

# Grammars and Pushdown Automata

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- 1 Context-Free Grammar
- 2 Chomsky Normal Form
- 3 Pushdown Automata
- 4 CFL and PDA
- 5 Next Class



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# A Simple Example

**Grammar:**

$A$

$$A \rightarrow 0A1$$

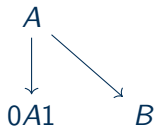
$$A \rightarrow B$$

$$B \rightarrow \#$$



# A Simple Example

## Grammar:

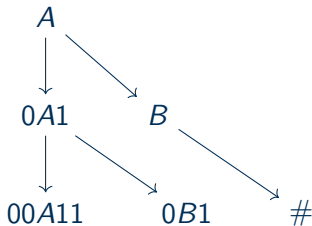
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# A Simple Example

## Grammar:

$$A \rightarrow 0A1$$

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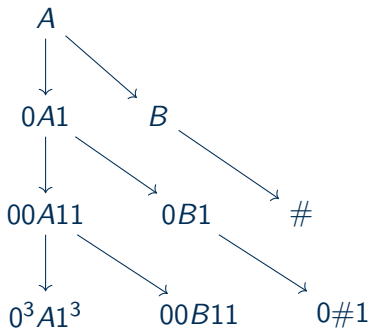
$$B \rightarrow \#$$


# A Simple Example

## Grammar:

$$A \rightarrow 0A1$$

$$A \rightarrow B$$

$$B \rightarrow \#$$


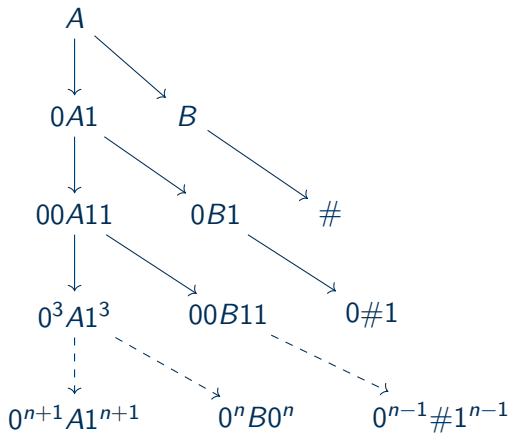
# A Simple Example

## Grammar:

$$A \rightarrow 0A1$$

$$A \rightarrow B$$

$$B \rightarrow \#$$





# A Simple Example: Derivation and Parse Tree

**Grammar:**

$$A \rightarrow 0A1$$

$$A \rightarrow B$$

$$B \rightarrow \#$$

**Parse Tree** for 000#111:

A

**Derivation** of 000#111:

$$A \Rightarrow$$



# A Simple Example: Derivation and Parse Tree

## Grammar:

$$A \rightarrow 0A1$$

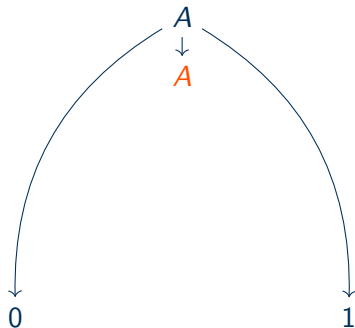
$$A \rightarrow B$$

$$B \rightarrow \#$$

## Derivation of 000#111:

$$A \Rightarrow 0A1$$

## Parse Tree for 000#111:



# A Simple Example: Derivation and Parse Tree

## Grammar:

$$A \rightarrow 0A1$$

$$A \rightarrow B$$

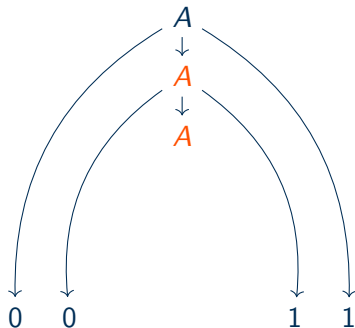
$$B \rightarrow \#$$

## Derivation of 000#111:

$$A \Rightarrow 0A1$$

$$\Rightarrow 00A11$$

## Parse Tree for 000#111:



# A Simple Example: Derivation and Parse Tree

## Grammar:

$$A \rightarrow 0A1$$

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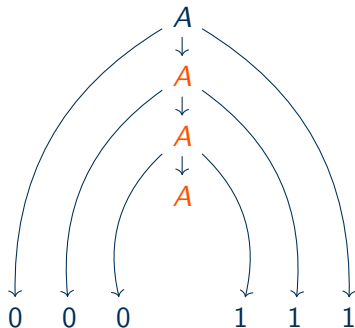
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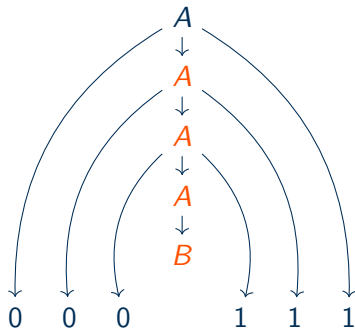
$$A \Rightarrow 0A1$$

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# A Simple Example: Derivation and Parse Tree

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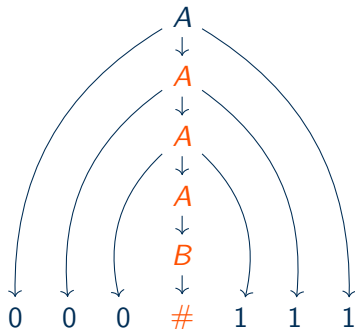
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## Parse Tree for 000#111:



# Formal Definition

## Definition (Context-Free Grammar)

A **context-free grammar** is a 4-tuple  $(V, \Sigma, R, S)$ , where

- 1  $V$  is a finite set called the **variables**,

**Grammar:**

$$A \rightarrow 0A1$$

$$A \rightarrow B$$

$$B \rightarrow \#$$

$$\text{Variables: } V = \{A, B\}$$



# Formal Definition

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A **context-free grammar** is a 4-tuple  $(V, \Sigma, R, S)$ , where

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**Grammar:**

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$$B \rightarrow \#$$

**Terminals:**  $\Sigma = \{0, 1, \#\}$





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- 3  $R$  is a finite set of **rules**, with each rule associating a variable to a string of variables and/or terminals, and

**Grammar:**

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**Rules**



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- ③  $R$  is a finite set of **rules**, with each rule associating a variable to a string of variables and/or terminals, and
- ④  $S \in V$  is a **start variable**.

**Grammar:**

$$A \rightarrow 0A1$$

$$A \rightarrow B$$

$$B \rightarrow \#$$

**Start Variable:  $S = A$**



# Designing a CFG

 $w * w^R$ 

Give a CFG what recognizes the language of all words in  $\Sigma = \{0, 1\}$  which end with the reverse of their beginnings, e.g. **10110** 0101010 **01101**.

