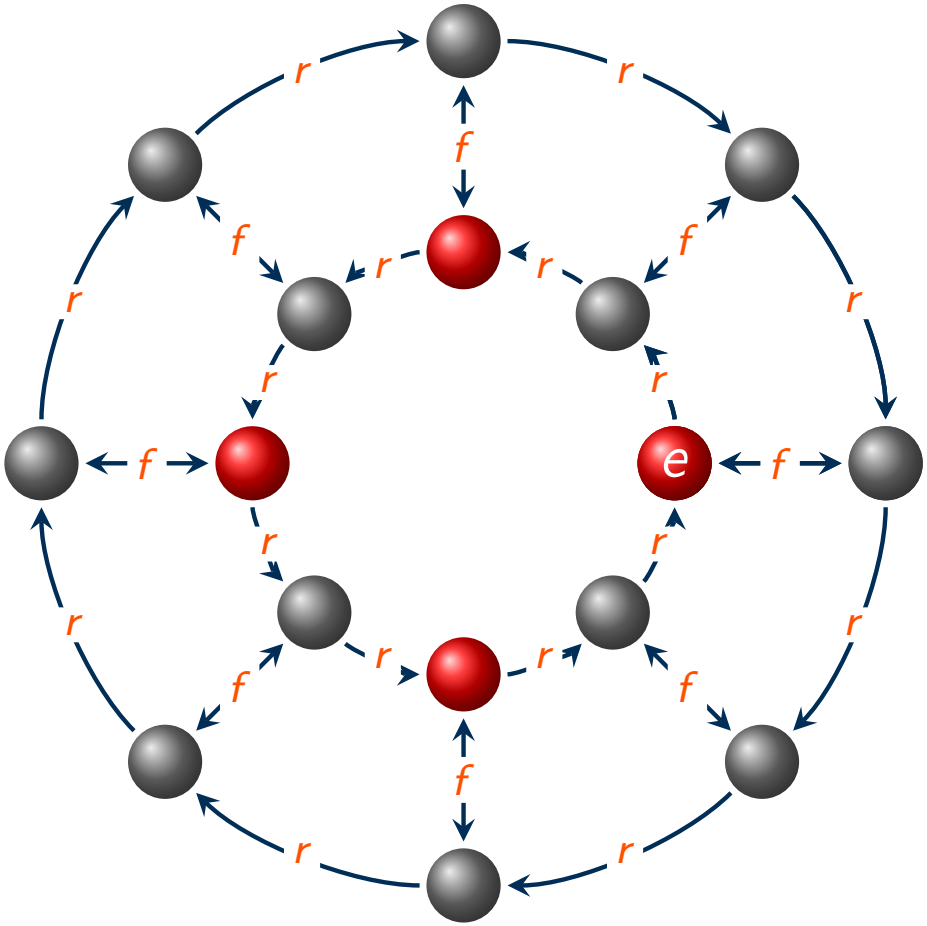
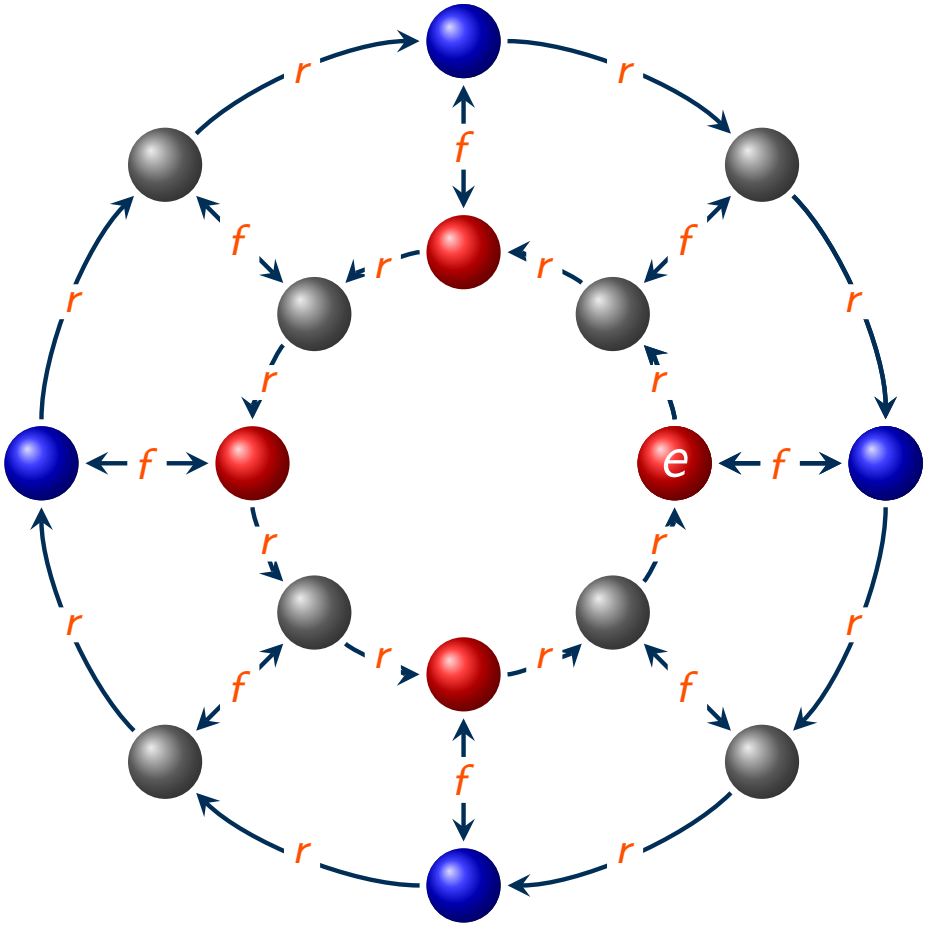
$D_8 = \langle r, f \,|\, r^8 = f^2 = e, rf = fr^{-1} \rangle$



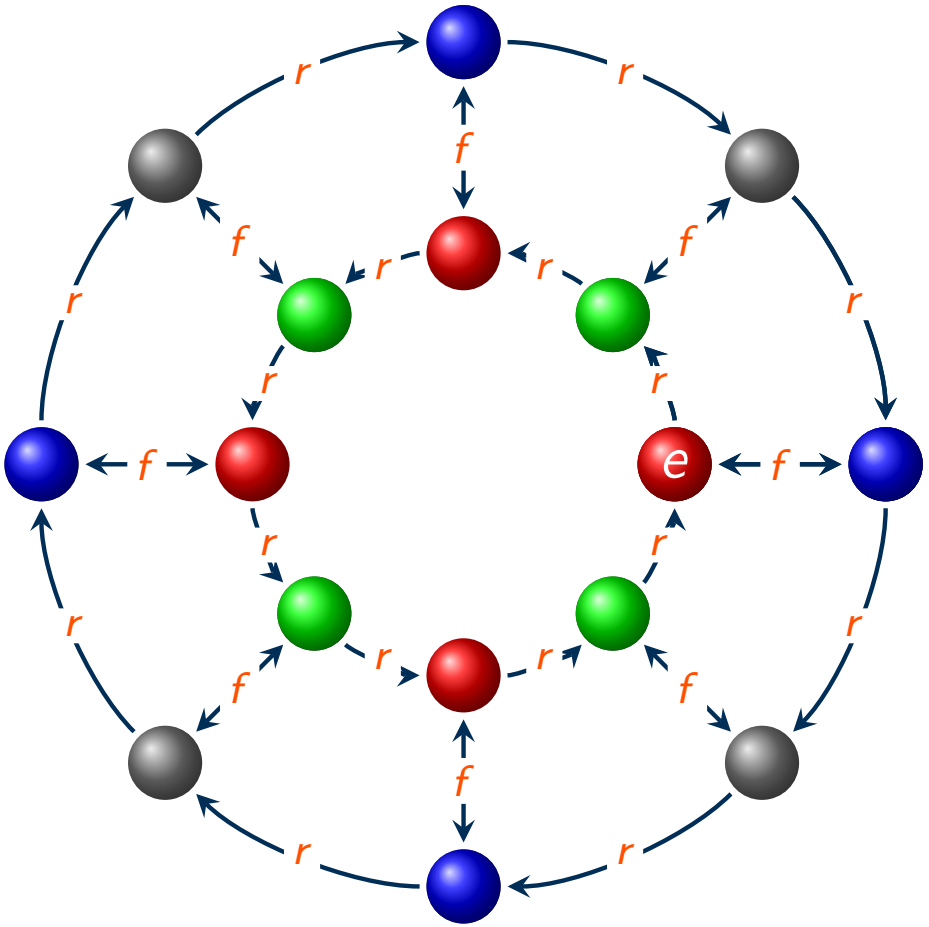
- $\langle r^2 \rangle = \{r^2, r^4, r^6, e\}$

$D_8 = \langle r, f | r^8 = f^2 = e, rf = fr^{-1} \rangle$



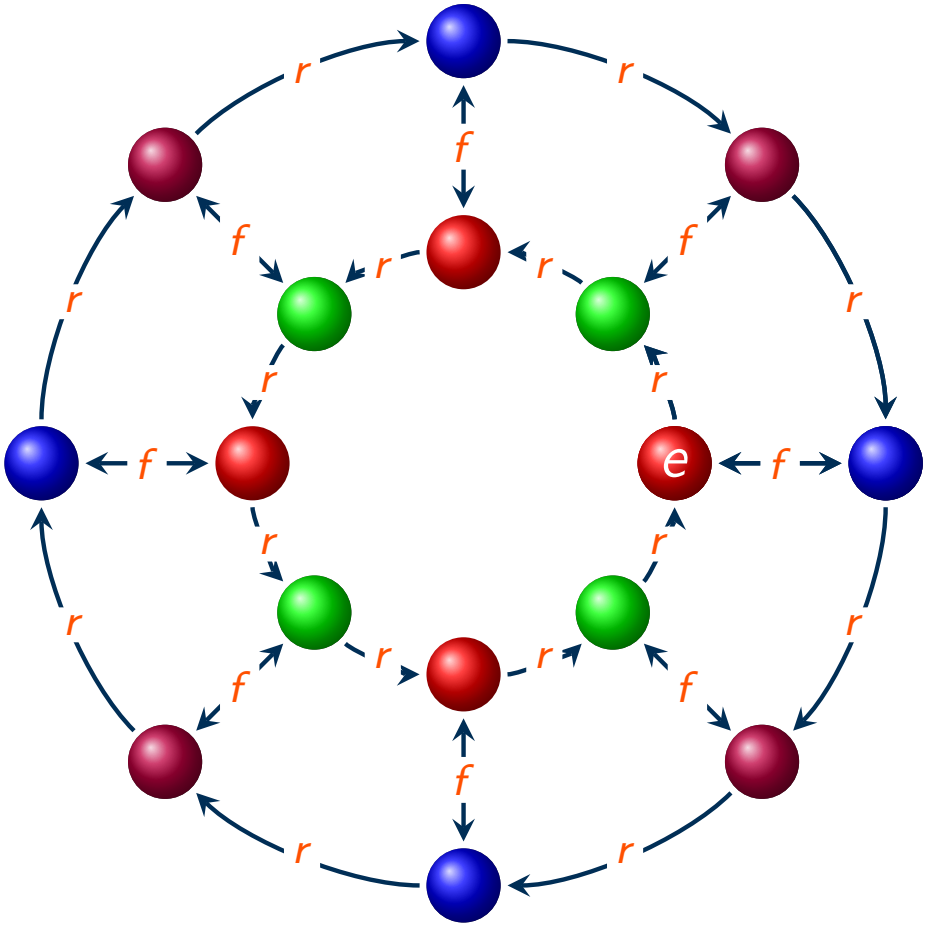
- $\langle r^2 \rangle = \{r^2, r^4, r^6, e\}$
- $f\langle r^2 \rangle = \{fr^2, fr^4, fr^6, f\}$

$D_8 = \langle r, f | r^8 = f^2 = e, rf = fr^{-1} \rangle$



- $\langle r^2 \rangle = \{r^2, r^4, r^6, e\}$
- $f\langle r^2 \rangle = \{fr^2, fr^4, fr^6, f\}$
- $r\langle r^2 \rangle = \{r^3, r^5, r^7, r\}$

$D_8 = \langle r, f | r^8 = f^2 = e, rf = fr^{-1} \rangle$



- $\langle r^2 \rangle = \{r^2, r^4, r^6, e\}$
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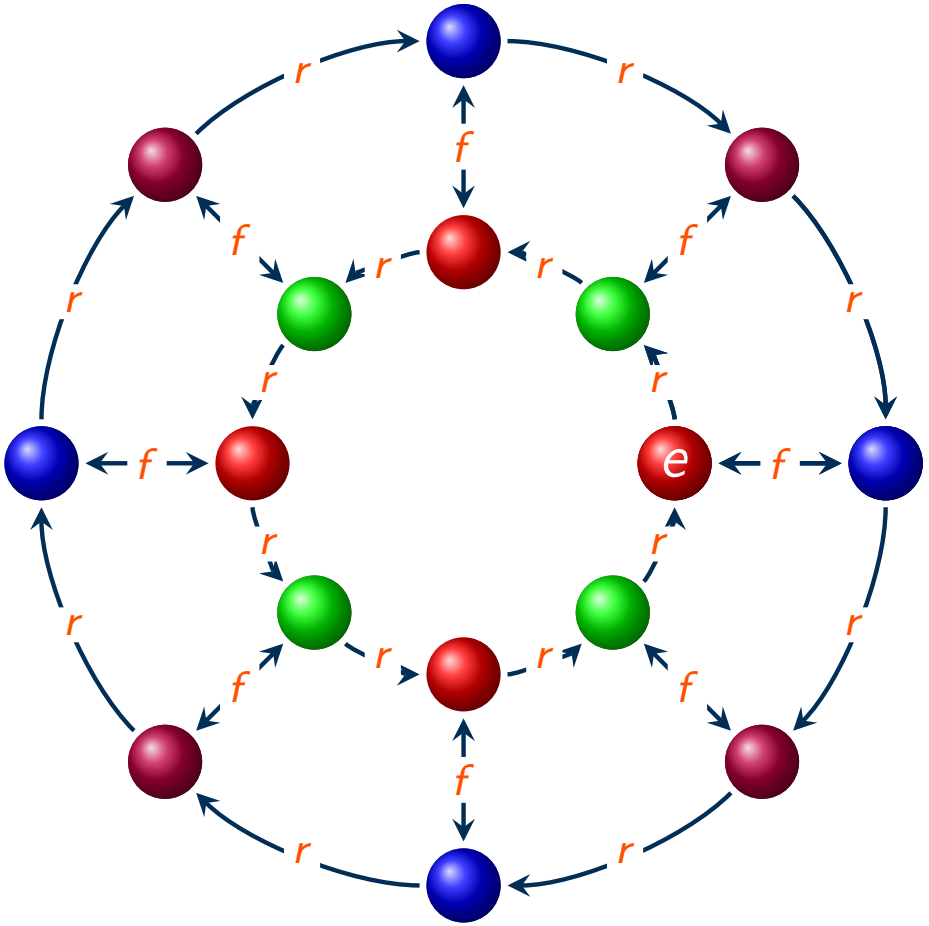
- $r\langle r^2 \rangle = \{r^3, r^5, r^7, r\}$
- $fr\langle r^2 \rangle = \{fr^3, fr^5, fr^7, fr\}$

