

Turing Machines

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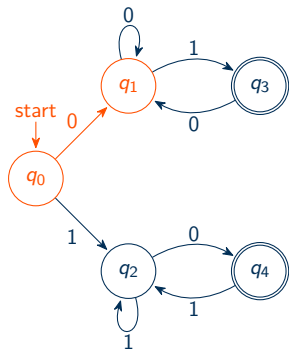
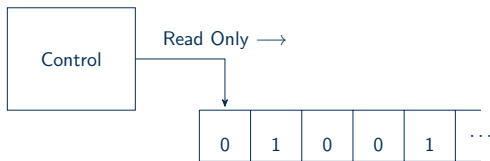


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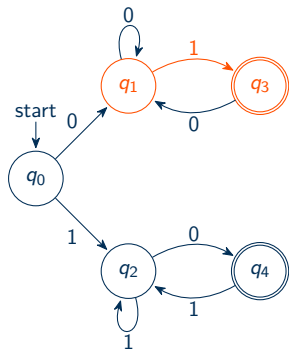
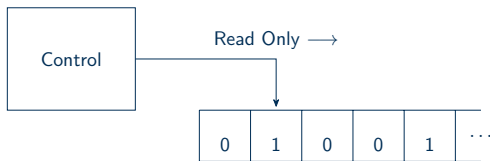
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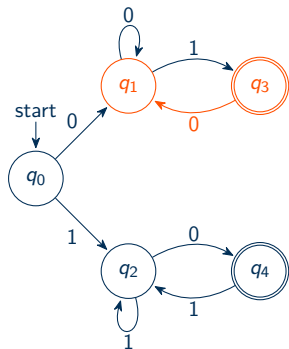
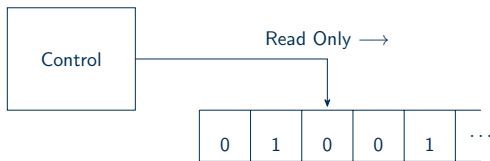
Finite Automata