Arbitrary Word:

$$S_1 \rightarrow 0S_1 | 1S_1 | \epsilon$$



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Even Length Palindrome,  $ww^R$ :

$$S_2 \rightarrow 0X0 | 1X1$$

$$X \rightarrow S_2 | \varepsilon$$



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$$S_2 \rightarrow 0X0|1X1$$

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Combining Them,  $w * w^R$ :

$$S \rightarrow S_2$$

$$S_2 \rightarrow 0X0|1X1$$

$$X \rightarrow S_2|S_1$$

$$S_1 \rightarrow 0S_1|1S_1|\epsilon$$



# A Couple Variations

## Combining Grammars

$$S_1 \rightarrow 0S_1|1S_1|\epsilon$$

$$S_2 \rightarrow 0X0|1X1$$

$$X \rightarrow S_2|\epsilon$$

$$S \rightarrow S_2$$

$$S_2 \rightarrow 0X0|1X1$$

$$X \rightarrow S_2|S_1$$

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$$X \rightarrow S_2|S_1$$

$$X \rightarrow S_2|\epsilon$$

$$S_1 \rightarrow 0S_1|1S_1|\epsilon$$

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$$S \rightarrow S_1|S_2$$

$$S \rightarrow S_1S_2$$

$$S_2 \rightarrow 0X0|1X1$$

$$S_2 \rightarrow 0X0|1X1$$

$$S_2 \rightarrow 0X0|1X1$$

$$X \rightarrow S_2|S_1$$

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$$S_2 \rightarrow 0X0|1X1$$

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$$S_2 \rightarrow 0X0|1X1$$

$$S_2 \rightarrow 0X0|1X1|S$$

$$X \rightarrow S_2|S_1$$

$$X \rightarrow S_2|\epsilon$$

$$X \rightarrow S_2|\epsilon$$

$$X \rightarrow S_2|\epsilon$$

$$S_1 \rightarrow 0S_1|1S_1|\epsilon$$

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# Example with Ambiguity

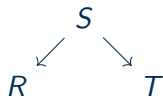
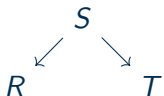
Given the grammar:

$$S \rightarrow RT$$

$$R \rightarrow aR | Rb | \epsilon$$

$$T \rightarrow Tb | \epsilon$$

there are multiple ways to derive the string  $aab$ . This is an **ambiguous** grammar.



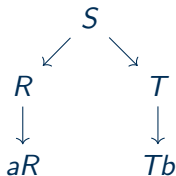
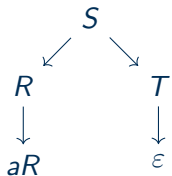
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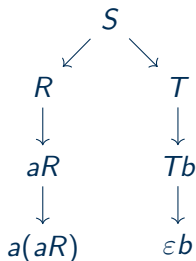
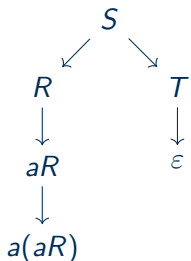
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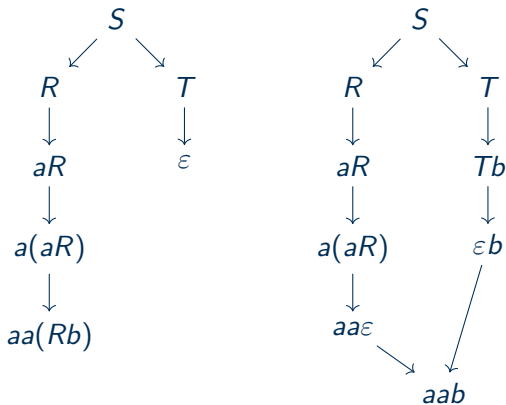
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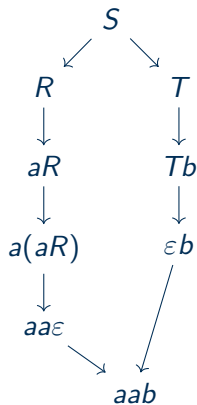
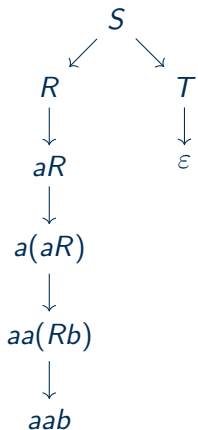
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# Chomsky Normal Form

## Definition

A context-free grammar is in **Chomsky Normal Form** if every rule is of the form

$$A \rightarrow BC \text{ or}$$

$$A \rightarrow a$$

where  $a$  is a terminal,  $A$ ,  $B$ , and  $C$  are variables,  $B$  and  $C$  are not start variables, and only a start variable may point to the empty string  $\epsilon$





# Converting to Chomsky Normal Form

New Start:

Replace

$$S \rightarrow A|b$$

$$A \rightarrow aS|a$$

With

$$S_0 \rightarrow S$$

$$S \rightarrow A|b$$

$$A \rightarrow aS|a$$



# Converting to Chomsky Normal Form

New Start:

No  $\epsilon$ :

Replace

Replace

$$S \rightarrow A|b$$

$$A \rightarrow XBY$$

$$A \rightarrow aS|a$$

$$B \rightarrow \epsilon$$

With

With

$$S_0 \rightarrow S$$

$$A \rightarrow XBY$$

$$S \rightarrow A|b$$

$$A \rightarrow XY$$

$$A \rightarrow aS|a$$



# Converting to Chomsky Normal Form

New Start:  
Replace

$$S \rightarrow A|b$$

$$A \rightarrow aS|a$$

No  $\varepsilon$ :  
Replace

$$A \rightarrow XBY$$

$$B \rightarrow \varepsilon$$

No Units:  
Replace

$$A \rightarrow B$$

$$B \rightarrow XY|a$$

With

$$S_0 \rightarrow S$$

$$S \rightarrow A|b$$

$$A \rightarrow aS|a$$

With

$$A \rightarrow XBY$$

$$A \rightarrow XY$$

With

$$A \rightarrow XY|a$$

$$B \rightarrow XY|a$$



# Converting to Chomsky Normal Form

New Start:  
Replace

$$S \rightarrow A|b$$

$$A \rightarrow aS|a$$

No  $\varepsilon$ :  
Replace

$$A \rightarrow XBY$$

$$B \rightarrow \varepsilon$$

No Units:  
Replace

$$A \rightarrow B$$

$$B \rightarrow XY|a$$

New Variables:  
Replace:

$$A \rightarrow BXY$$

$$B \rightarrow bX$$

With

$$S_0 \rightarrow S$$

$$S \rightarrow A|b$$

$$A \rightarrow aS|a$$

With

$$A \rightarrow XBY$$

$$A \rightarrow XY$$

With

$$A \rightarrow XY|a$$

$$B \rightarrow XY|a$$

With:

$$A \rightarrow BU$$

$$U \rightarrow XY$$

$$B \rightarrow VX$$

$$V \rightarrow b$$



## Chomsky Example Steps 1-3

$$S_0 \rightarrow 0X0|1X1$$

$$X \rightarrow S_0|S_1|\epsilon$$

$$S_1 \rightarrow 0S_1|1S_1|\epsilon$$

$$S \rightarrow S_0$$

$$S_0 \rightarrow 0X0|1X1$$

$$X \rightarrow S_0|S_1|\epsilon$$

$$S_1 \rightarrow 0S_1|1S_1|\epsilon$$

$$S \rightarrow S_0$$

$$S_0 \rightarrow 0X0|1X1$$

$$S_0 \rightarrow 00|11$$

$$X \rightarrow S_0|S_1$$

$$S_1 \rightarrow 0S_1|1S_1$$

$$S_1 \rightarrow 0|1$$



## Chomsky Example Steps 3-5

$$S \rightarrow S_0$$

$$S_0 \rightarrow 0X0|1X1$$

$$S_0 \rightarrow 00|11$$

$$X \rightarrow S_0|S_1$$

$$S_1 \rightarrow 0S_1|1S_1$$

$$S_1 \rightarrow 0|1$$

$$S \rightarrow S_0$$

$$S_0 \rightarrow 0W_0|1W_1$$

$$W_0 \rightarrow X0$$

$$W_1 \rightarrow X1$$

$$S_0 \rightarrow 00|11$$

$$X \rightarrow S_0|S_1$$

$$S_1 \rightarrow U_0S_1|U_1S_1$$

$$U_0 \rightarrow 0$$

$$U_1 \rightarrow 1$$

$$S_1 \rightarrow 0|1$$

$$S \rightarrow S_0$$

$$S_0 \rightarrow 0W_0|1W_1$$

$$W_0 \rightarrow X0$$

$$W_1 \rightarrow X1$$

$$S_0 \rightarrow 00|11$$

$$X \rightarrow S_0|S_1$$

$$S_1 \rightarrow U_0S_1|U_1S_1$$

$$U_0 \rightarrow 0$$

$$U_1 \rightarrow 1$$

$$S_1 \rightarrow 0|1$$


## Chomsky Example Steps 6-7

$$S \rightarrow S_0$$

$$S_0 \rightarrow U_0 W_0 | U_1 W_1$$

$$W_0 \rightarrow XU_0$$

$$W_1 \rightarrow XU_1$$

$$S_0 \rightarrow U_0 U_0 | U_1 U_1$$

$$X \rightarrow S_0 | S_1$$

$$S_1 \rightarrow U_0 S_1 | U_1 S_1$$

$$U_0 \rightarrow 0$$

$$U_1 \rightarrow 1$$

$$S_1 \rightarrow 0 | 1$$

$$S \rightarrow U_0 W_0 | U_1 W_1 | U_0 U_0 | U_1 U_1$$

$$S_0 \rightarrow U_0 W_0 | U_1 W_1$$

$$W_0 \rightarrow XU_0$$

$$W_1 \rightarrow XU_1$$

$$S_0 \rightarrow U_0 U_0 | U_1 U_1$$

$$X \rightarrow U_0 U_0 | U_1 U_1 | U_0 S_1 | U_1 S_1$$

$$S_1 \rightarrow U_0 S_1 | U_1 S_1$$

$$U_0 \rightarrow 0$$

$$U_1 \rightarrow 1$$

$$S_1 \rightarrow 0 | 1$$



## Chomsky Example Steps 7-8

$$S \rightarrow U_0 W_0 | U_1 W_1 | U_0 U_0 | U_1 U_1$$

$$S_0 \rightarrow U_0 W_0 | U_1 W_1$$

$$W_0 \rightarrow XU_0$$

$$W_1 \rightarrow XU_1$$

$$S_0 \rightarrow U_0 U_0 | U_1 U_1$$

$$X \rightarrow U_0 U_0 | U_1 U_1 | U_0 S_1 | U_1 S_1$$

$$S_1 \rightarrow U_0 S_1 | U_1 S_1$$

$$U_0 \rightarrow 0$$

$$U_1 \rightarrow 1$$

$$S_1 \rightarrow 0|1$$

$$S \rightarrow U_0 W_0 | U_1 W_1 | U_0 U_0 | U_1 U_1$$

$$W_0 \rightarrow XU_0$$

$$W_1 \rightarrow XU_1$$

$$X \rightarrow U_0 U_0 | U_1 U_1 | U_0 S_1 | U_1 S_1$$

$$S_1 \rightarrow U_0 S_1 | U_1 S_1 | 0|1$$

$$U_0 \rightarrow 0$$

$$U_1 \rightarrow 1$$





# Converting to Chomsky Normal Form

New Start:  
Replace

$$S \rightarrow A|b$$

$$A \rightarrow aS|a$$

No  $\epsilon$ :  
Replace

$$A \rightarrow XBY$$

$$B \rightarrow \epsilon$$

No Units:  
Replace

$$A \rightarrow B$$

$$B \rightarrow XY|a$$

New Variables:  
Replace:

$$A \rightarrow BXY$$

$$B \rightarrow bX$$

With

$$S_0 \rightarrow S$$

$$S \rightarrow A|b$$

$$A \rightarrow aS|a$$

With

$$A \rightarrow XBY$$

$$A \rightarrow XY$$

With

$$A \rightarrow XY|a$$

$$B \rightarrow XY|a$$

With:

$$A \rightarrow BU$$

$$U \rightarrow XY$$

$$B \rightarrow VX$$

$$V \rightarrow b$$

Practice Grammar:

$$S \rightarrow XY|Z$$

$$X \rightarrow aXb|\epsilon$$

$$Y \rightarrow Yc|c|\epsilon$$

$$Z \rightarrow aZc|W|\epsilon$$

$$W \rightarrow bW|b|\epsilon$$

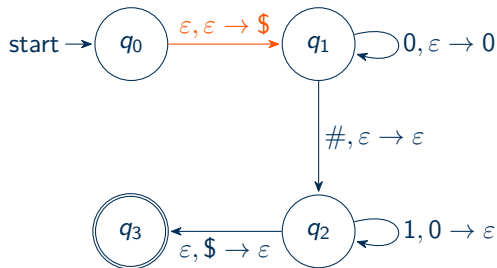


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# A Simple Example: $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$



$\$$  – Empty Stack Symbol

$a, \epsilon \rightarrow \epsilon$  – read

$\epsilon, \epsilon \rightarrow b$  – push

$\epsilon, b \rightarrow \epsilon$  – pop

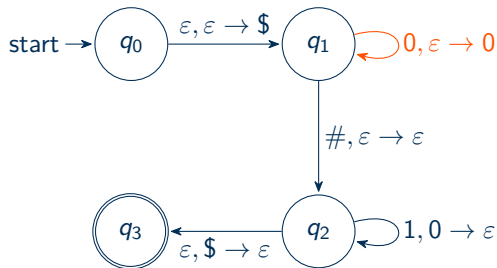
$a, \epsilon \rightarrow b$  – read and push