$D_8 = \langle r, f | r^8 = f^2 = e, rf = fr^{-1} \rangle$



- $\langle r^2 \rangle = \{r^2, r^4, r^6, e\}$
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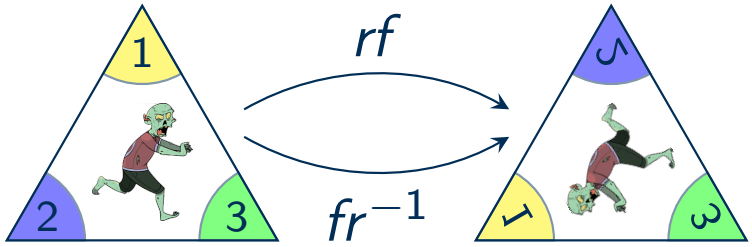
- $r\langle r^2 \rangle = \{r^3, r^5, r^7, r\}$
- $fr\langle r^2 \rangle = \{fr^3, fr^5, fr^7, fr\}$
- $\langle r^2 \rangle f = \{r^2 f, r^4 f, r^6 f, f\}$

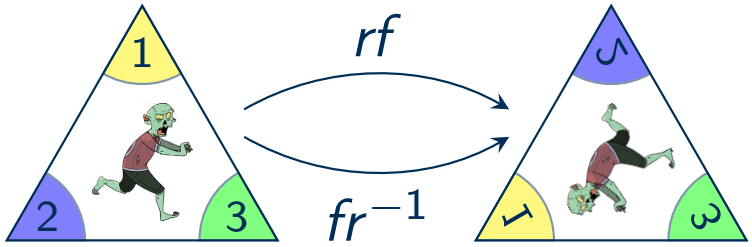
Calculating in D_n

$$D_n = \langle r, f \,|\, r^n = f^2 = e, rf = fr^{-1} \rangle$$



Calculating in D_n

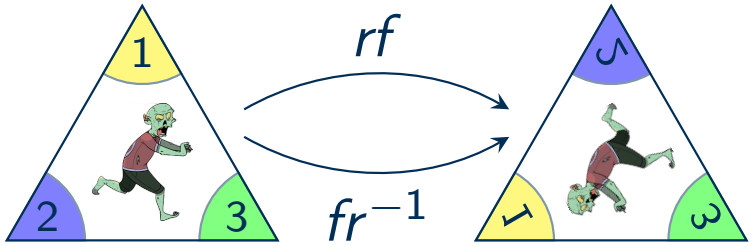
$$D_n = \langle r, f \,|\, r^n = f^2 = e, rf = fr^{-1} \rangle$$



$$r^3 f$$

Calculating in D_n

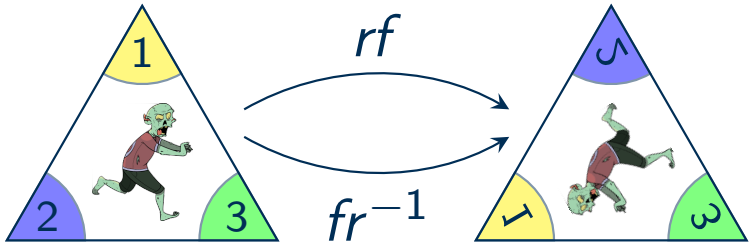
$$D_n = \left\langle r, f \middle| r^n = f^2 = e, rf = fr^{-1} \right\rangle$$



$$r^3 f = r^2(rf)$$

Calculating in D_n

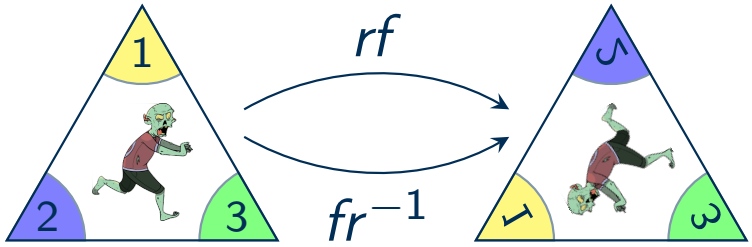
$$D_n = \left\langle r, f \middle| r^n = f^2 = e, rf = fr^{-1} \right\rangle$$



$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \end{aligned}$$

Calculating in D_n

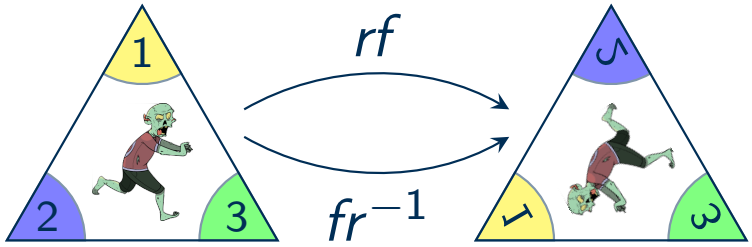
$$D_n = \langle r, f \,|\, r^n = f^2 = e, rf = fr^{-1} \rangle$$



$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \\ &= r(rf)r^{n-1} \end{aligned}$$

Calculating in D_n

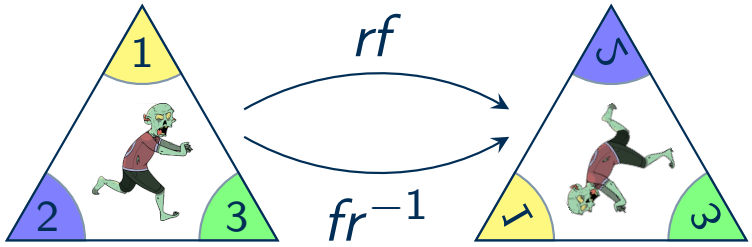
$$D_n = \langle r, f \,|\, r^n = f^2 = e, rf = fr^{-1} \rangle$$



$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \\ &= r(rf)r^{n-1} \\ &= r(fr^{-1})r^{n-1} \end{aligned}$$

Calculating in D_n

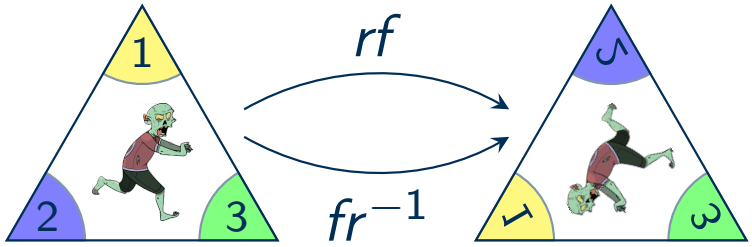
$$D_n = \langle r, f \,|\, r^n = f^2 = e, rf = fr^{-1} \rangle$$



$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \\ &= r(rf)r^{n-1} \\ &= r(fr^{-1})r^{n-1} \\ &= (rf)r^{n-2} \end{aligned}$$

Calculating in D_n

$$D_n = \langle r, f \,|\, r^n = f^2 = e, rf = fr^{-1} \rangle$$



$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \\ &= r(rf)r^{n-1} \\ &= r(fr^{-1})r^{n-1} \\ &= (rf)r^{n-2} \\ &= (fr^{-1})r^{n-2} \end{aligned}$$

Calculating in D_n

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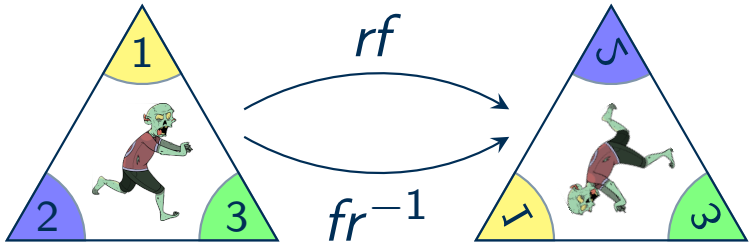

$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \\ &= r(rf)r^{n-1} \\ &= r(fr^{-1})r^{n-1} \\ &= (rf)r^{n-2} \\ &= (fr^{-1})r^{n-2} \\ &= fr^{n-3} \end{aligned}$$

Calculating in D_n

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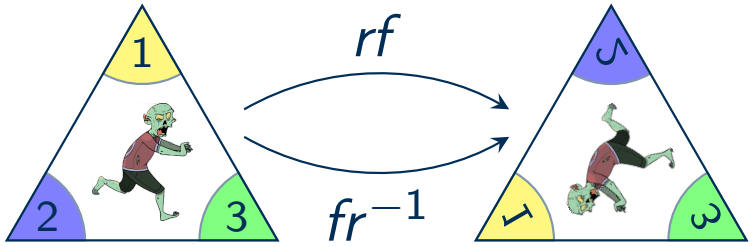

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$$r^k f = fr^{-k} = fr^{n-k}$$

Calculating in D_n

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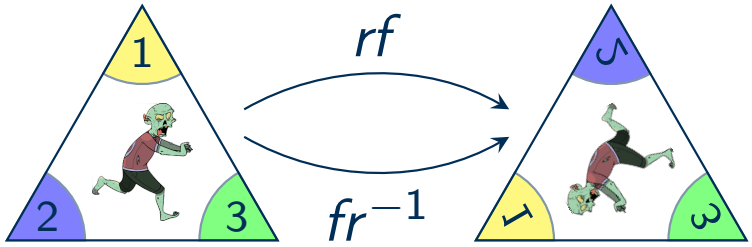
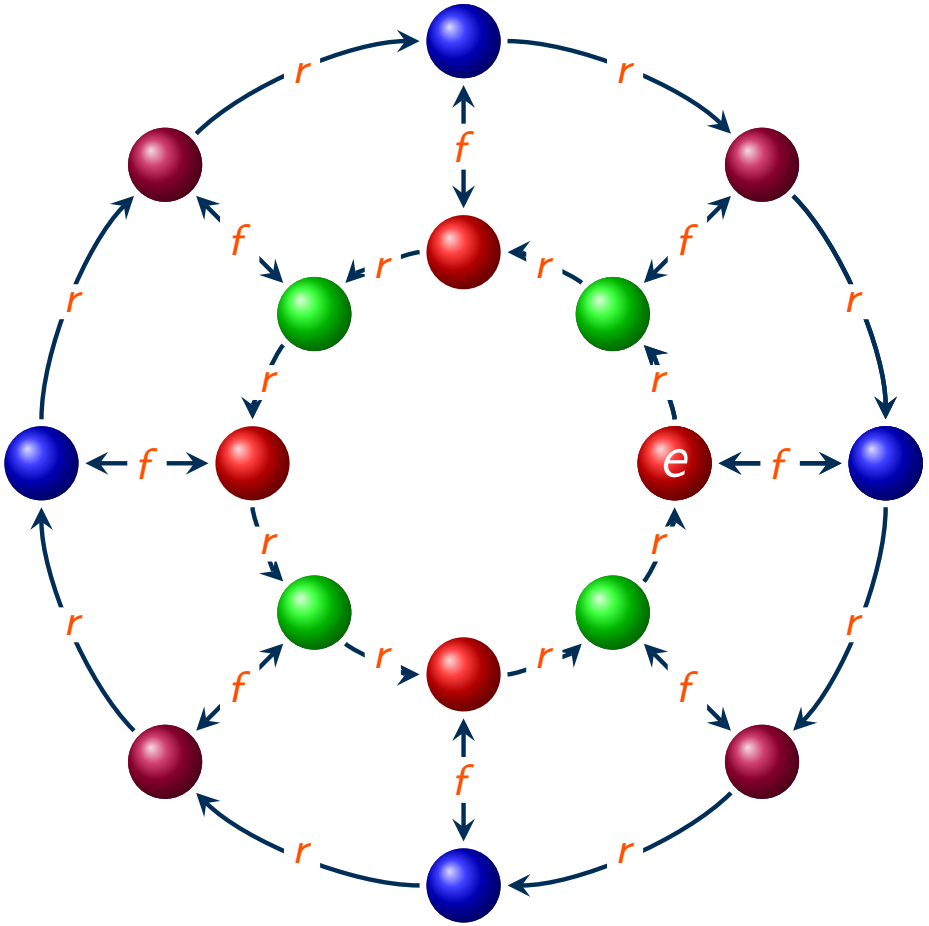


$$\begin{aligned} r^3 f &= r^2(rf) \\ &= r^2(fr^{-1}) \\ &= r(rf)r^{n-1} \\ &= r(fr^{-1})r^{n-1} \\ &= (rf)r^{n-2} \\ &= (fr^{-1})r^{n-2} \\ &= fr^{n-3} \end{aligned}$$

$$r^k f = fr^{-k} = fr^{n-k}$$

$$fr^k = r^{-k} f = r^{n-k} f$$

$D_8 = \langle r, f \mid r^8 = f^2 = e, rf = fr^{-1} \rangle$



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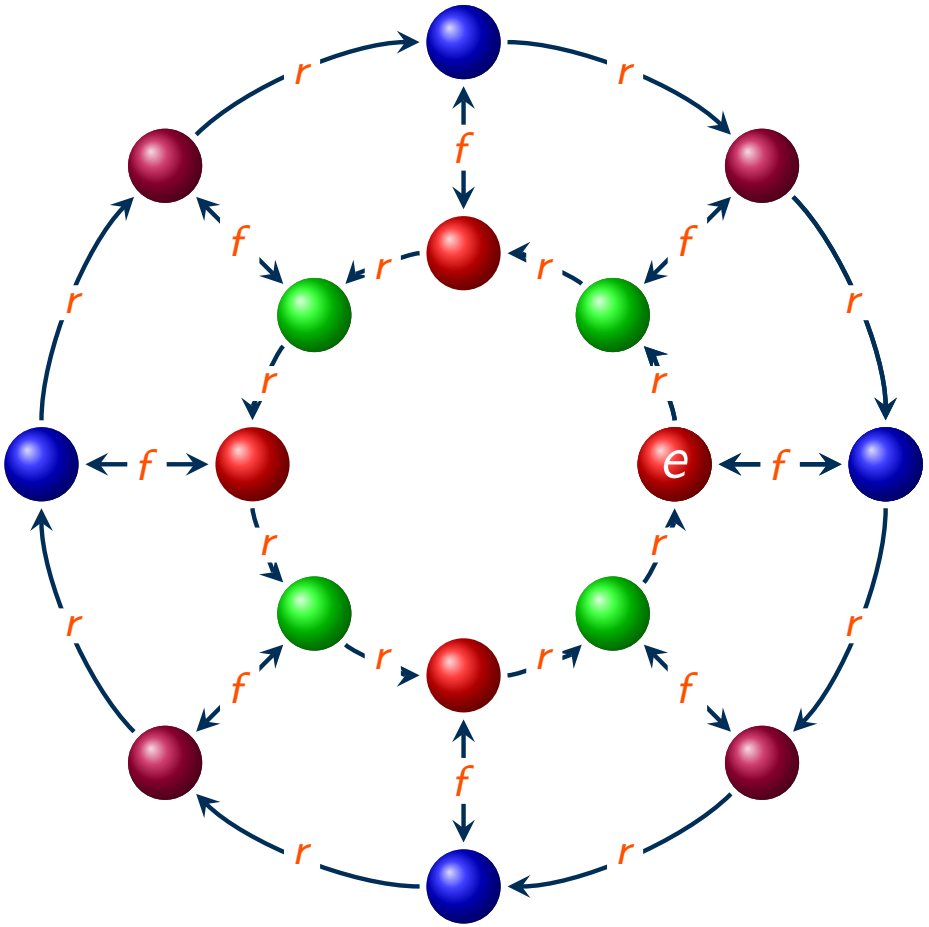