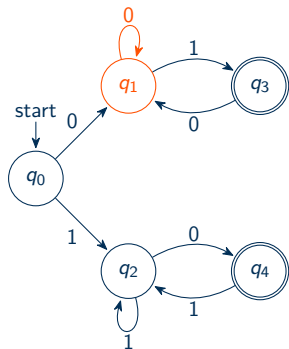
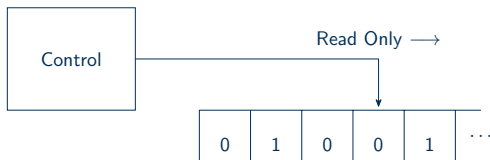
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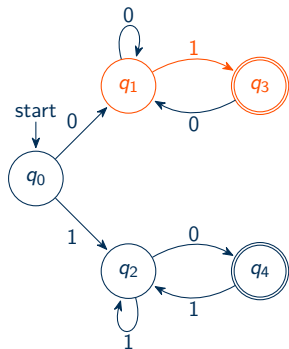
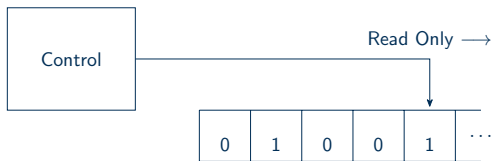
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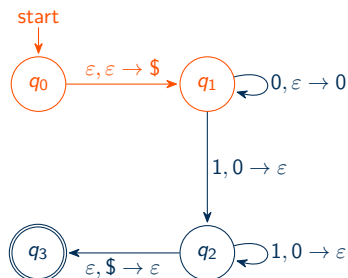
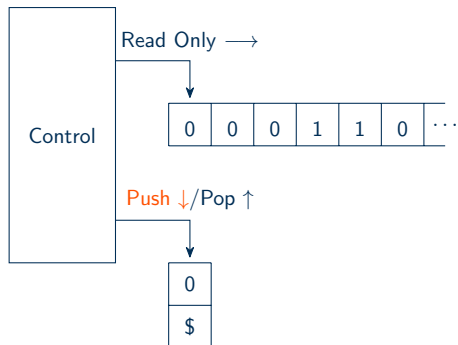
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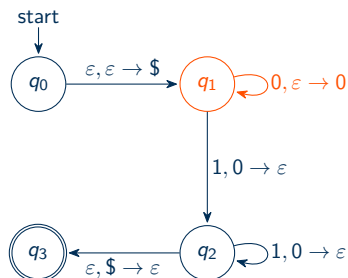
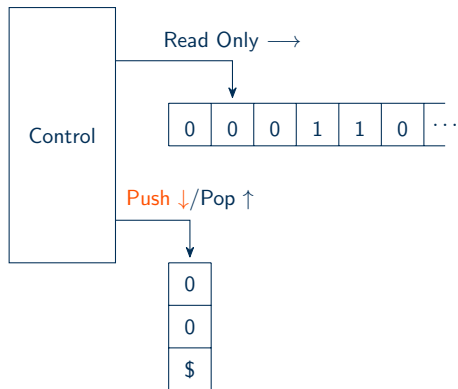
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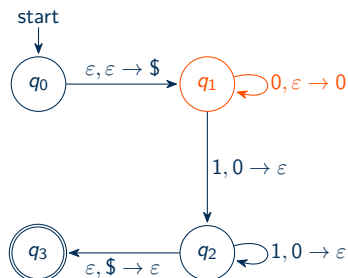
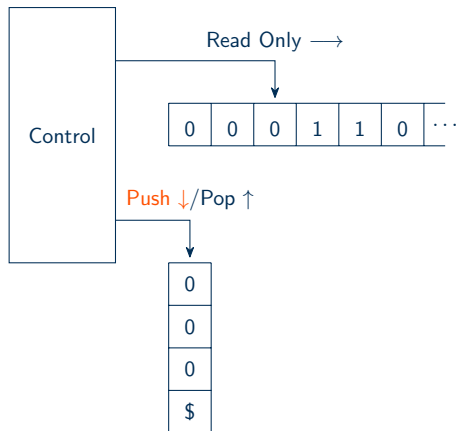
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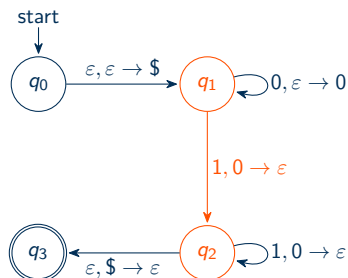
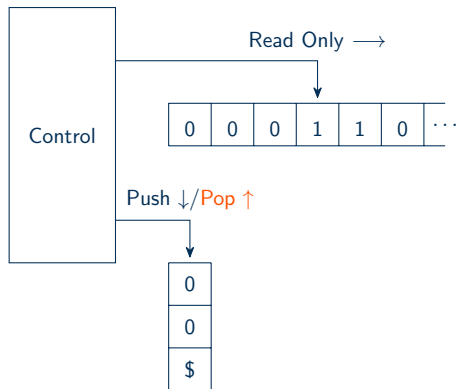
Pushdown Automata



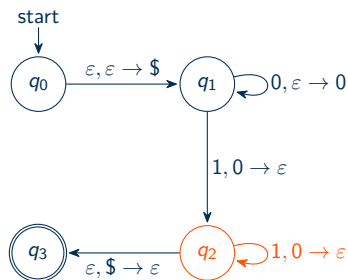
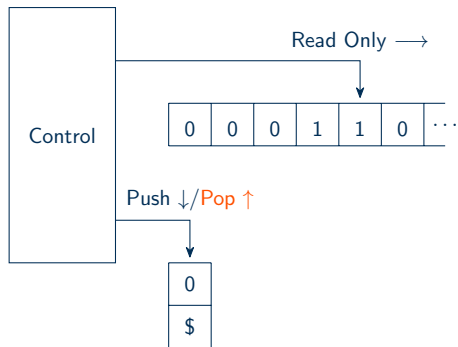
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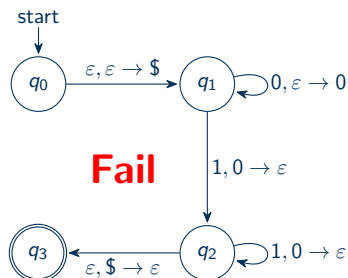
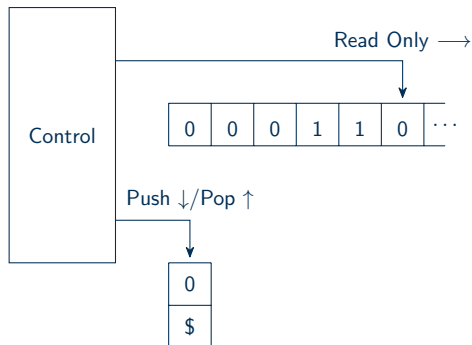
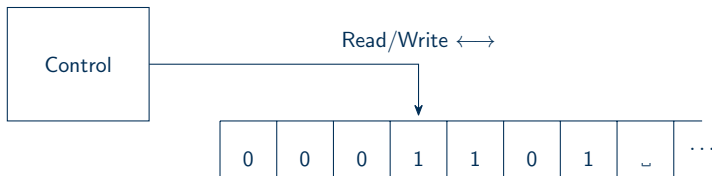