$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push



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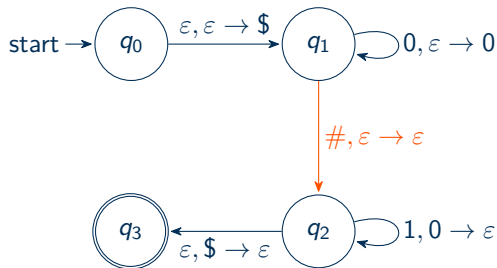
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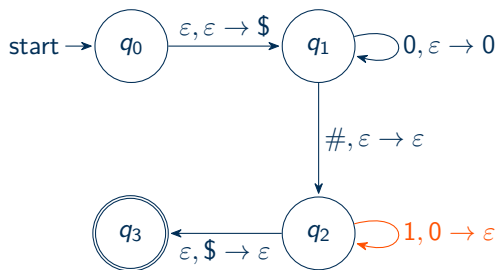
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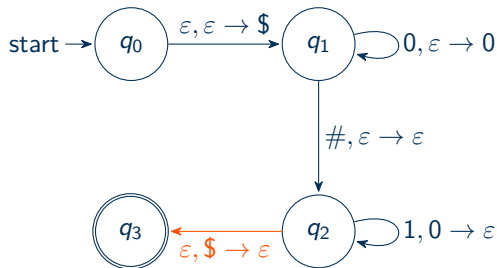
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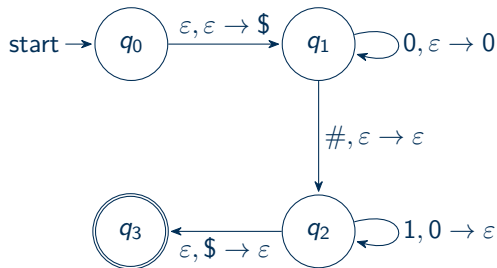
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$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push



# A Simple Example: $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$



$\$$  – Empty Stack Symbol

$a, \epsilon \rightarrow \epsilon$  – read

$\epsilon, \epsilon \rightarrow b$  – push

$\epsilon, b \rightarrow \epsilon$  – pop

$a, \epsilon \rightarrow b$  – read and push

$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

Input Tape:

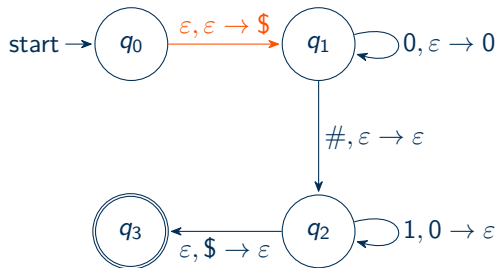
000#111

Stack:  $\sqcup$





# A Simple Example: $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$



$\$$  – Empty Stack Symbol

$a, \epsilon \rightarrow \epsilon$  – read

$\epsilon, \epsilon \rightarrow b$  – push

$\epsilon, b \rightarrow \epsilon$  – pop

$a, \epsilon \rightarrow b$  – read and push

$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

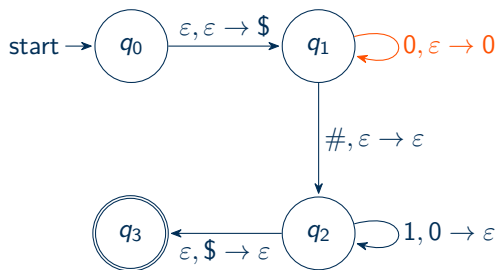
Input Tape:

000#111

Stack:  $\$$



# A Simple Example: $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$



\$ – Empty Stack Symbol

$a, \epsilon \rightarrow \epsilon$  – read

$\epsilon, \epsilon \rightarrow b$  – push

$\epsilon, b \rightarrow \epsilon$  – pop

$a, \epsilon \rightarrow b$  – read and push

$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

Input Tape:

000#111

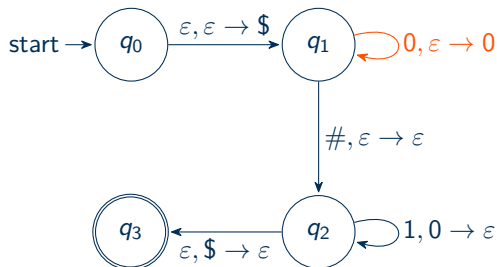
Stack:

0

\$



# A Simple Example: $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$



$\$$  – Empty Stack Symbol

$a, \epsilon \rightarrow \epsilon$  – read

$\epsilon, \epsilon \rightarrow b$  – push

$\epsilon, b \rightarrow \epsilon$  – pop

$a, \epsilon \rightarrow b$  – read and push

$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

Input Tape:

000#111

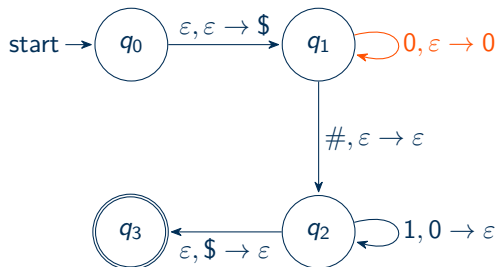
Stack:

0

0

\$



A Simple Example:  $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$ 

$\$$  – Empty Stack Symbol  
 $a, \epsilon \rightarrow \epsilon$  – read  
 $\epsilon, \epsilon \rightarrow b$  – push  
 $\epsilon, b \rightarrow \epsilon$  – pop  
 $a, \epsilon \rightarrow b$  – read and push  
 $a, b \rightarrow \epsilon$  – read and pop  
 $a, b \rightarrow c$  – read, pop, and push

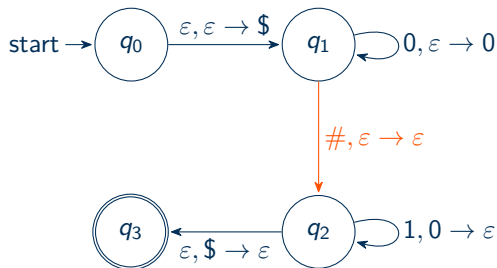
Input Tape:

000#111

Stack:

0  
0  
0  
\$



A Simple Example:  $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$ 

\$ – Empty Stack Symbol

$a, \epsilon \rightarrow \epsilon$  – read

$\epsilon, \epsilon \rightarrow b$  – push

$\epsilon, b \rightarrow \epsilon$  – pop

$a, \epsilon \rightarrow b$  – read and push

$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

Input Tape:

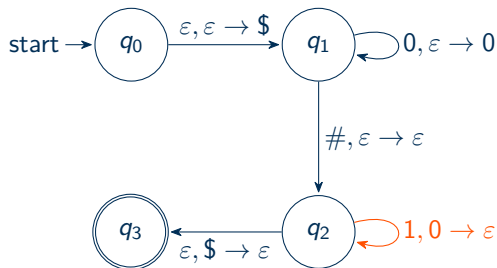
000#111

Stack:

0  
0  
0  
\$



# A Simple Example: $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$



\$ – Empty Stack Symbol

$a, \varepsilon \rightarrow \varepsilon$  – read

$\varepsilon, \varepsilon \rightarrow b$  – push

$\varepsilon, b \rightarrow \varepsilon$  – pop

$a, \varepsilon \rightarrow b$  – read and push

$a, b \rightarrow \varepsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

Input Tape:

000#111

Stack:

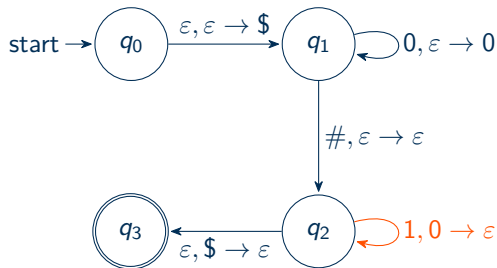
0

0

\$



# A Simple Example: $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$



$\$$  – Empty Stack Symbol

$a, \epsilon \rightarrow \epsilon$  – read

$\epsilon, \epsilon \rightarrow b$  – push

$\epsilon, b \rightarrow \epsilon$  – pop

$a, \epsilon \rightarrow b$  – read and push

$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

Input Tape:

000#111

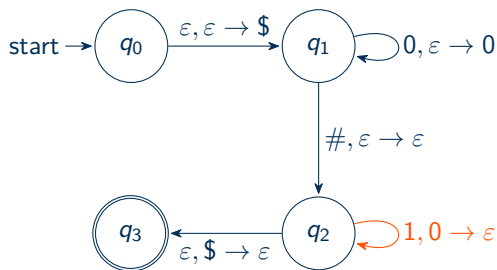
Stack:

0

\$



# A Simple Example: $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$



\$ – Empty Stack Symbol

$a, \epsilon \rightarrow \epsilon$  – read

$\epsilon, \epsilon \rightarrow b$  – push

$\epsilon, b \rightarrow \epsilon$  – pop

$a, \epsilon \rightarrow b$  – read and push

$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

Input Tape:

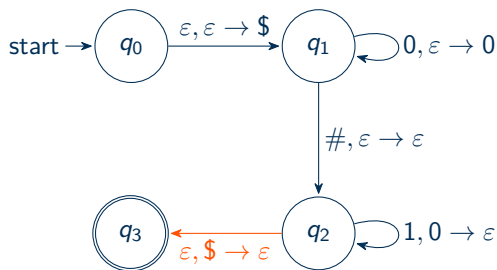
000#111

Stack: \$





# A Simple Example: $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$



\$ – Empty Stack Symbol

$a, \epsilon \rightarrow \epsilon$  – read

$\epsilon, \epsilon \rightarrow b$  – push

$\epsilon, b \rightarrow \epsilon$  – pop

$a, \epsilon \rightarrow b$  – read and push

$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

Input Tape:

000#111

Stack:  $\sqcup$



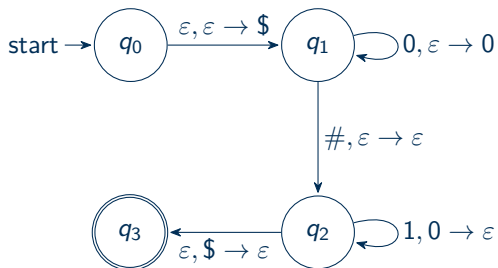
# Formal Definition

## Definition (Pushdown Automaton)

A **pushdown automaton** is a 6-tuple  $(Q, \Sigma, \Gamma, \delta, q_0, F)$ , where  $Q, \Sigma, \Gamma$ , and  $F$  are all finite sets, and

- 1  $Q$  is the set of states,
- 2  $\Sigma$  is the input alphabet,
- 3  $\Gamma$  is the stack alphabet,
- 4  $\delta : Q \times \Sigma_{\epsilon} \times \Gamma_{\epsilon} \rightarrow \mathcal{P}(Q \times \Gamma_{\epsilon})$  is the transition function,
- 5  $q_0 \in Q$  is the start state, and
- 6  $F \subseteq Q$  is the set of accept states.



Second Look at  $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$ 

\$ – Empty Stack Symbol

$a, \epsilon \rightarrow \epsilon$  – read

$\epsilon, \epsilon \rightarrow b$  – push

$\epsilon, b \rightarrow \epsilon$  – pop

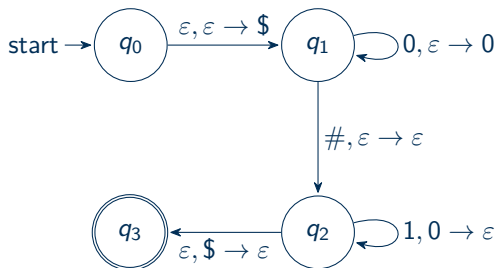
$a, \epsilon \rightarrow b$  – read and push

$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

- $\delta(q_0, \epsilon, \epsilon) = \{(q_1, \$)\}$



Second Look at  $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$ 

\$ – Empty Stack Symbol

$a, \varepsilon \rightarrow \varepsilon$  – read

$\varepsilon, \varepsilon \rightarrow b$  – push

$\varepsilon, b \rightarrow \varepsilon$  – pop

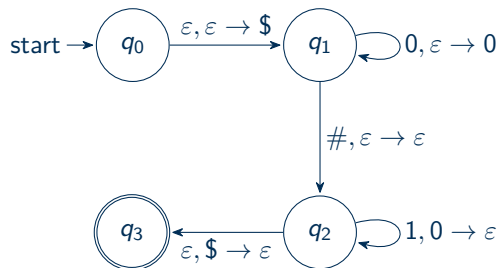
$a, \varepsilon \rightarrow b$  – read and push

$a, b \rightarrow \varepsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

- $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$
- $\delta(q_1, 0, \varepsilon) = \{(q_1, 0)\}$



Second Look at  $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$ 

\$ – Empty Stack Symbol

$a, \varepsilon \rightarrow \varepsilon$  – read

$\varepsilon, \varepsilon \rightarrow b$  – push

$\varepsilon, b \rightarrow \varepsilon$  – pop

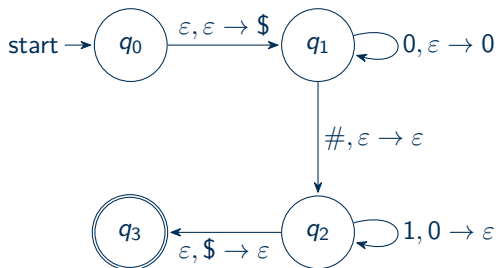
$a, \varepsilon \rightarrow b$  – read and push

$a, b \rightarrow \varepsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

- $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$
- $\delta(q_1, 0, \varepsilon) = \{(q_1, 0)\}$
- $\delta(q_1, \#, \varepsilon) = \{(q_2, \varepsilon)\}$