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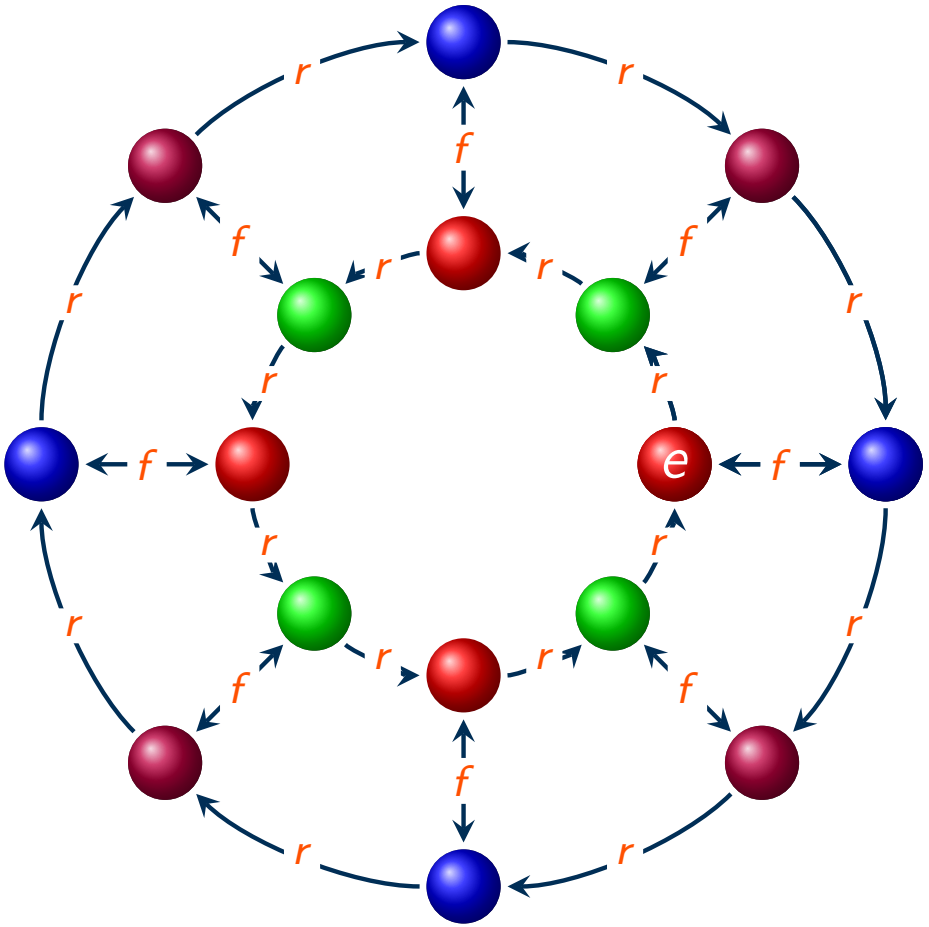
- $r\langle r^2 \rangle = \{r^3, r^5, r^7, r\}$
- $fr\langle r^2 \rangle = \{fr^3, fr^5, fr^7, fr\}$
- $\langle r^2 \rangle f = \{fr^6, fr^4, fr^2, f\}$

$D_8 = \langle r, f \,|\, r^8 = f^2 = e, rf = fr^{-1} \rangle$



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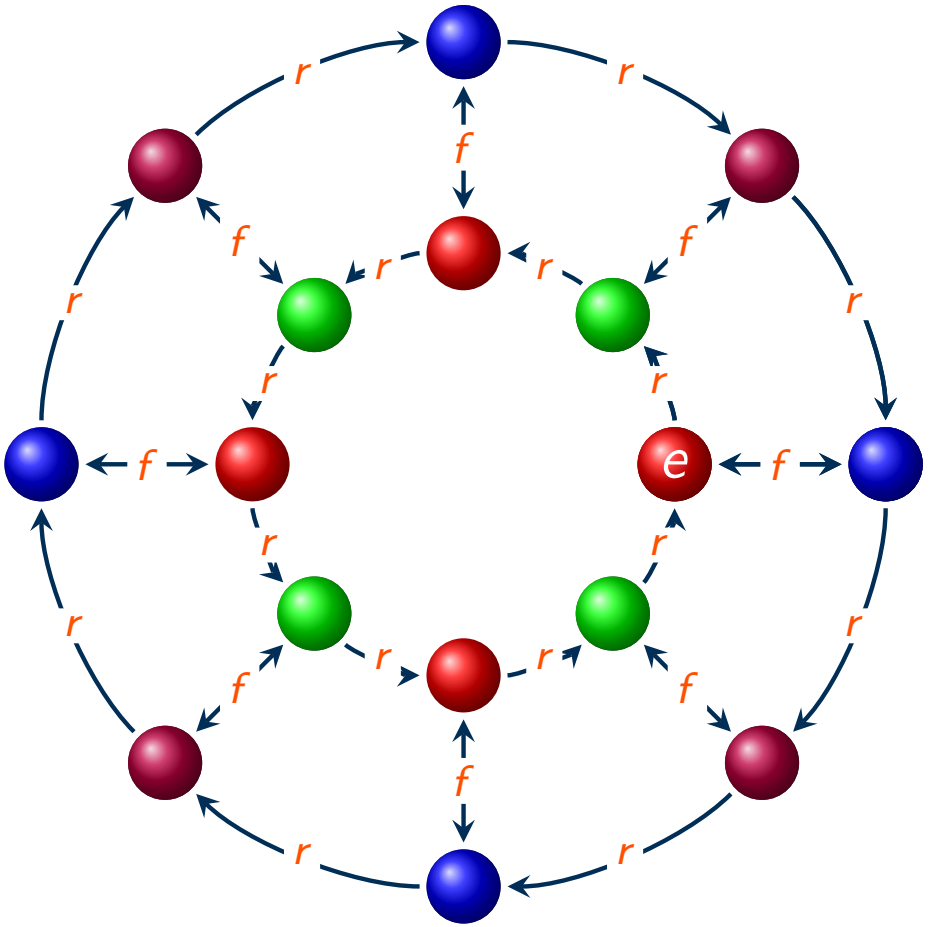
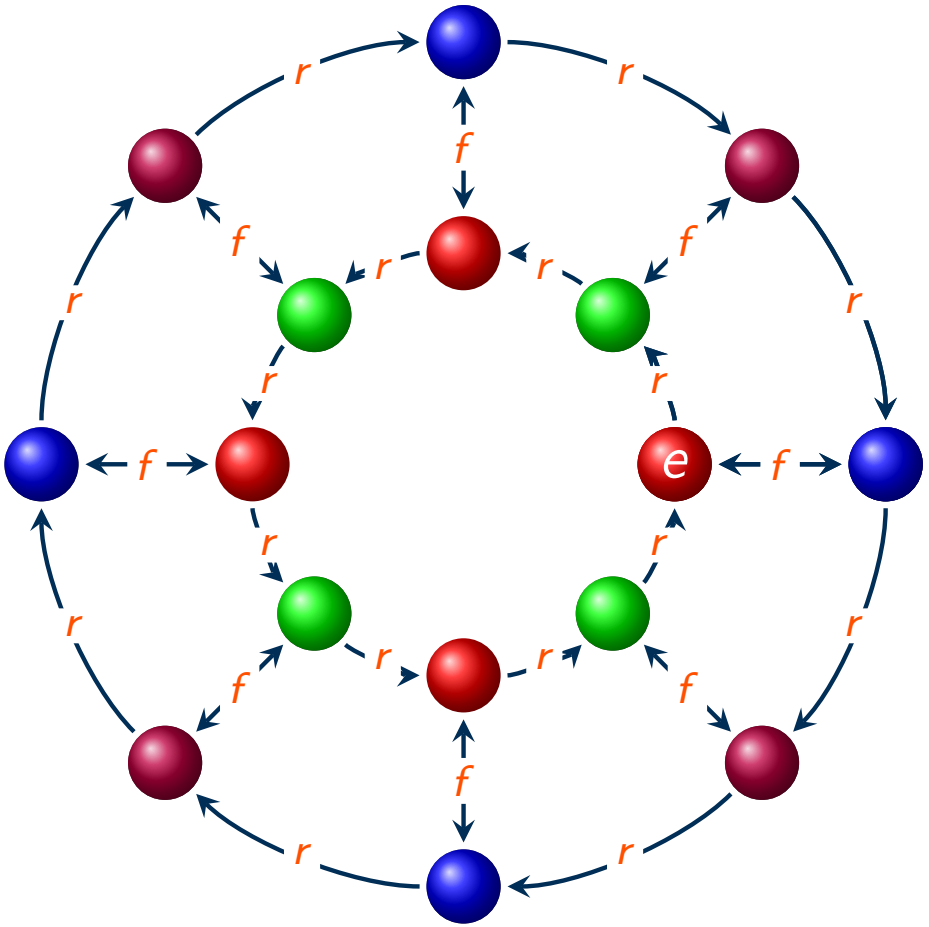
- $f\langle r^2 \rangle = \{fr^2, fr^4, fr^6, f\}$
- $r\langle r^2 \rangle = \{r^3, r^5, r^7, r\}$
- $fr\langle r^2 \rangle = \{fr^3, fr^5, fr^7, fr\}$
- $\langle r^2 \rangle f = \{fr^6, fr^4, fr^2, f\}$
- $\langle r^2 \rangle r = \{r^3, r^5, r^7, r\}$

$D_8 = \langle r, f \mid r^8 = f^2 = e, rf = fr^{-1} \rangle$



- $\langle r^2 \rangle = \{r^2, r^4, r^6, e\}$
- $f\langle r^2 \rangle = \{fr^2, fr^4, fr^6, f\}$
- $r\langle r^2 \rangle = \{r^3, r^5, r^7, r\}$
- $fr\langle r^2 \rangle = \{fr^3, fr^5, fr^7, fr\}$
- $\langle r^2 \rangle f = \{fr^6, fr^4, fr^2, f\}$
- $\langle r^2 \rangle r = \{r^3, r^5, r^7, r\}$
- $\langle r^2 \rangle fr = \{r^2fr, r^4fr, r^6fr, fr\}$

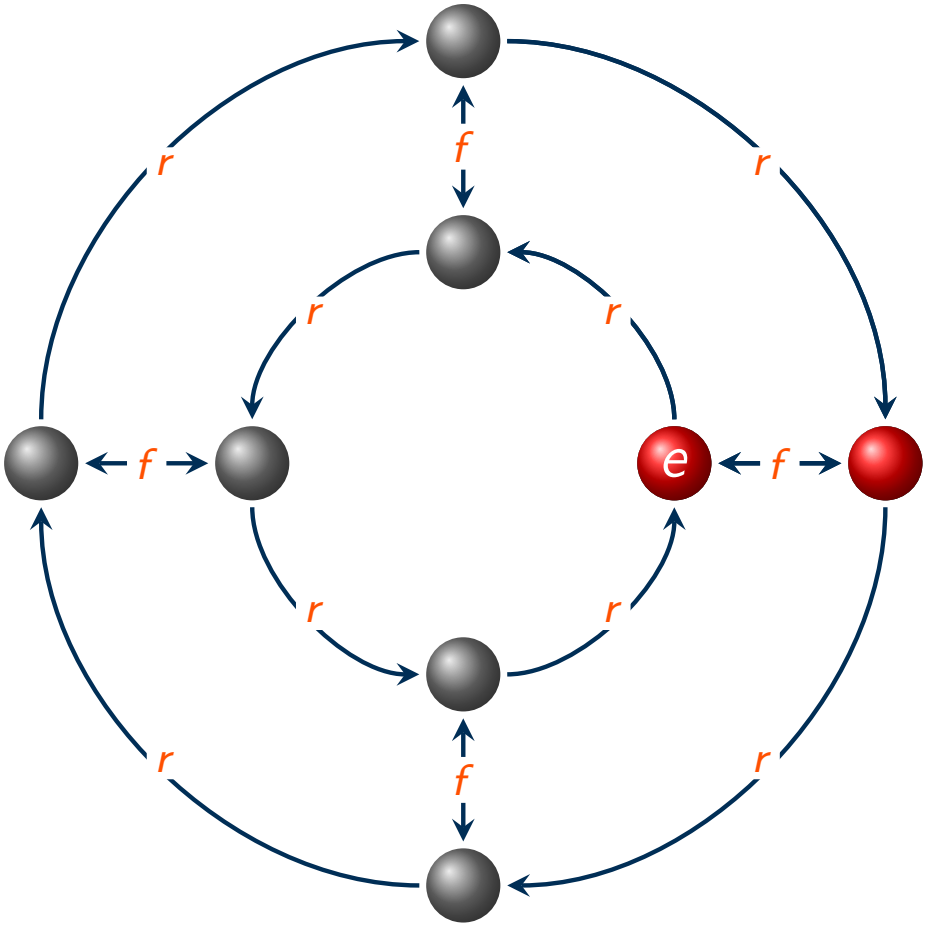
$D_8 = \left\langle r, f \middle| r^8 = f^2 = e, rf = fr^{-1} \right\rangle$



- $\left\langle r^2 \right\rangle = \left\{ r^2, r^4, r^6, e \right\}$
- $f\left\langle r^2 \right\rangle = \left\{ fr^2, fr^4, fr^6, f \right\}$
- $r\left\langle r^2 \right\rangle = \left\{ r^3, r^5, r^7, r \right\}$
- $fr\left\langle r^2 \right\rangle = \left\{ fr^3, fr^5, fr^7, fr \right\}$
- $\left\langle r^2 \right\rangle f = \left\{ fr^6, fr^4, fr^2, f \right\}$
- $\left\langle r^2 \right\rangle r = \left\{ r^3, r^5, r^7, r \right\}$
- $\left\langle r^2 \right\rangle fr = \left\{ fr^7, fr^5, fr^3, fr \right\}$

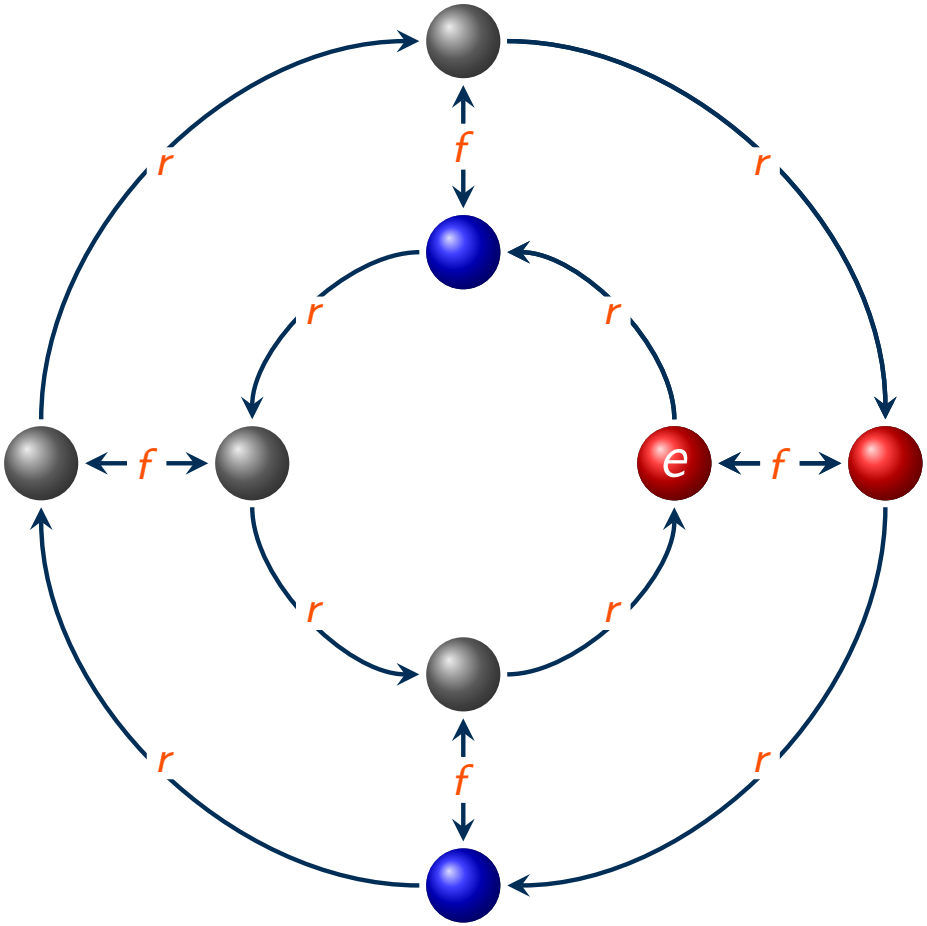
$$D_8 = \left\langle r^2 \right\rangle \cup f\left\langle r^2 \right\rangle \cup r\left\langle r^2 \right\rangle \cup fr\left\langle r^2 \right\rangle$$