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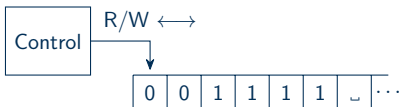


Turing Machine Example



Turing Machine Example

Given $L = \{0^n 1^n \mid n \geq 0\}$, is $w \in L$?



● 001111

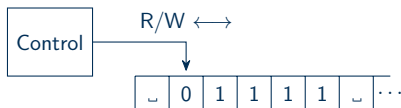
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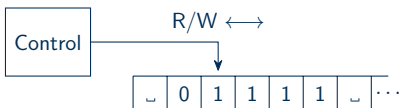
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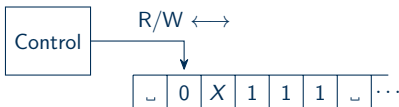
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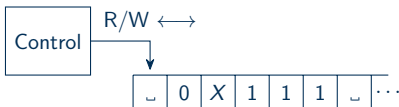
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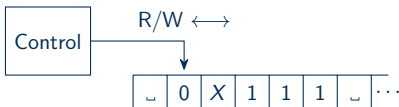
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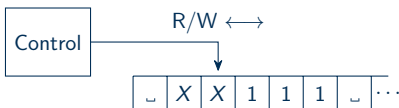
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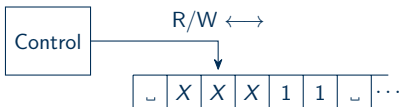
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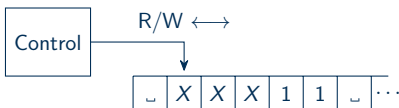
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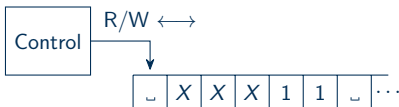
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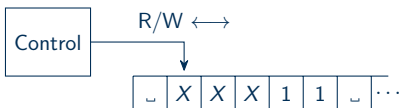
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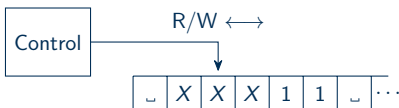
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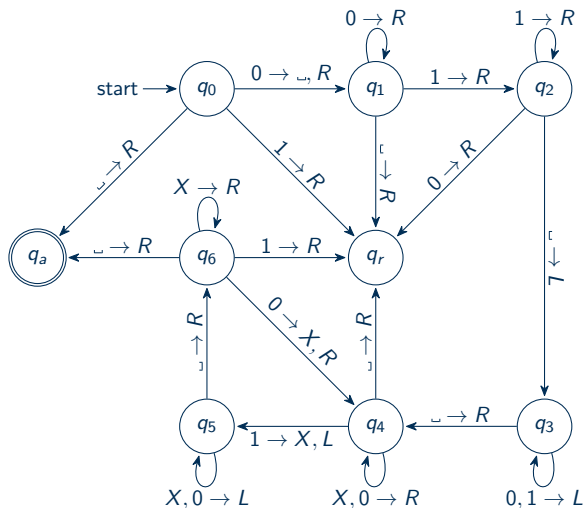
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- 001111
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- 0X111
- XX111
- XXX11
- **Fail!**