Second Look at  $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$ 

\$ – Empty Stack Symbol

$a, \varepsilon \rightarrow \varepsilon$  – read

$\varepsilon, \varepsilon \rightarrow b$  – push

$\varepsilon, b \rightarrow \varepsilon$  – pop

$a, \varepsilon \rightarrow b$  – read and push

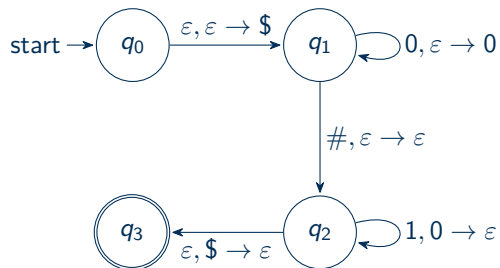
$a, b \rightarrow \varepsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

- $\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$
- $\delta(q_1, 0, \varepsilon) = \{(q_1, 0)\}$
- $\delta(q_1, \#, \varepsilon) = \{(q_2, \varepsilon)\}$

- $\delta(q_2, 1, 0) = \{(q_2, \varepsilon)\}$



Second Look at  $L = \{0^n \# 1^n \mid n \in \mathbb{N}\}$ 

$\$$  – Empty Stack Symbol

$a, \epsilon \rightarrow \epsilon$  – read

$\epsilon, \epsilon \rightarrow b$  – push

$\epsilon, b \rightarrow \epsilon$  – pop

$a, \epsilon \rightarrow b$  – read and push

$a, b \rightarrow \epsilon$  – read and pop

$a, b \rightarrow c$  – read, pop, and push

- $\delta(q_0, \epsilon, \epsilon) = \{(q_1, \$)\}$
- $\delta(q_1, 0, \epsilon) = \{(q_1, 0)\}$
- $\delta(q_1, \#, \epsilon) = \{(q_2, \epsilon)\}$
- $\delta(q_2, 1, 0) = \{(q_2, \epsilon)\}$
- $\delta(q_2, \epsilon, \$) = \{(q_3, \epsilon)\}$



# A More Complicated Example:

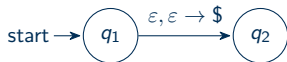
$$\{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$$





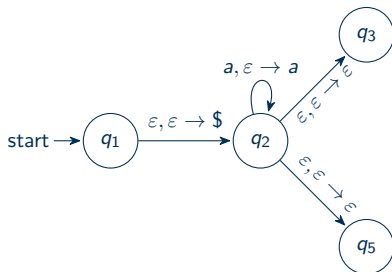
# A More Complicated Example:

$$\{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$$



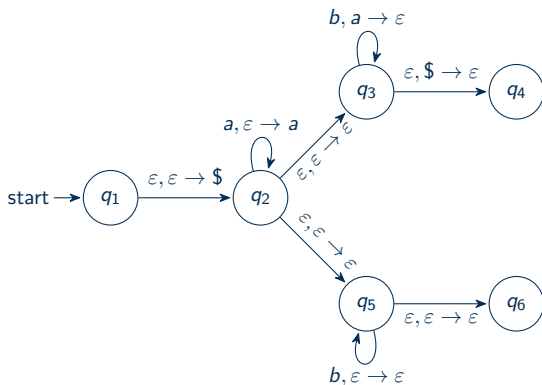
## A More Complicated Example:

$$\{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$$



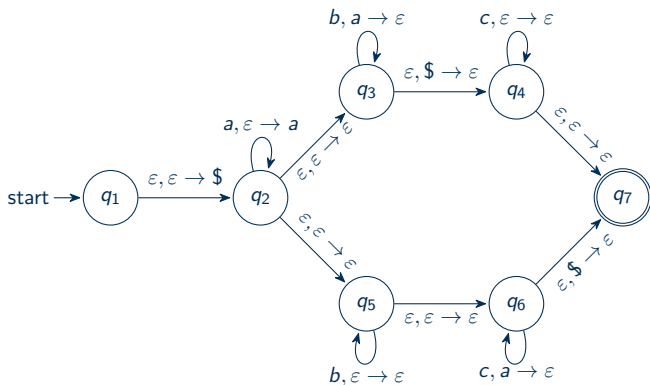
## A More Complicated Example:

$$\{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$$



## A More Complicated Example:

$$\{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$$



# Table of Contents

- 1 Context-Free Grammar
- 2 Chomsky Normal Form
- 3 Pushdown Automata
- 4 CFL and PDA**
- 5 Next Class



# CFL implies PDA

## Context Free Grammar

$$X \rightarrow XX$$

$$X \rightarrow aXb \mid bXa$$

$$X \rightarrow \varepsilon$$



# CFL implies PDA

## Context Free Grammar

$$X \rightarrow XX$$

$$X \rightarrow aXb|bXa$$

$$X \rightarrow \varepsilon$$

OR

$$X \rightarrow XX|aB|bA$$

$$A \rightarrow Xa|a$$

$$B \rightarrow Xb|b$$

$$X \rightarrow \varepsilon$$



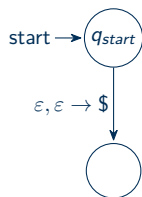
# CFL implies PDA

## Context Free Grammar

$$X \rightarrow XX$$

$$X \rightarrow aXb \mid bXa$$

$$X \rightarrow \varepsilon$$



OR

$$X \rightarrow XX \mid aB \mid bA$$

$$A \rightarrow Xa \mid a$$

$$B \rightarrow Xb \mid b$$

$$X \rightarrow \varepsilon$$





# CFL implies PDA

## Context Free Grammar

$$X \rightarrow XX$$

$$X \rightarrow aXb \mid bXa$$

$$X \rightarrow \varepsilon$$

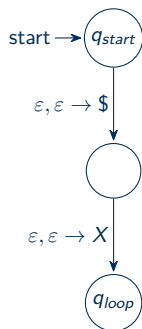
OR

$$X \rightarrow XX \mid aB \mid bA$$

$$A \rightarrow Xa \mid a$$

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# CFL implies PDA

## Context Free Grammar

$$X \rightarrow XX$$

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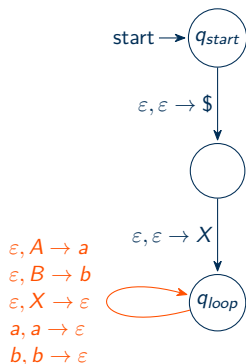
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$$X \rightarrow XX \mid aB \mid bA$$