$D_4 = \langle r, f \,|\, r^4 = f^2 = e, rf = fr^{-1} \rangle$



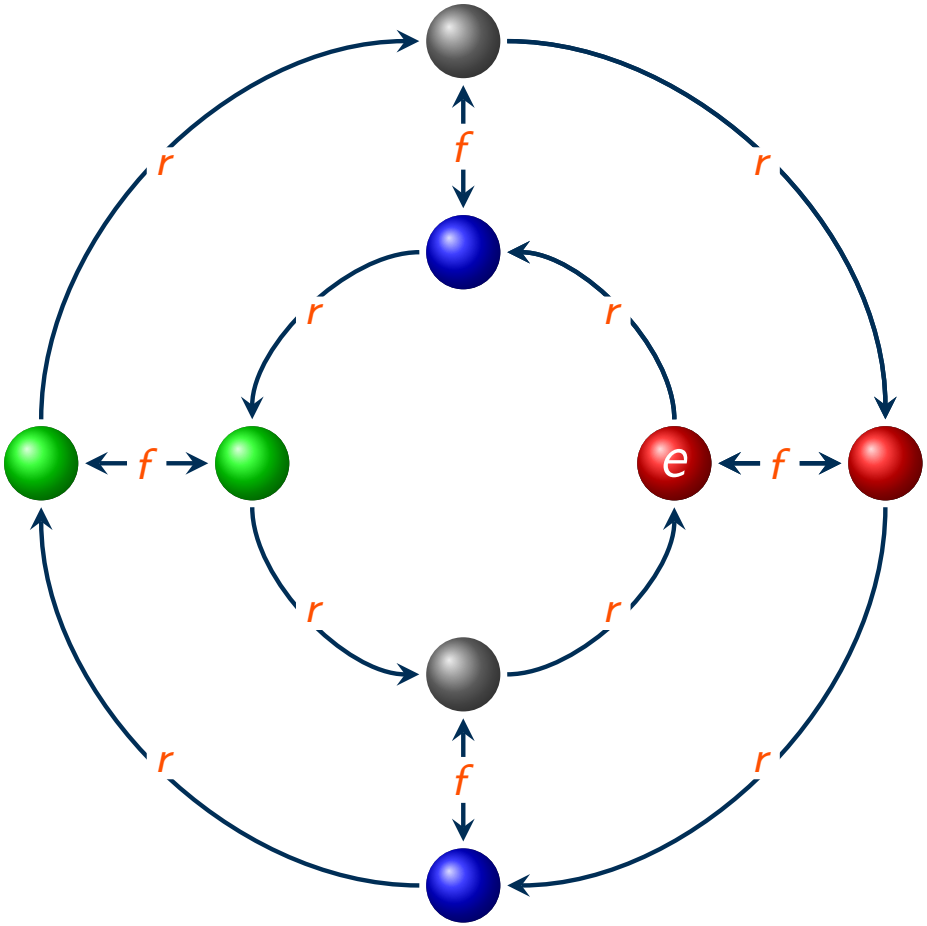
- $\langle f \rangle = \{f, e\}$

$D_4 = \langle r, f \,|\, r^4 = f^2 = e, rf = fr^{-1} \rangle$



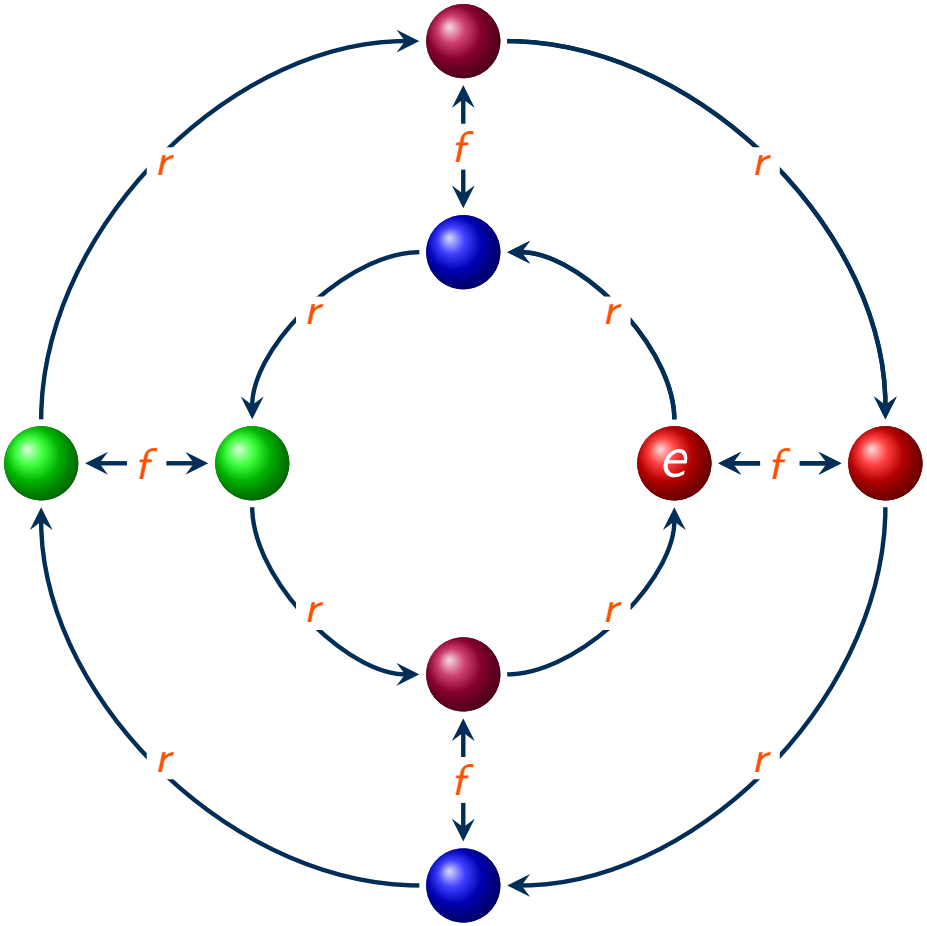
- $\langle f \rangle = \{f, e\}$
- $r\langle f \rangle = \{rf, r\}$

$D_4 = \langle r, f \,|\, r^4 = f^2 = e, rf = fr^{-1} \rangle$



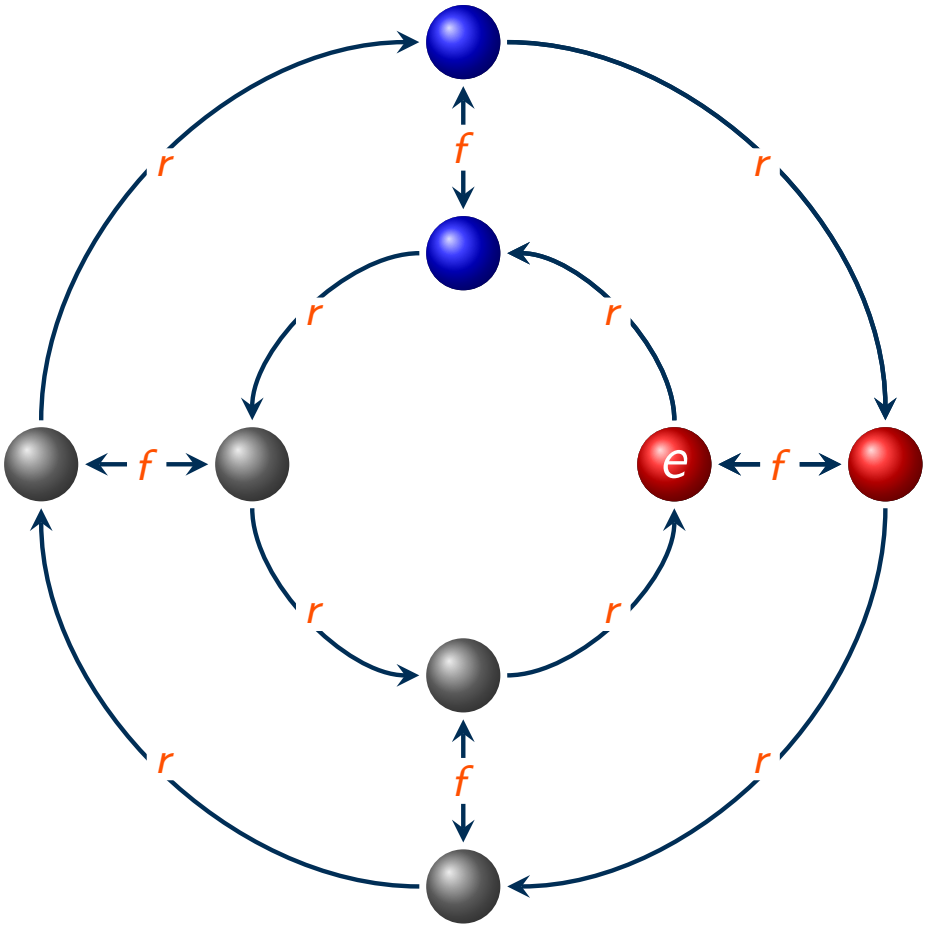
- $\langle f \rangle = \{f, e\}$
- $r\langle f \rangle = \{rf, r\}$
- $r^2\langle f \rangle = \{r^2 f, r^2\}$

$D_4 = \langle r, f \mid r^4 = f^2 = e, rf = fr^{-1} \rangle$



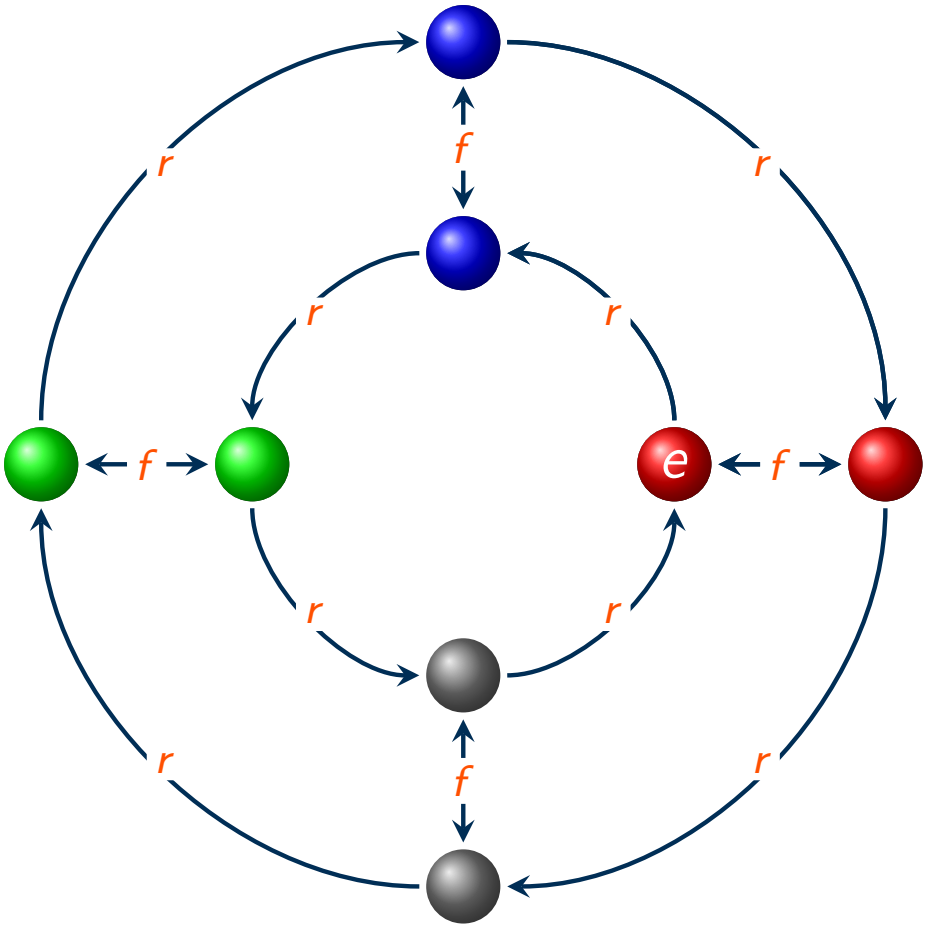
- $\langle f \rangle = \{f, e\}$
- $r\langle f \rangle = \{rf, r\}$
- $r^2\langle f \rangle = \{r^2 f, r^2\}$
- $r^3\langle f \rangle = \{r^3 f, r^3\}$

$D_4 = \langle r, f \,|\, r^4 = f^2 = e, rf = fr^{-1} \rangle$



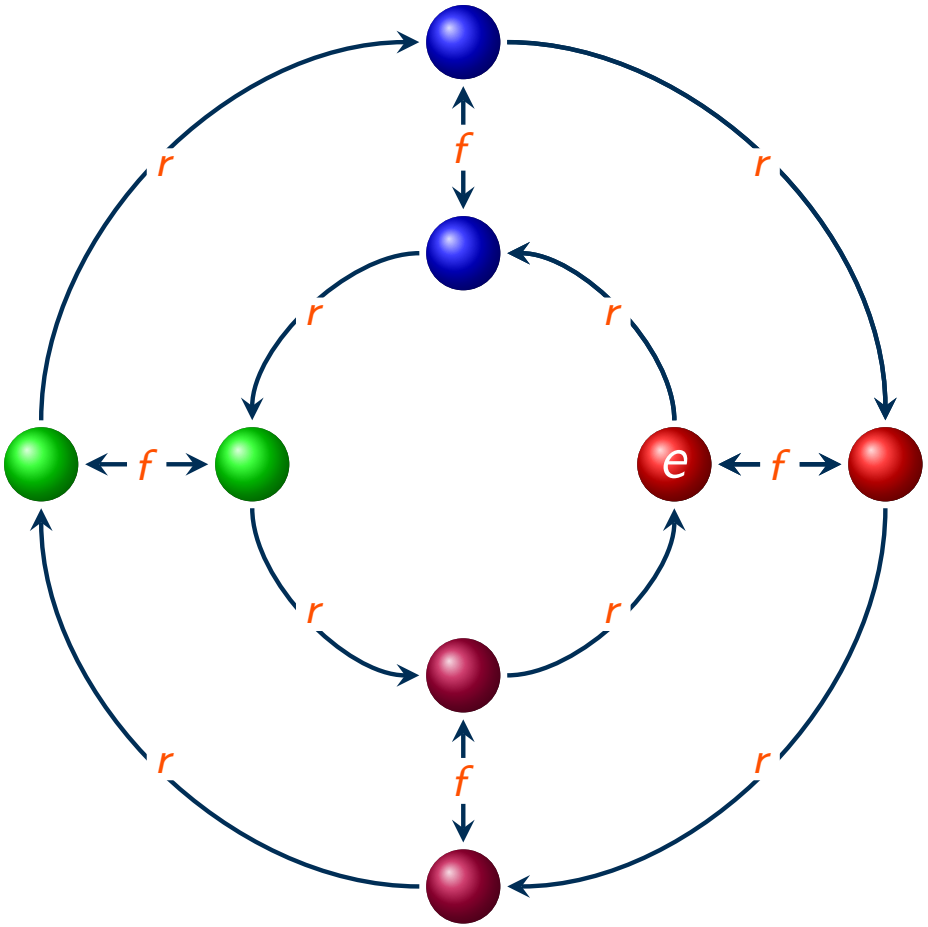
- $\langle f \rangle = \{f, e\}$
- $r\langle f \rangle = \{rf, r\}$
- $r^2\langle f \rangle = \{r^2 f, r^2\}$
- $r^3\langle f \rangle = \{r^3 f, r^3\}$
- $\langle f \rangle r = \{fr, r\} = \{r^3 f, r\}$