Turing Machine Diagram



$$L = \{0^n 1^n \mid n \geq 0\}$$

- $X \rightarrow R$
- $X, \sqcup \rightarrow L$
- $0 \rightarrow X, R$



Formal Definitions

Definition (Turing Machine)

A **Turing machine** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$, where Q, Σ, Γ are finite sets and

- 1 Q is the set of states,
- 2 Σ is the input alphabet not containing \sqcup ,
- 3 Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subset \Gamma$,
- 4 $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ is the transition function,
- 5 q_0 is the start state,
- 6 q_a is the accept state, and
- 7 q_r is the reject state, where $q_r \neq q_a$.



Level of Description

- **Formal Description:** This spells out all seven components of the formal definition of the Turing machine.



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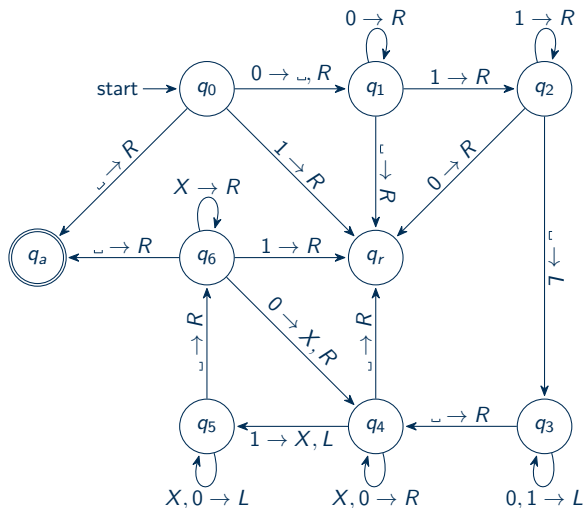


Level of Description

- **Formal Description:** This spells out all seven components of the formal definition of the Turing machine.
- **Implementation Description:** Describes in words the way the Turing machine moves its head and stores data on the tape.
- **High-Level Description:** Describes in words an algorithm for how the Turing machine recognizes a language ignoring implementation details.



Turing Machine Diagram



$$L = \{0^n 1^n \mid n \geq 0\}$$

- $X \rightarrow R$
- $X, \sqcup \rightarrow L$
- $0 \rightarrow X, R$

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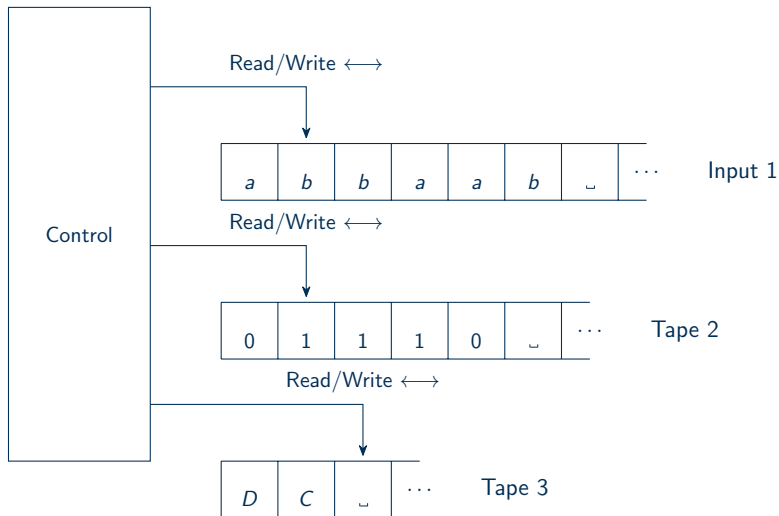


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Multitape Turing Machines



Multiple Tapes to A Single Tape

Theorem

A language is Turing recognizable if and only if some multiple tape Turing machine recognizes it.