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# CFL implies PDA

## Context Free Grammar

$$X \rightarrow XX$$

$$X \rightarrow aXb \mid bXa$$

$$X \rightarrow \varepsilon$$

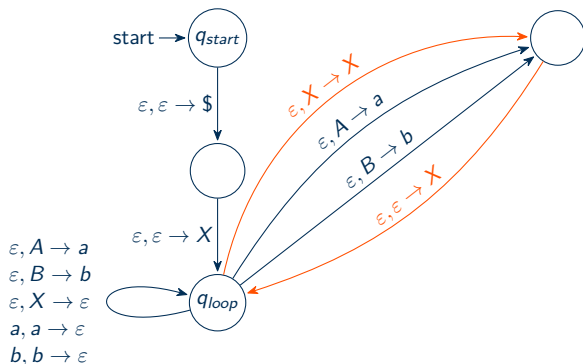
OR

$$X \rightarrow \color{red}XX \mid aB \mid bA$$

$$A \rightarrow \color{red}Xa \mid a$$

$$B \rightarrow \color{red}Xb \mid b$$

$$X \rightarrow \varepsilon$$



# CFL implies PDA

## Context Free Grammar

$$X \rightarrow XX$$

$$X \rightarrow aXb \mid bXa$$

$$X \rightarrow \varepsilon$$

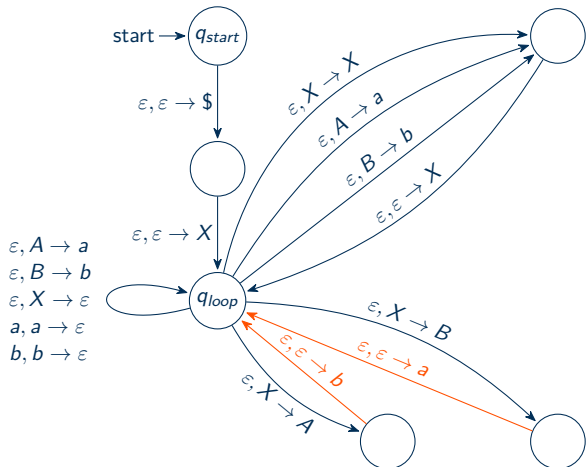
OR

$$X \rightarrow XX \mid aB \mid bA$$

$$A \rightarrow Xa \mid a$$

$$B \rightarrow Xb \mid b$$

$$X \rightarrow \varepsilon$$



# CFL implies PDA

## Context Free Grammar

$$X \rightarrow XX$$

$$X \rightarrow aXb \mid bXa$$

$$X \rightarrow \epsilon$$

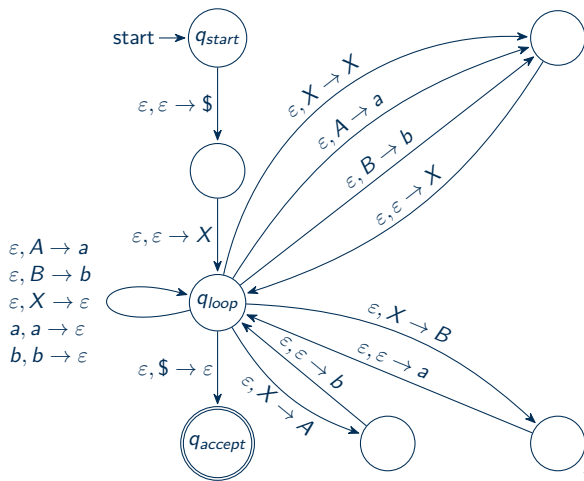
OR

$$X \rightarrow XX \mid aB \mid bA$$

$$A \rightarrow Xa \mid a$$

$$B \rightarrow Xb \mid b$$

$$X \rightarrow \epsilon$$



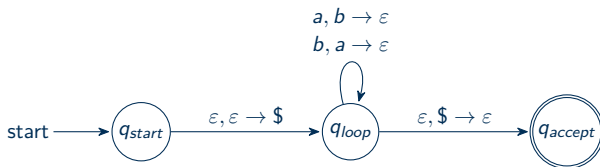
# PDA implies CFL (Part 1)



# PDA implies CFL (Part 1)

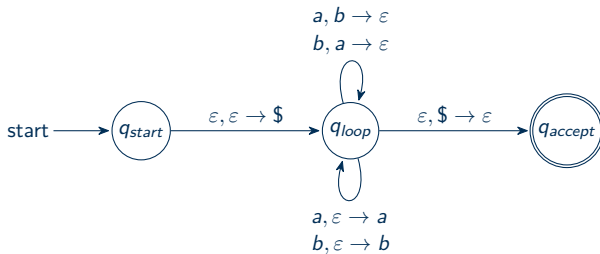


# PDA implies CFL (Part 1)

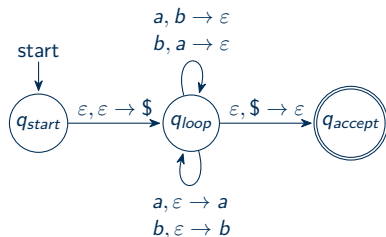




# PDA implies CFL (Part 1)



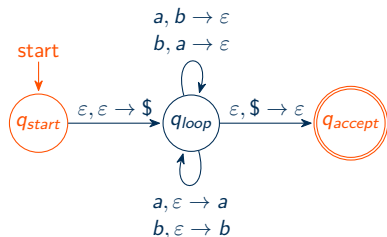
# PDA implies CFL (Part 2)



- 1 Add  $S \rightarrow A_{start, accept}$  as the start variable.
- 2 If  $(r, u) \in \delta(p, a, \epsilon) \wedge (q, \epsilon) \in \delta(s, b, u)$  then add  $A_{pq} \rightarrow aA_{rs}b$ .
- 3 For each  $p, q, r$  add  $A_{pq} \rightarrow A_{pr}A_{rq}$ .
- 4 For each  $p$  add  $A_{pp} \rightarrow \epsilon$ .



# PDA implies CFL (Part 2)



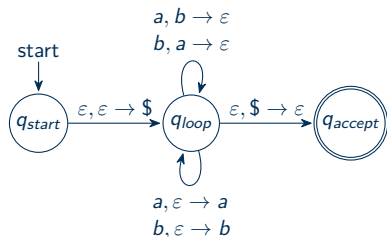
New Rules:

- $S \rightarrow R_{SA}$
- $R_{SS} \rightarrow \epsilon$
- $R_{LL} \rightarrow \epsilon$
- $R_{AA} \rightarrow \epsilon$

- 1 Add  $S \rightarrow A_{start, accept}$  as the start variable.
- 2 If  $(r, u) \in \delta(p, a, \epsilon) \wedge (q, \epsilon) \in \delta(s, b, u)$  then add  $A_{pq} \rightarrow aA_{rs}b$ .
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# PDA implies CFL (Part 2)

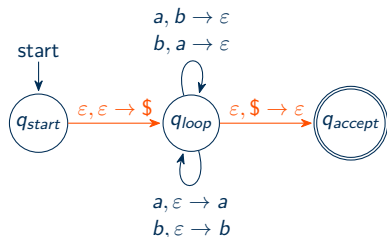


New Rules:

- $S \rightarrow R_{SA}$
  - $R_{SS} \rightarrow \epsilon | R_{SS}R_{SS}$
  - $R_{LL} \rightarrow \epsilon | R_{LL}R_{LL}$
  - $R_{AA} \rightarrow \epsilon | R_{AA}R_{AA}$
  - $R_{SA} \rightarrow R_{SL}R_{LA}$
  - $R_{SL} \rightarrow R_{SS}R_{SL} | R_{SL}R_{LL}$
  - $R_{LA} \rightarrow R_{LL}R_{LA} | R_{LA}R_{AA}$
- 1 Add  $S \rightarrow A_{start, accept}$  as the start variable.
  - 2 If  $(r, u) \in \delta(p, a, \epsilon) \wedge (q, \epsilon) \in \delta(s, b, u)$  then add  $A_{pq} \rightarrow aA_{rs}b$ .
  - 3 For each  $p, q, r$  add  $A_{pq} \rightarrow A_{pr}A_{rq}$ .
  - 4 For each  $p$  add  $A_{pp} \rightarrow \epsilon$ .