$D_4 = \langle r, f \,|\, r^4 = f^2 = e, rf = fr^{-1} \rangle$



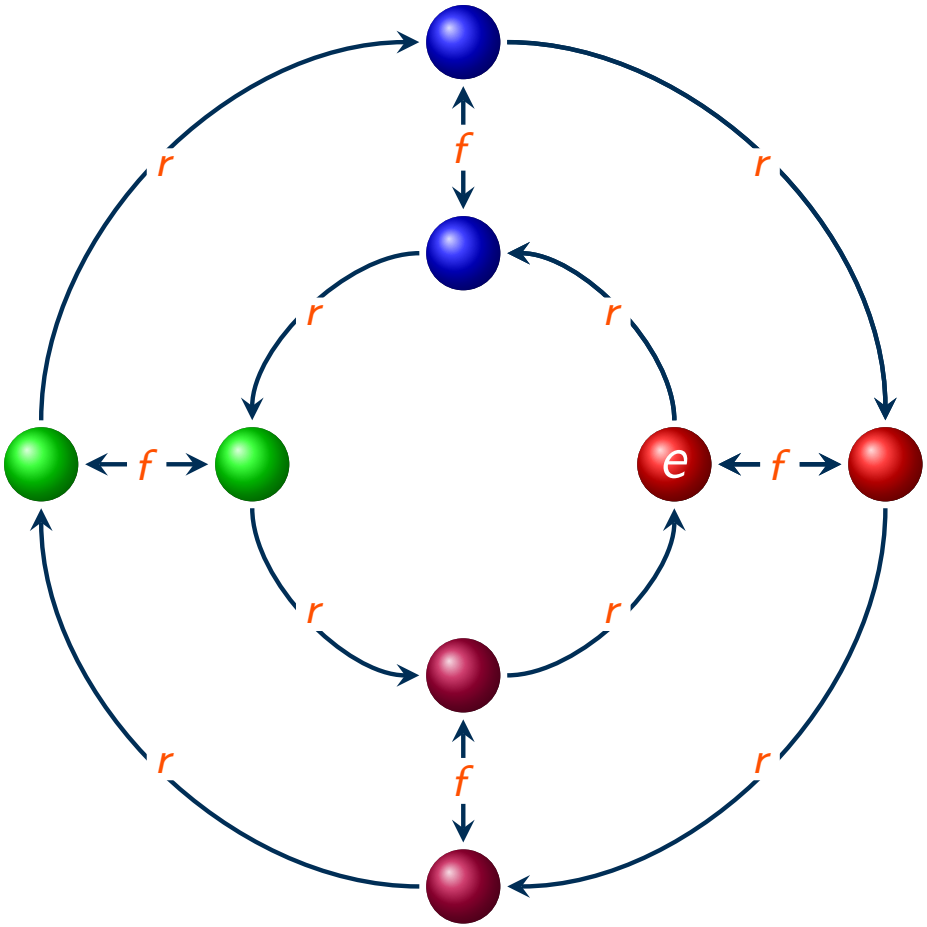
- $\langle f \rangle = \{f, e\}$
- $r\langle f \rangle = \{rf, r\}$
- $r^2\langle f \rangle = \{r^2 f, r^2\}$
- $r^3\langle f \rangle = \{r^3 f, r^3\}$
- $\langle f \rangle r = \{fr, r\} = \{r^3 f, r\}$
- $\langle f \rangle r^2 = \{fr^2, r^2\} = \{r^2 f, r^2\}$

$D_4 = \langle r, f \,|\, r^4 = f^2 = e, rf = fr^{-1} \rangle$



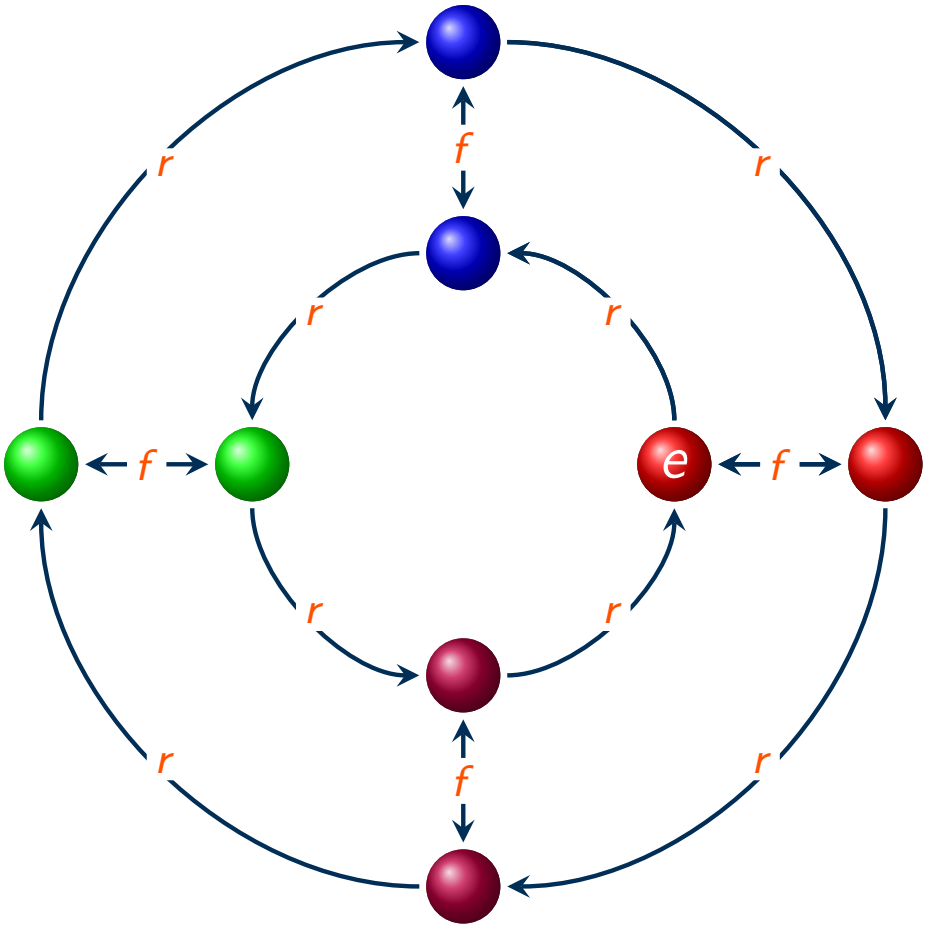
- $\langle f \rangle = \{f, e\}$
- $r\langle f \rangle = \{rf, r\}$
- $r^2\langle f \rangle = \{r^2 f, r^2\}$
- $r^3\langle f \rangle = \{r^3 f, r^3\}$
- $\langle f \rangle r = \{fr, r\} = \{r^3 f, r\}$
- $\langle f \rangle r^2 = \{fr^2, r^2\} = \{r^2 f, r^2\}$
- $\langle f \rangle r^3 = \{fr^3, r^3\} = \{rf, r^3\}$

$D_4 = \left\langle r, f \middle| r^4 = f^2 = e, rf = fr^{-1} \right\rangle$



- $\langle f \rangle = \{f, e\}$
- $r\langle f \rangle = \{rf, r\}$
- $r^2\langle f \rangle = \{r^2 f, r^2\}$
- $r^3\langle f \rangle = \{r^3 f, r^3\}$
- $\langle f \rangle r = \{fr, r\} = \{r^3 f, r\}$
- $\langle f \rangle r^2 = \{fr^2, r^2\} = \{r^2 f, r^2\}$
- $\langle f \rangle r^3 = \{fr^3, r^3\} = \{rf, r^3\}$
- $\exists g \in D_4 : g\langle f \rangle \neq \langle f \rangle g$

$D_4 = \left\langle r, f \middle| r^4 = f^2 = e, rf = fr^{-1} \right\rangle$



- $\langle f \rangle = \{f, e\}$
- $r\langle f \rangle = \{rf, r\}$
- $r^2\langle f \rangle = \{r^2 f, r^2\}$
- $r^3\langle f \rangle = \{r^3 f, r^3\}$
- $\langle f \rangle r = \{fr, r\} = \{r^3 f, r\}$
- $\langle f \rangle r^2 = \{fr^2, r^2\} = \{r^2 f, r^2\}$
- $\langle f \rangle r^3 = \{fr^3, r^3\} = \{rf, r^3\}$
- $\exists g \in D_4 : g\langle f \rangle \neq \langle f \rangle g$

$$D_4 = \langle f \rangle \cup r\langle f \rangle \cup r^2\langle f \rangle \cup r^3\langle f \rangle$$

Definition of Cosets

Definition (Coset)

Given a group G, subgroup H, and element $g \in G$,

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is a **left coset of** H and

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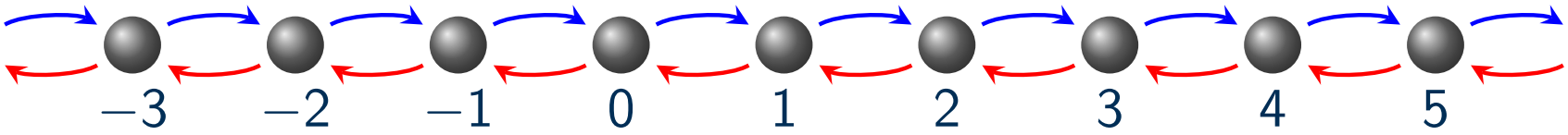
Definition (Normal Subgroup)

Given a group G and subgroup H, if for all $g \in G$,

$$gH = Hg,$$

then we say that H is a **normal subgroup** of G.

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$



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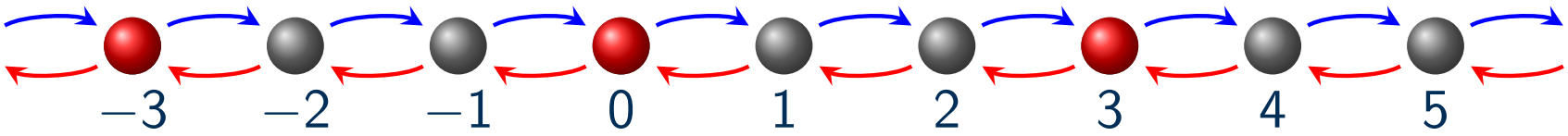

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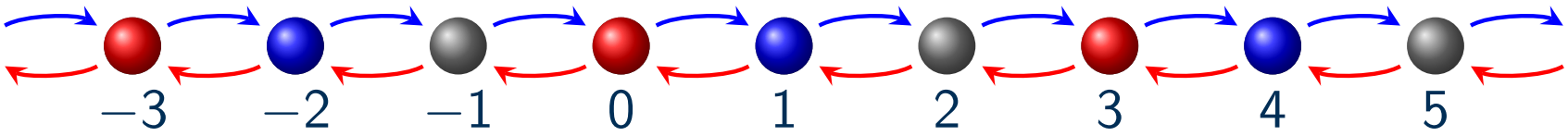

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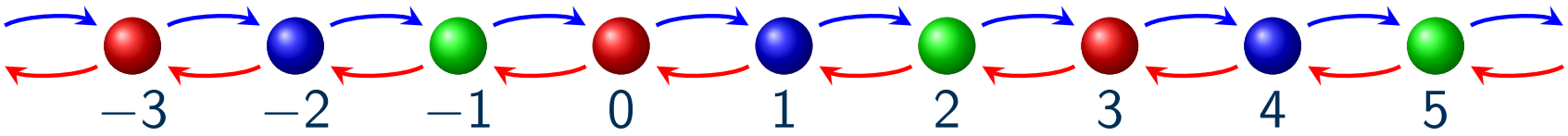

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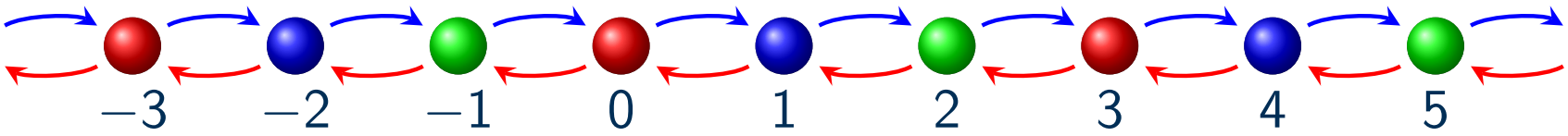

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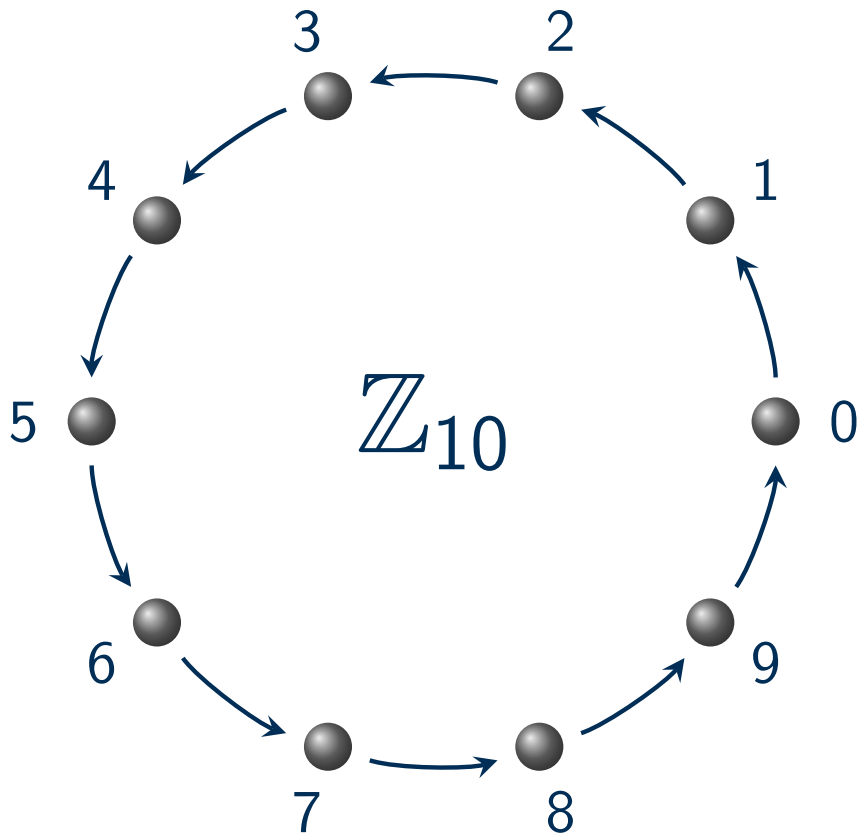