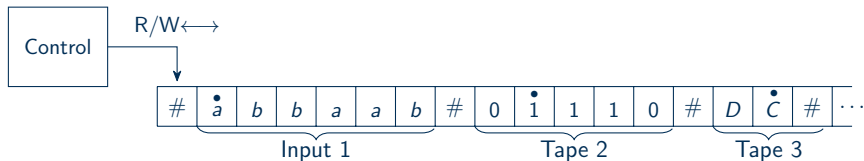
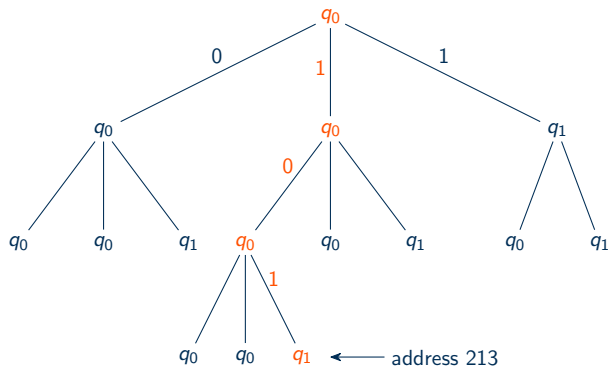
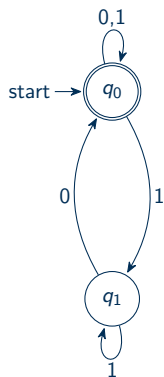


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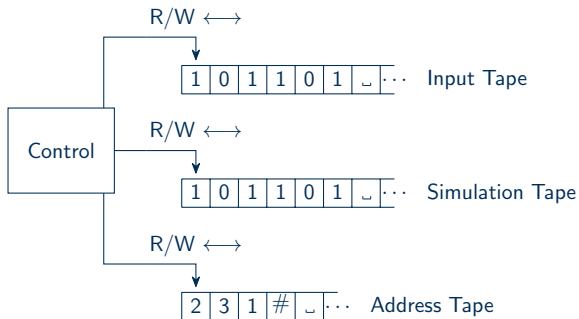


Nondeterministic Turing Machines



Simulating a Nondeterministic Turing Machine

Simulating a Nondeterministic T.M. N with a Deterministic One D :



Simulating a Nondeterministic Turing Machine

Simulating a Nondeterministic T.M. N with a Deterministic One D :

- 1 Place the input word w on tape 1, the input tape.



Simulating a Nondeterministic Turing Machine

Simulating a Nondeterministic T.M. N with a Deterministic One D :

- 1 Place the input word w on tape 1, the input tape.
- 2 Initialize tape 3, the address tape, to ϵ .



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- 1 Place the input word w on tape 1, the input tape.
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- 3 Copy tape 1 to tape 2, the simulation tape.



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 - If you reach a fail state, the end of an address, or an invalid address, go to step 5.



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- 3 Copy tape 1 to tape 2, the simulation tape.
- 4 Simulate N using tape 2 following the address on tape 3.
 - If you reach a fail state, the end of an address, or an invalid address, go to step 5.
 - If you reach an accept state, then accept.



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- 2 Initialize tape 3, the address tape, to ε .
- 3 Copy tape 1 to tape 2, the simulation tape.
- 4 Simulate N using tape 2 following the address on tape 3.
 - If you reach a fail state, the end of an address, or an invalid address, go to step 5.
 - If you reach an accept state, then accept.
- 5 Replace the address on tape 3 with the next address in line and go to step 3.