# PDA implies CFL (Part 2)

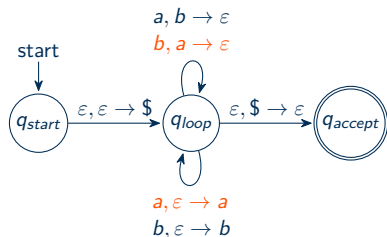


New Rules:

- $S \rightarrow R_{SA}$
  - $R_{SS} \rightarrow \epsilon | R_{SS}R_{SS}$
  - $R_{LL} \rightarrow \epsilon | R_{LL}R_{LL}$
  - $R_{AA} \rightarrow \epsilon | R_{AA}R_{AA}$
  - $R_{SA} \rightarrow R_{SL}R_{LA} | \epsilon R_{LL} \epsilon$
  - $R_{SL} \rightarrow R_{SS}R_{SL} | R_{SL}R_{LL}$
  - $R_{LA} \rightarrow R_{LL}R_{LA} | R_{LA}R_{AA}$
- 1 Add  $S \rightarrow A_{start, accept}$  as the start variable.
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# PDA implies CFL (Part 2)



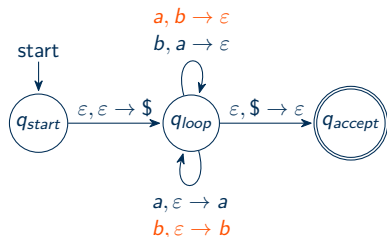
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- 3 For each  $p, q, r$  add  $A_{pq} \rightarrow A_{pr}A_{rq}$ .
- 4 For each  $p$  add  $A_{pp} \rightarrow \epsilon$ .

- $S \rightarrow R_{SA}$
- $R_{SS} \rightarrow \epsilon | R_{SS}R_{SS}$
- $R_{LL} \rightarrow \epsilon | R_{LL}R_{LL} | aR_{LL}b$
- $R_{AA} \rightarrow \epsilon | R_{AA}R_{AA}$
- $R_{SA} \rightarrow R_{SL}R_{LA} | \epsilon R_{LL} \epsilon$
- $R_{SL} \rightarrow R_{SS}R_{SL} | R_{SL}R_{LL}$
- $R_{LA} \rightarrow R_{LL}R_{LA} | R_{LA}R_{AA}$



# PDA implies CFL (Part 2)



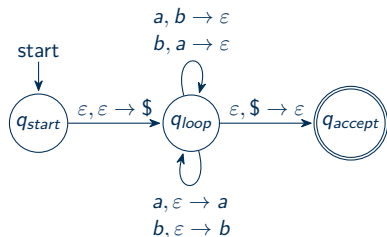
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- $S \rightarrow R_{SA}$
- $R_{SS} \rightarrow \epsilon | R_{SS}R_{SS}$
- $R_{LL} \rightarrow \epsilon | R_{LL}R_{LL} | aR_{LL}b | bR_{LL}a$
- $R_{AA} \rightarrow \epsilon | R_{AA}R_{AA}$
- $R_{SA} \rightarrow R_{SL}R_{LA} | \epsilon R_{LL} \epsilon$
- $R_{SL} \rightarrow R_{SS}R_{SL} | R_{SL}R_{LL}$
- $R_{LA} \rightarrow R_{LL}R_{LA} | R_{LA}R_{AA}$



# PDA implies CFL (Part 2)



- 1 Add  $S \rightarrow A_{start, accept}$  as the start variable.
- 2 If  $(r, u) \in \delta(p, a, \epsilon) \wedge (q, \epsilon) \in \delta(s, b, u)$  then add  $A_{pq} \rightarrow aA_{rs}b$ .
- 3 For each  $p, q, r$  add  $A_{pq} \rightarrow A_{pr}A_{rq}$ .
- 4 For each  $p$  add  $A_{pp} \rightarrow \epsilon$ .

New Rules:

- $S \rightarrow R_{SA}$
- $R_{SS} \rightarrow \epsilon | R_{SS}R_{SS}$
- $R_{LL} \rightarrow \epsilon | R_{LL}R_{LL} | aR_{LL}b | bR_{LL}a$
- $R_{AA} \rightarrow \epsilon | R_{AA}R_{AA}$
- $R_{SA} \rightarrow R_{SL}R_{LA} | \epsilon R_{LL} \epsilon$
- $R_{SL} \rightarrow R_{SS}R_{SL} | R_{SL}R_{LL}$
- $R_{LA} \rightarrow R_{LL}R_{LA} | R_{LA}R_{AA}$

Compared to:

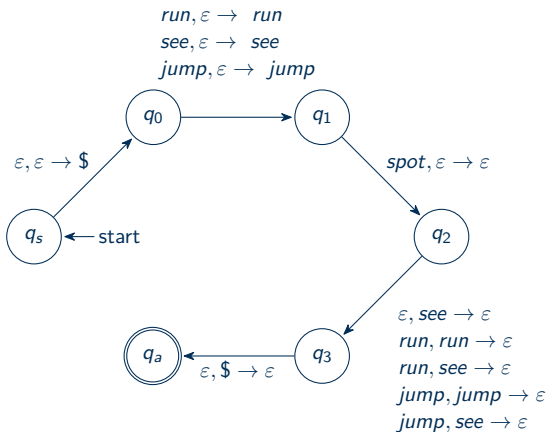
$$X \rightarrow XX | aXb | bXa | \epsilon.$$





## Practice

- 1  $S \rightarrow A_{start, accept}$
- 2 Add  $A_{pq} \rightarrow aA_{rs}b.$ , if  $(r, u) \in \delta(p, a, \epsilon)$  and  $(q, \epsilon) \in \delta(s, b, u)$
- 3 Add  $A_{pq} \rightarrow A_{pr}A_{rq}.$
- 4 Add  $A_{pp} \rightarrow \epsilon.$



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# Next Class

- Non-Context-Free Languages
- Pumping Lemma for CFL



# Next Class

- Non-Context-Free Languages
- Pumping Lemma for CFL
- Deterministic CFL



# Grammars and Pushdown Automata

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