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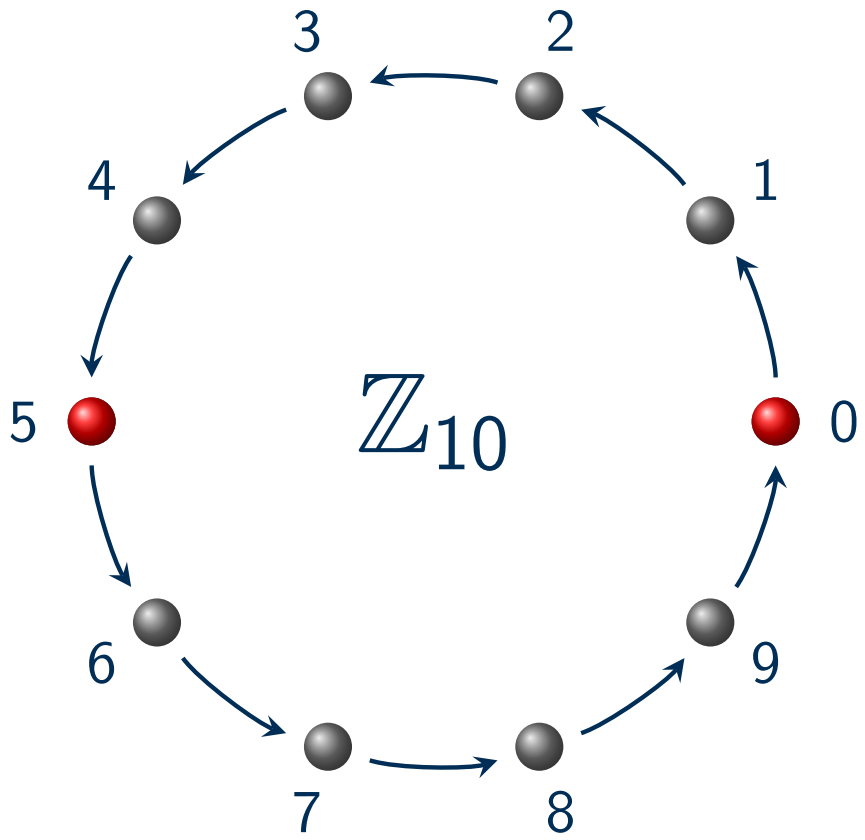

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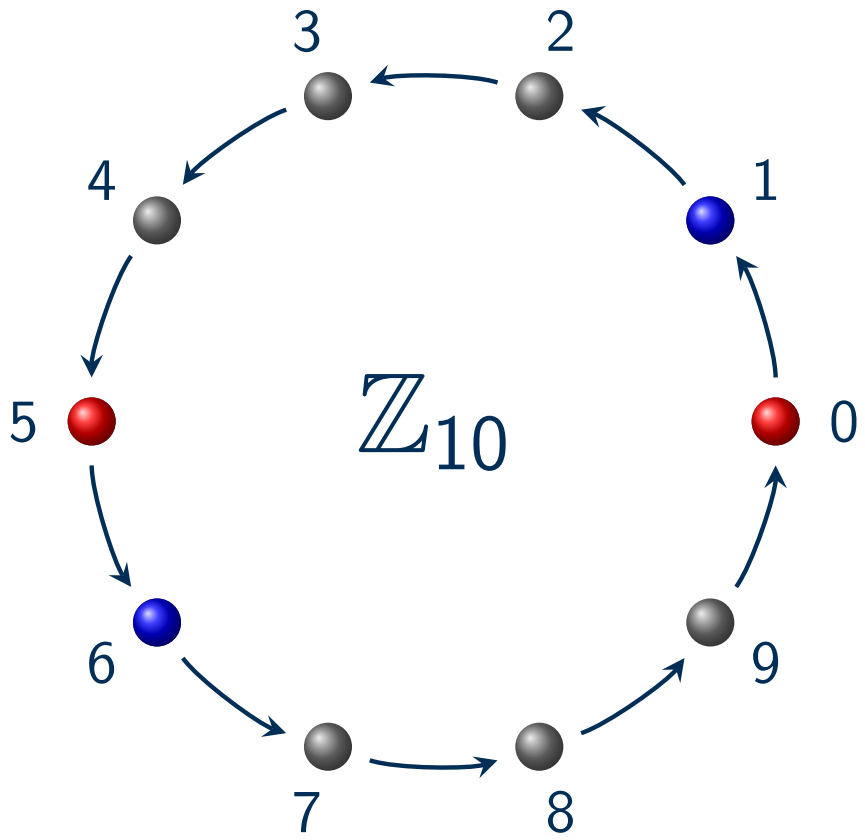

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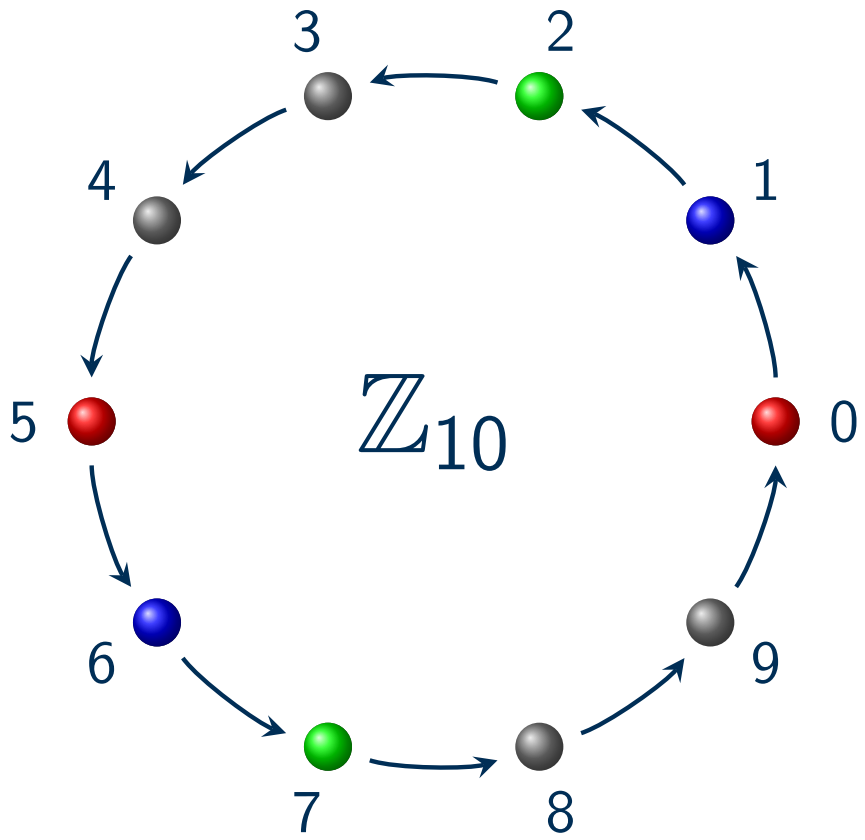

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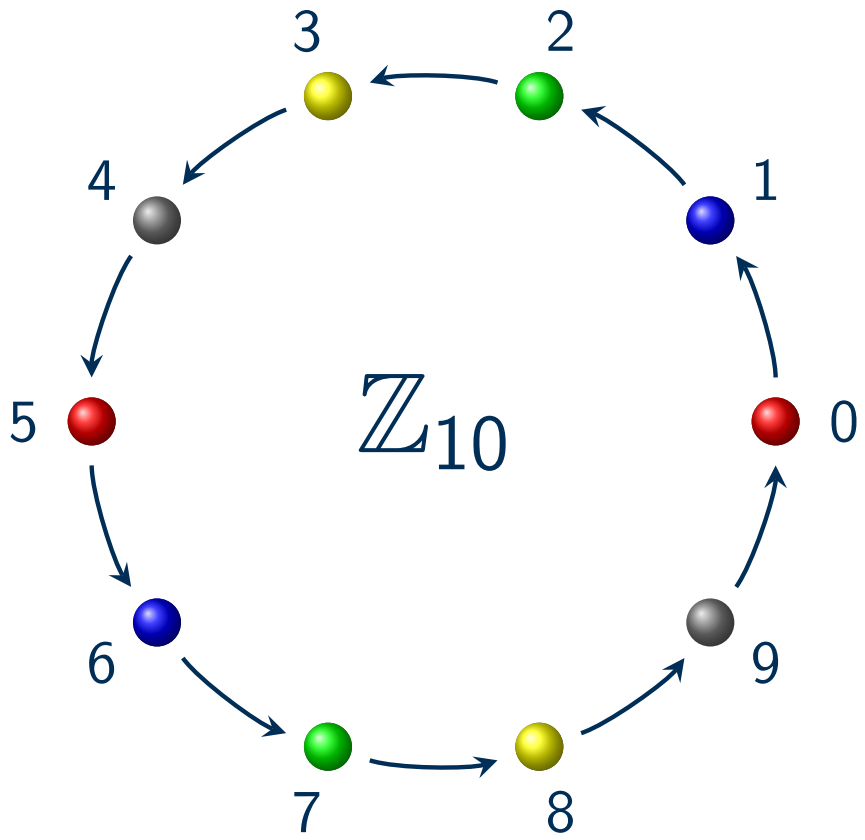

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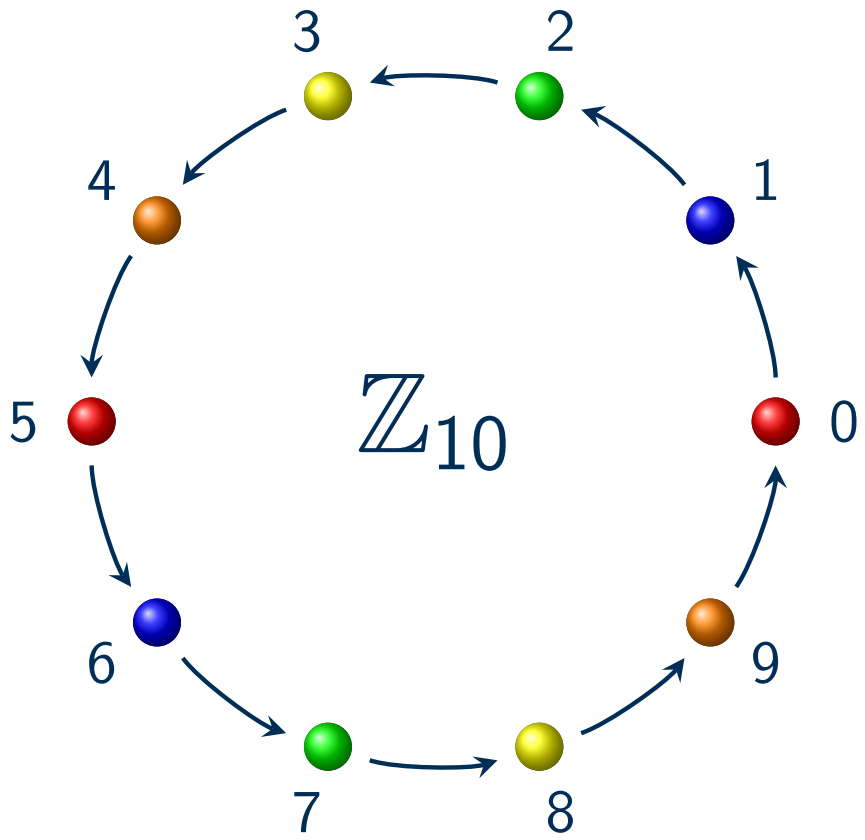

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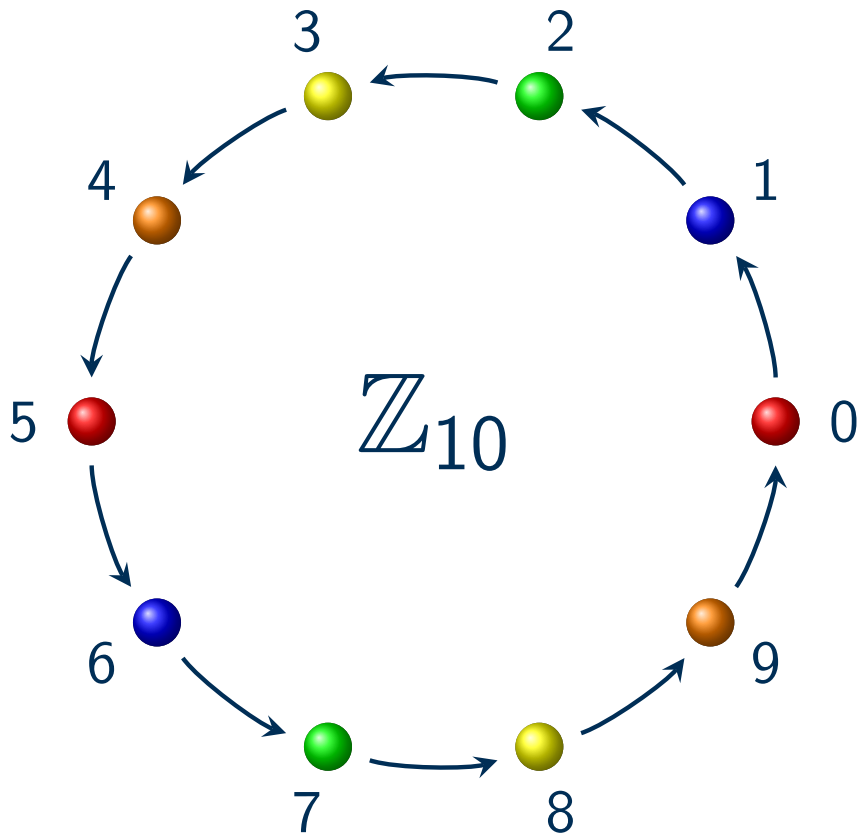

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Coset Properties

Theorem

Given a group G, subgroup $H \subseteq G$, and elements $a, b \in G$:

1. $|H| = |aH|$,
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3. $aH = bH$ *or* $aH \cap bH = \emptyset$, *and*
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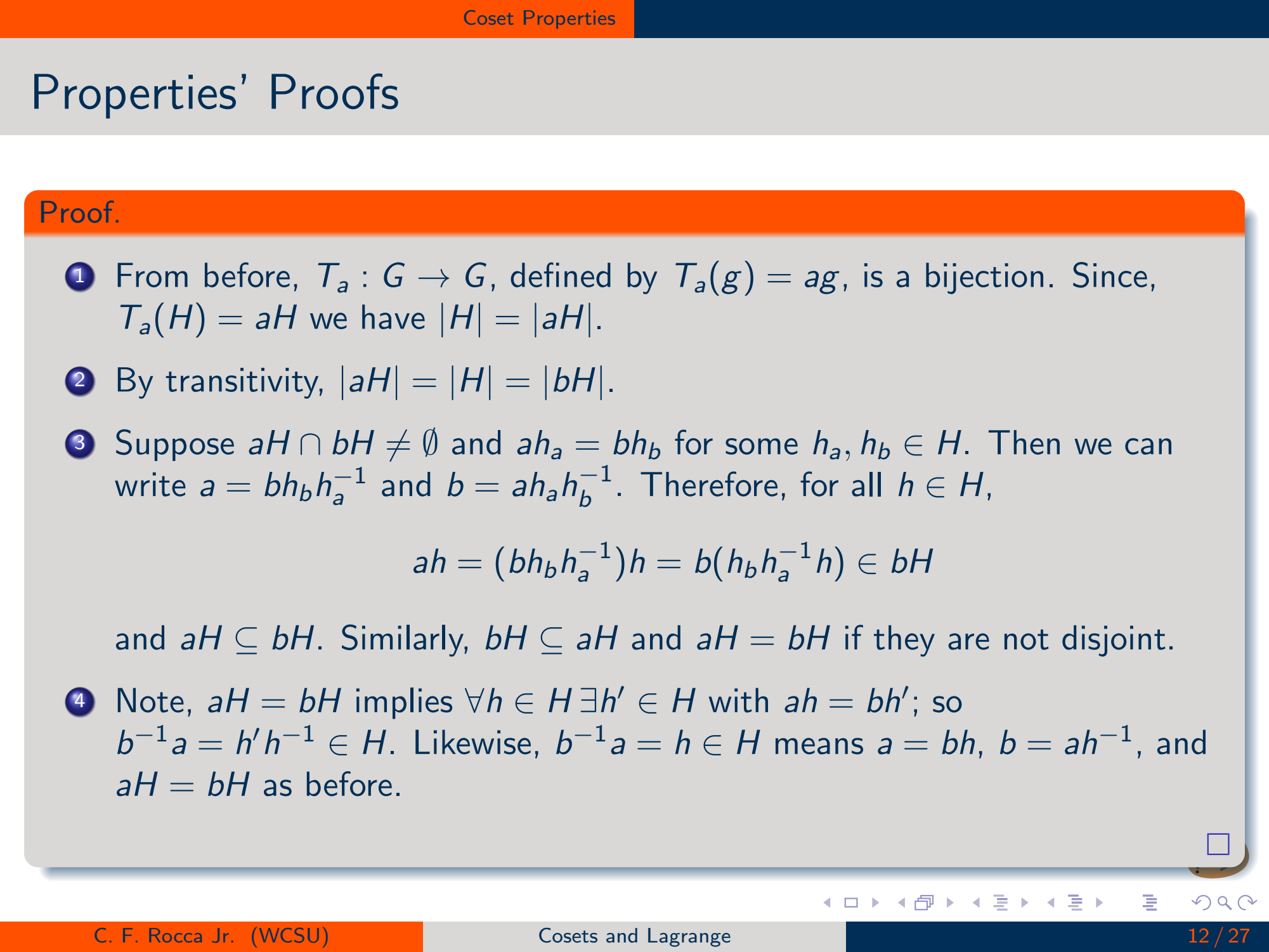

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Theorem

From the previous theorem, given a group G and subgroup $H \subseteq G$, the cosets of H partition G, e.g. for some set of $g_i \in G$

$$\bigcup_i g_i H = G$$

and $g_i H \cap g_j H = \emptyset$ when $i \neq j$.

Equivalence Classes and Partitions

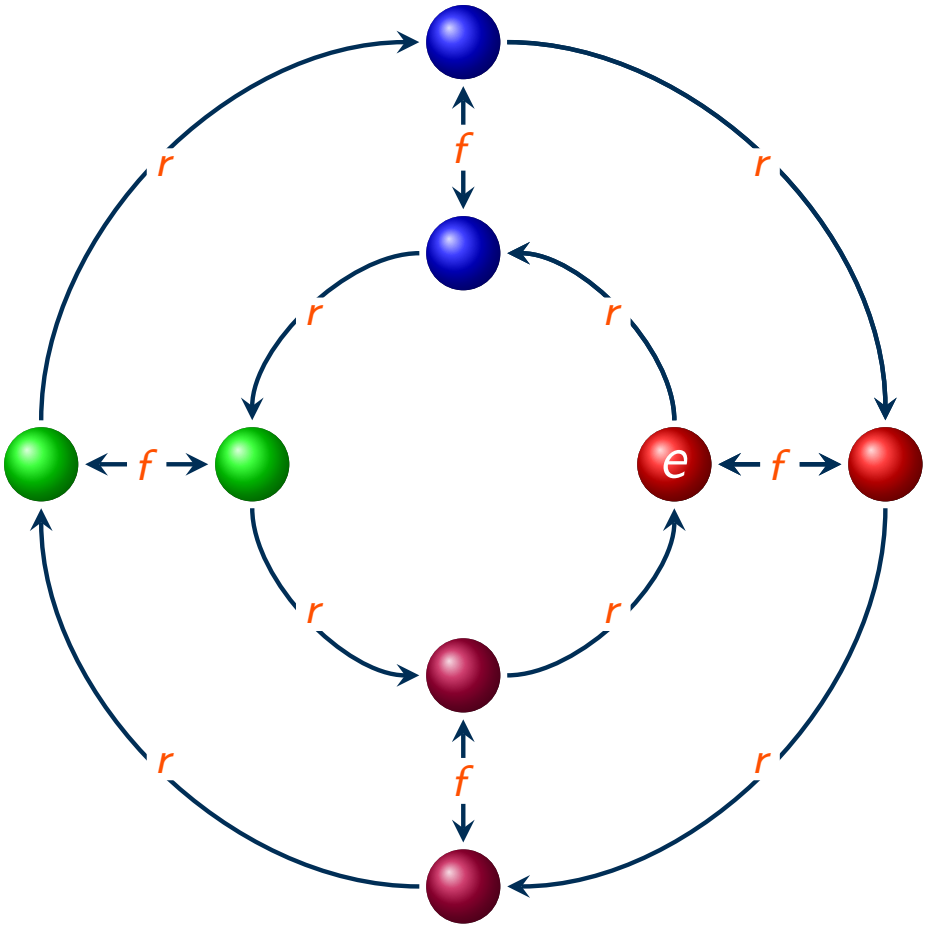
Definition

Given a group G, subgroup $H \subseteq G$, and elements $a, b \in G$ we say

$$a \equiv b \pmod{H}$$

if and only if $b^{-1}a \in H$ (i.e. $aH = bH$).
(We could show that this is an equivalence relation using the properties of cosets.)

$D_4 = \left\langle r, f \middle| r^4 = f^2 = e, rf = fr^{-1} \right\rangle$



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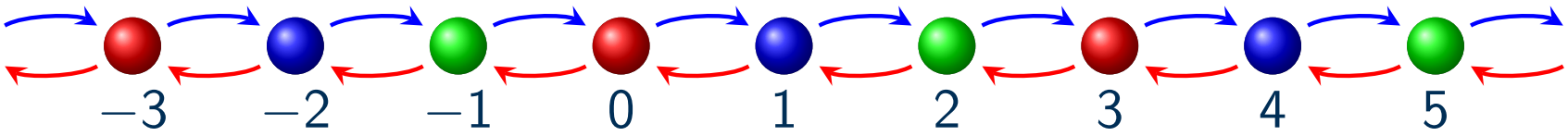

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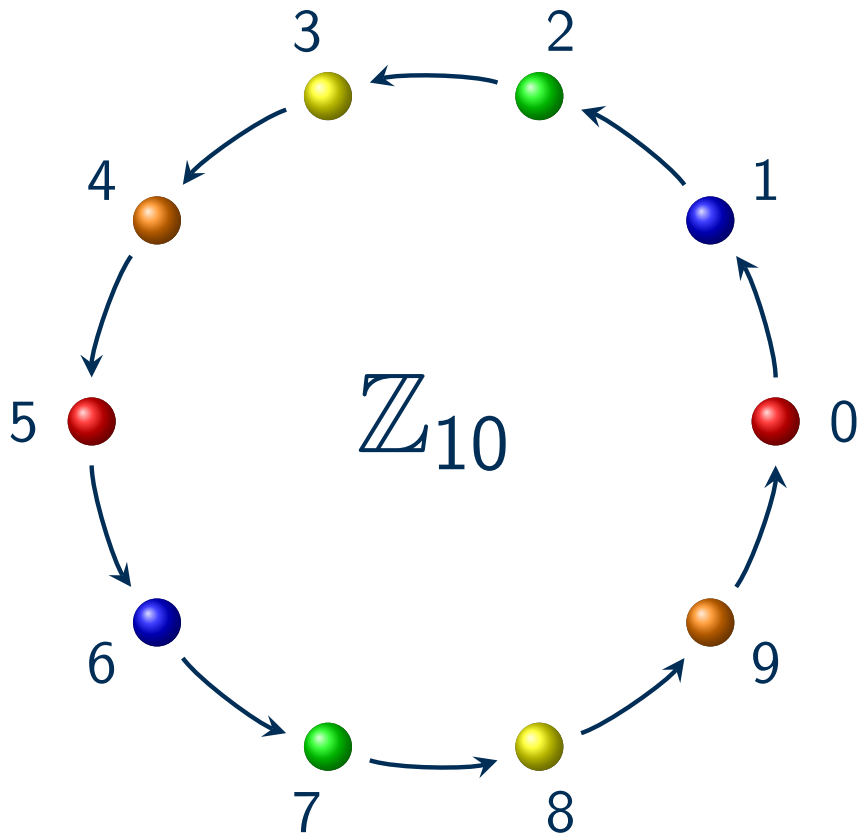

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Lagrange's Theorem

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Given a finite group G and subgroup $H \subseteq G$, the order of H divides the order of G, $|H| \big| |G|$.

Proof of Lagrange's Theorem

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From previous theorems we have that for some set of $g_i \in G$, we can write

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Therefore, $|G| = n|H|$ for some integer n. □

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Given a finite group G and subgroup $H \subseteq G$, the order of H divides the order of G, $|H| \mid |G|$.

Definition (Index of a Subgroup)

The number of distinct left cosets of $H \subseteq G$ is called the **index of H in G** and denoted $[G : H]$. If $|G|$ is finite then $|G| = [G : H]|H|$.

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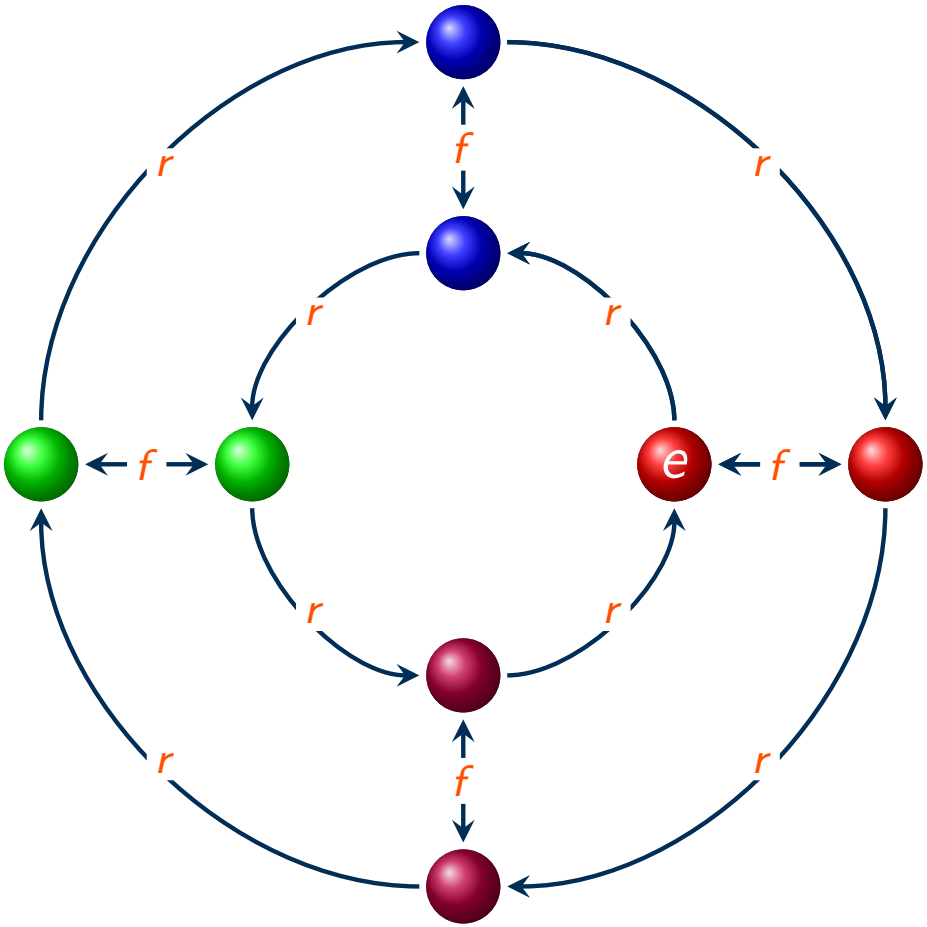

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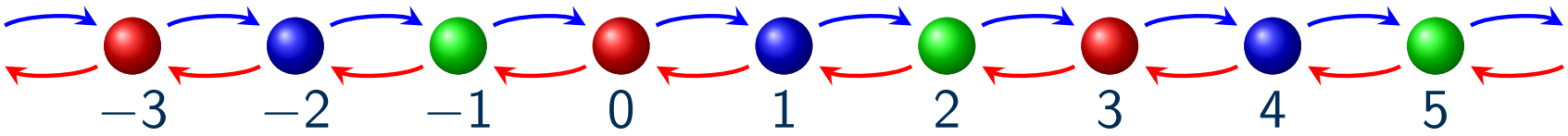

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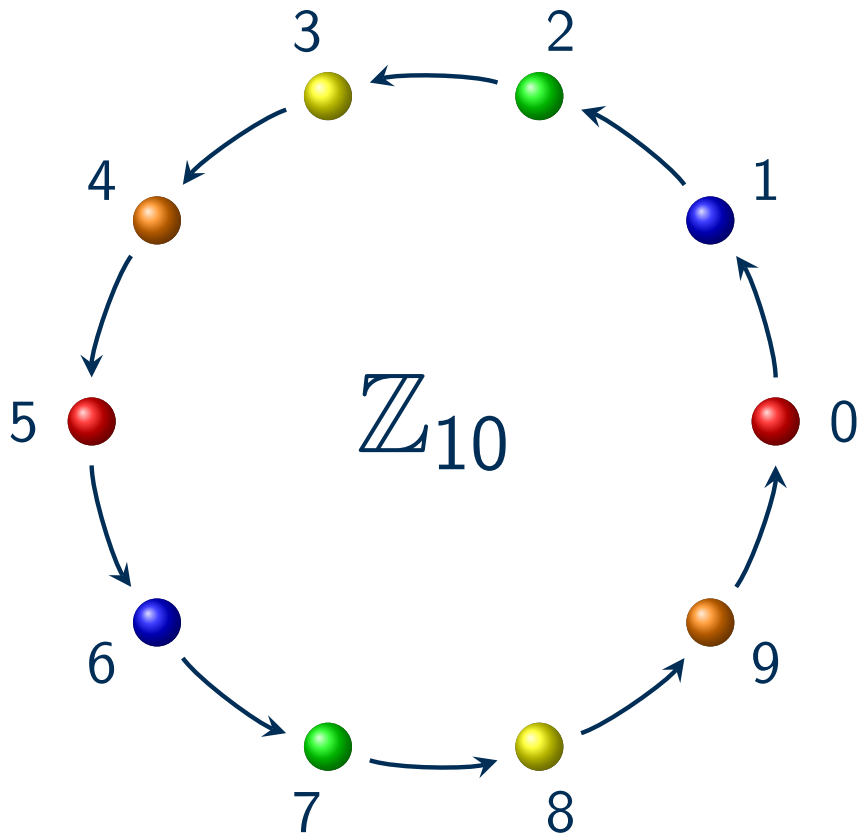

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Corollaries to Lagrange

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Given a finite group G with $k = |G|$, and $a \in G$:

1. $|a|$ *divides* $k = |G|$
2. $a^k = e \in G$

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The order of a is the least l such that $a^l = e$. This will also be the order of the cyclic subgroup $\langle a \rangle = \{a, a^2, \ldots, a^{l-1}, e\}$ and so divides $k = |G|$ by Lagrange's Theorem.