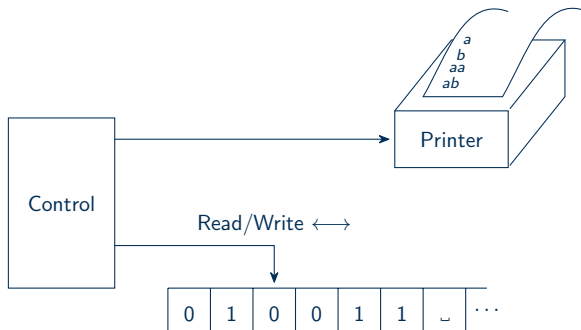


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- 6 Next Class

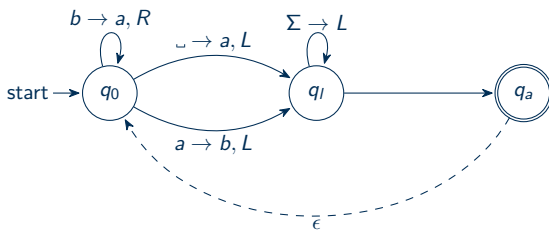


Enumerators



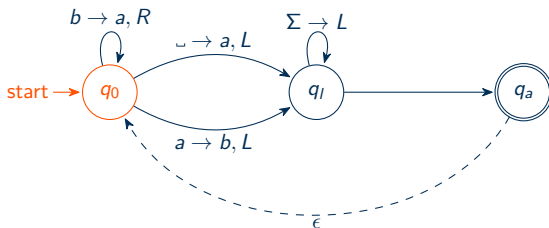
Sample Enumerator Diagram

The following produces $a, b, aa, ba, ab, bb, aaa, baa, aba, \dots$:



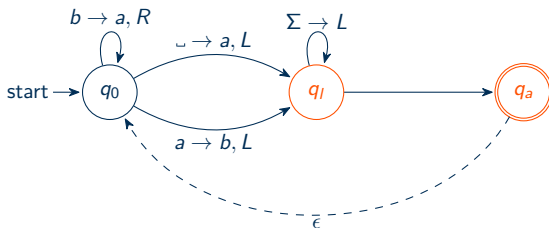
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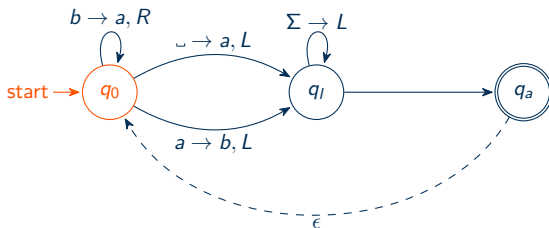


\downarrow **a** $_$



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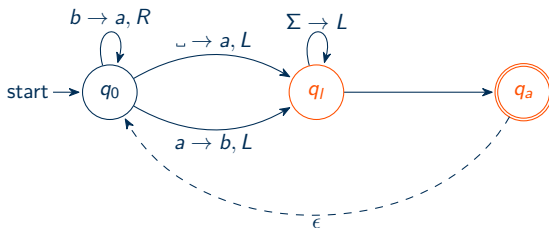


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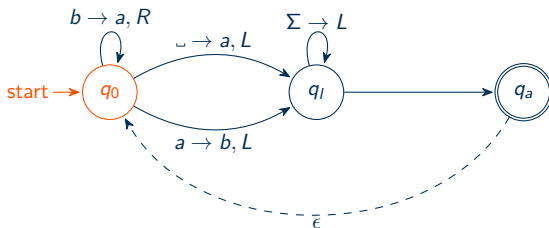


↓ **b** \sqcup



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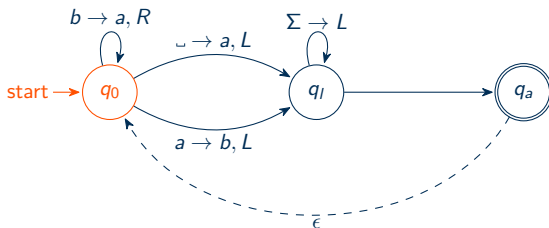


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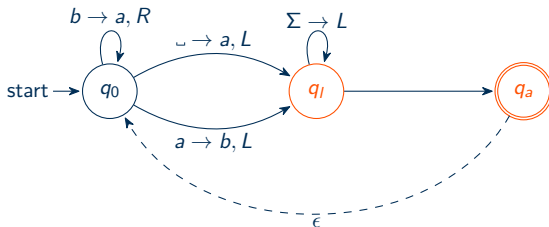


a ↓ ⊔



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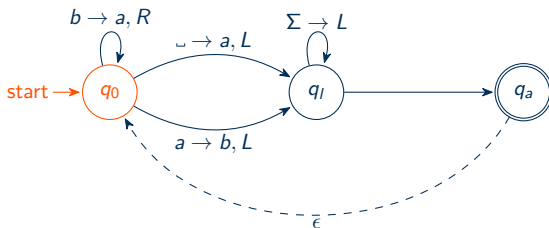


↓ **aa**␣



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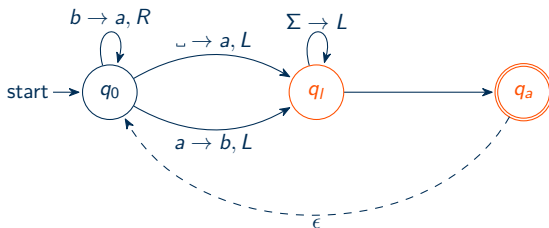


↓ **aa**␣



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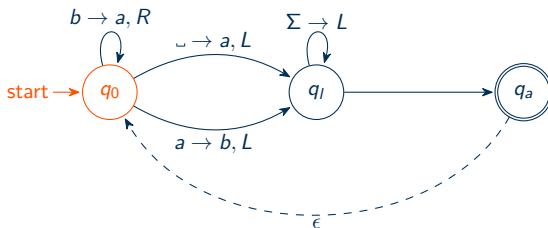


\downarrow **ba** $_$



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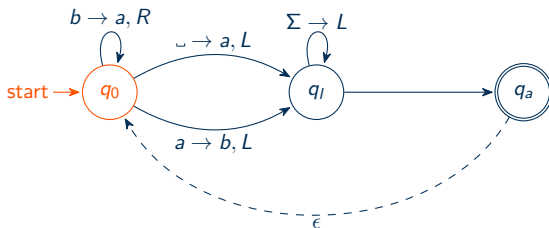


\downarrow **ba** $_$



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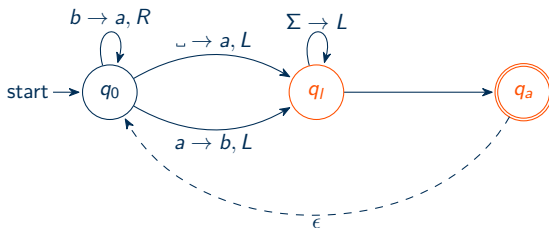


a ↓ **a** $_$



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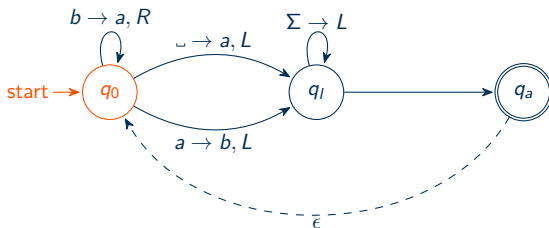


\downarrow **ab** $_$



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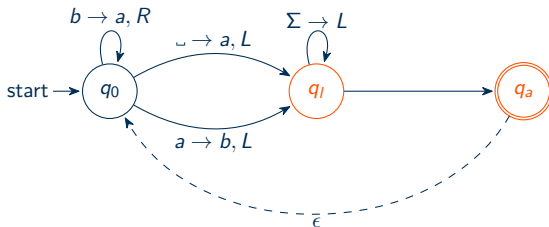


\downarrow **ab** $_$



Sample Enumerator Diagram

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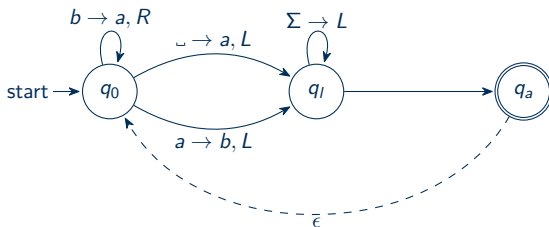


↓ **bb** ϵ



Sample Enumerator Diagram

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etc.



Equivalence of Enumerators and Turing Machines

Theorem

A language is Turing recognizable if and only if some enumerator enumerates it.

Proof.

- Define M : Given a word w , run the given enumerator and if w appears on its list, accept.



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 - 2 If any computation accepts, print the corresponding s_j .



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Next Class

- Algorithms



Next Class

- Algorithms
- Decidable Languages



Next Class

- Algorithms
- Decidable Languages
- Decidable Problems and Regular Languages



Next Class

- Algorithms
- Decidable Languages
- Decidable Problems and Regular Languages
- **Decidable Problems and Context-Free Languages**



Turing Machines

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