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The order of a is the least l such that $a^l = e$. This will also be the order of the cyclic subgroup $\langle a \rangle = \{a, a^2, \ldots, a^{l-1}, e\}$ and so divides $k = |G|$ by Lagrange's Theorem. Now we may write $k = ql$ for some unique q and therefore

$$a^k = a^{ql} = (a^l)^q = e.$$

□

Corollaries to Lagrange

Corollary

Given a finite group G with $k = |G|$, and $a \in G$:

1. $|a|$ *divides* $k = |G|$
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Theorem

Given a finite group G,if the order of G is prime, $|G| = p$, then G isomorphic to $\mathbb{Z}_p$.

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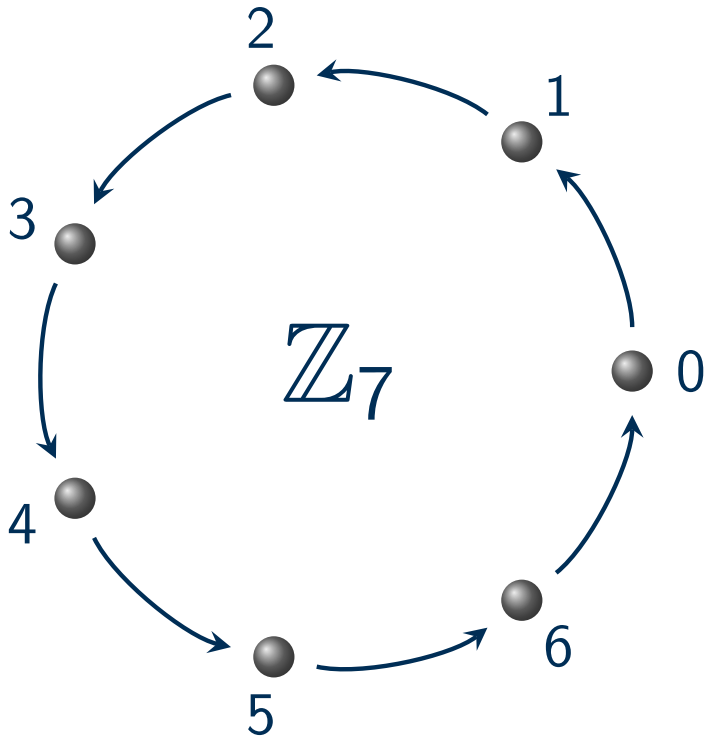
$$\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\} = \langle 3 \rangle$$



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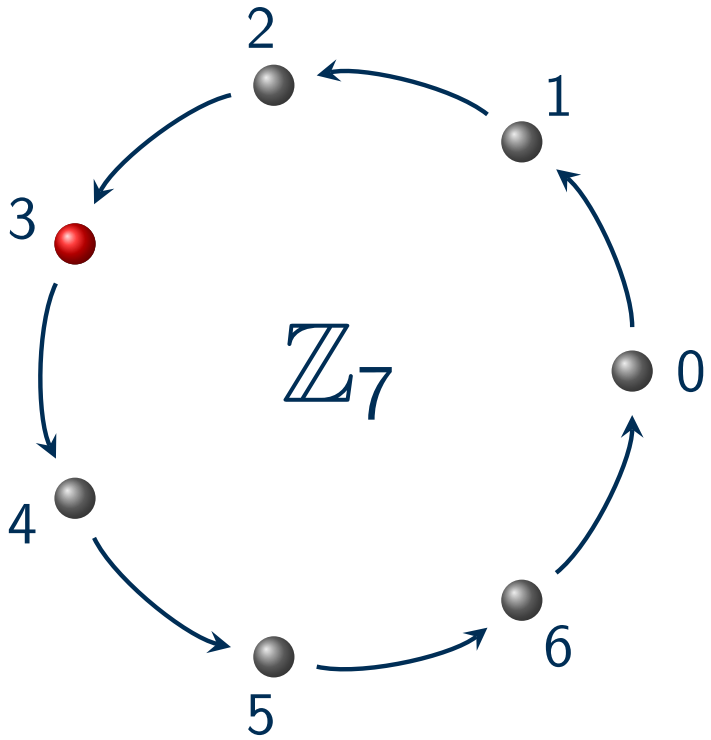

1. $1 \cdot 3 \equiv 3 \pmod 7$

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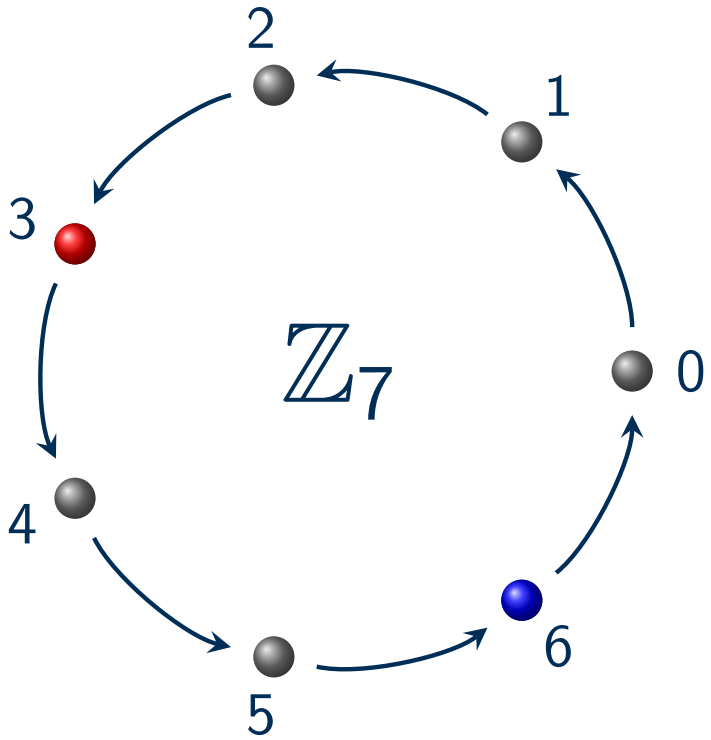

1. $1 \cdot 3 \equiv 3 \pmod 7$
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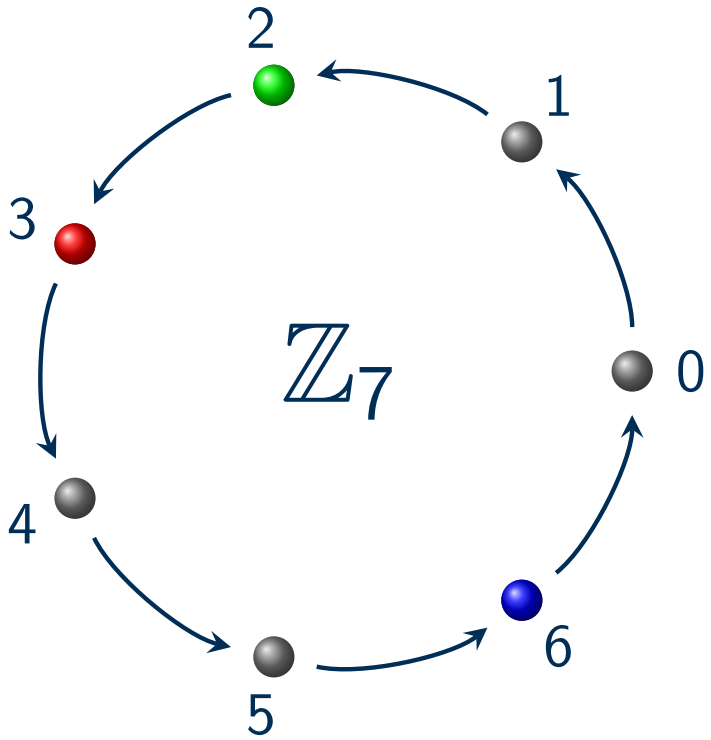

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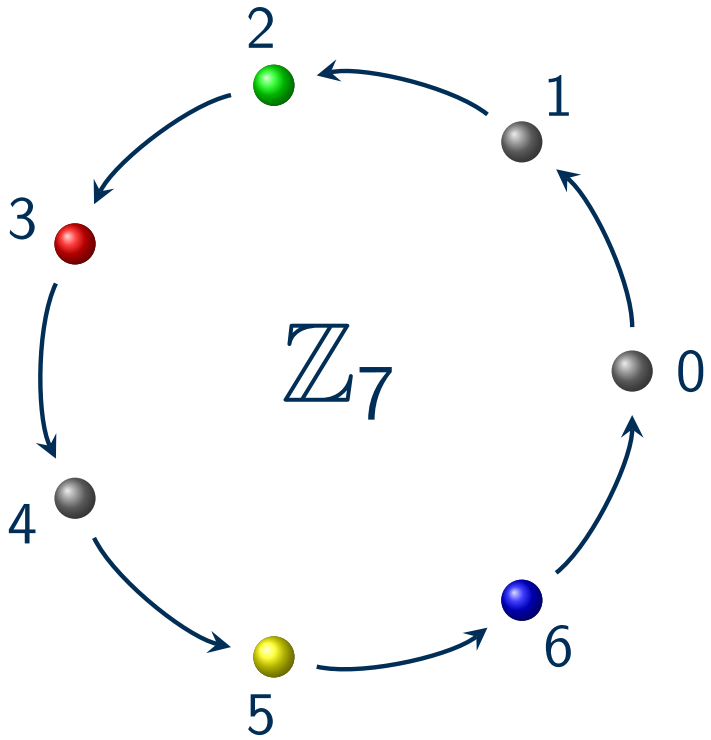

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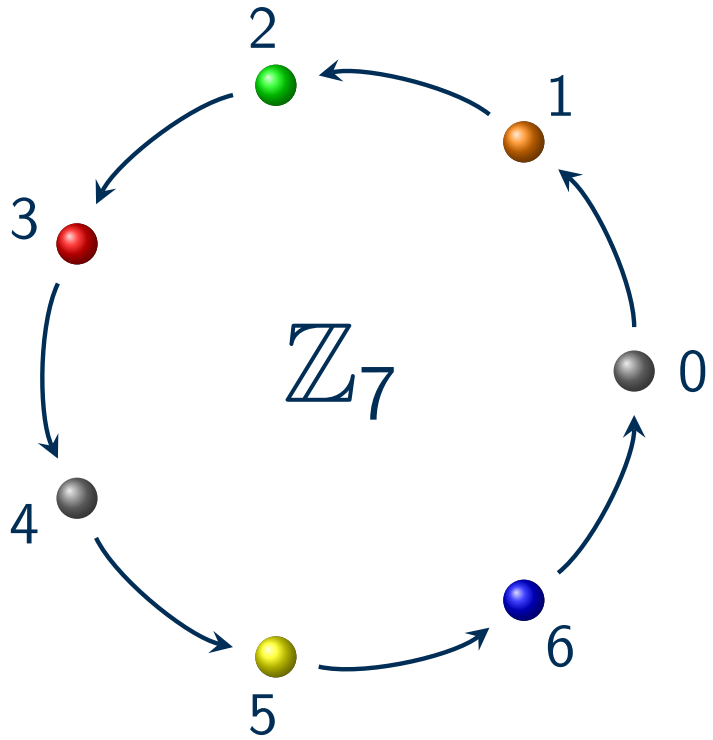

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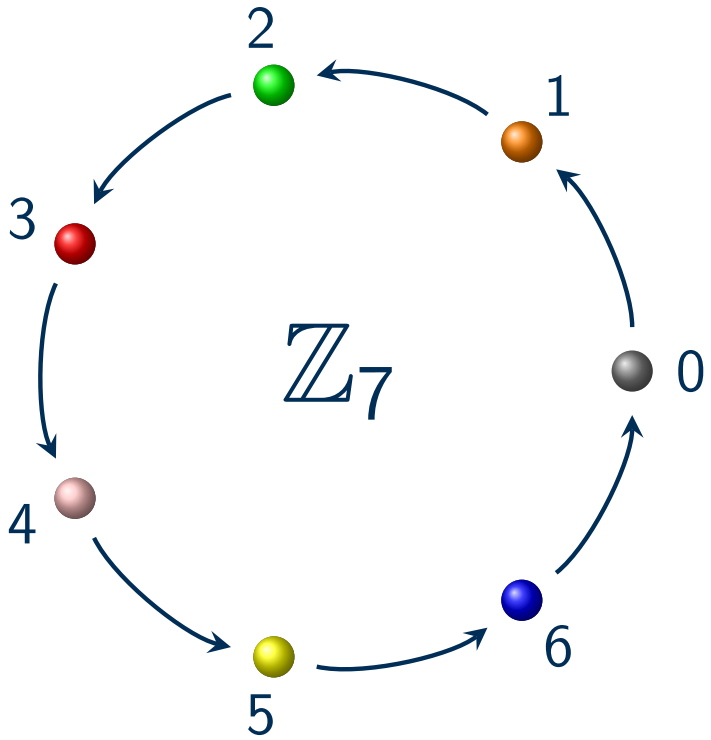

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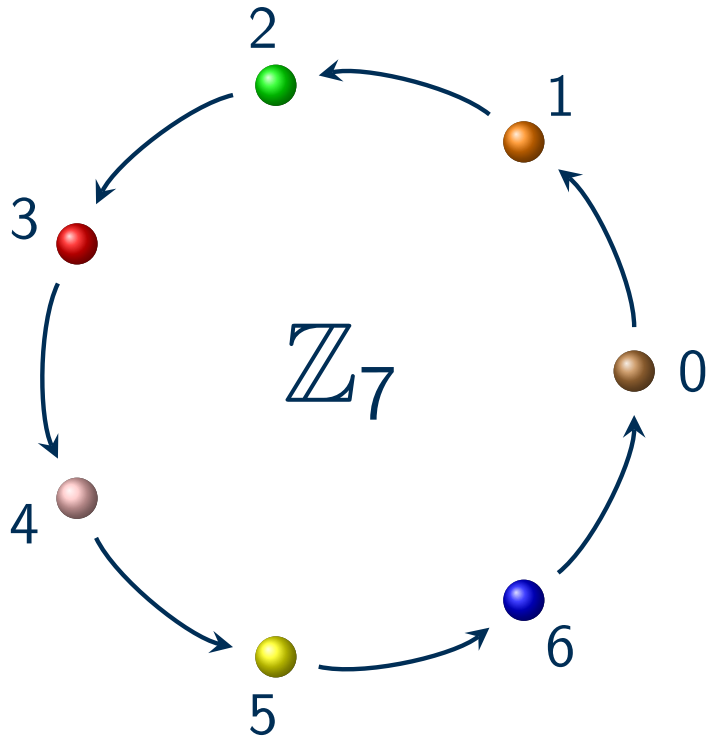

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6. $6 \cdot 3 \equiv 4 \pmod 7$
7. $7 \cdot 3 \equiv 0 \pmod 7$

Cosets and Lagrange

Dr. Chuck Rocca

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