Regular Expressions and the Pumping Lemma

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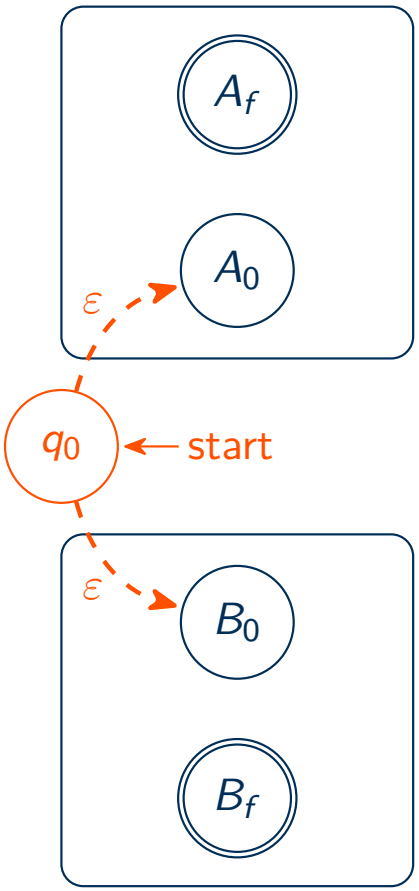
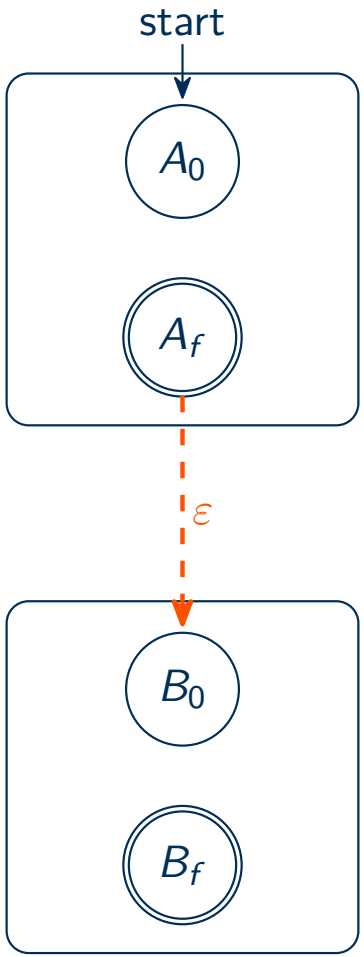
Review Combining NFAs

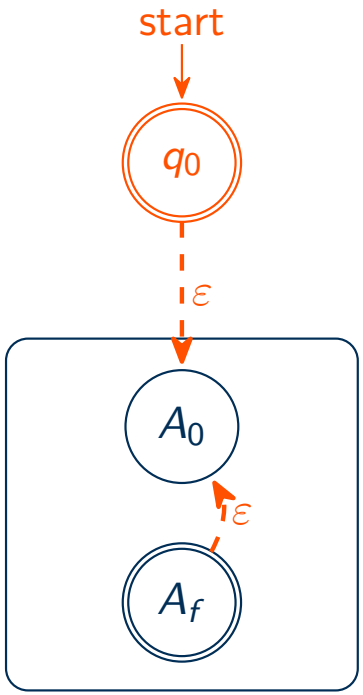
$A \cup B$



$A \circ B$



A^*



Basics of Regular Expressions

Definition (Regular Expression)

Given an alphabet Σ and regular languages A and B a *regular expression* is any combination of the following:

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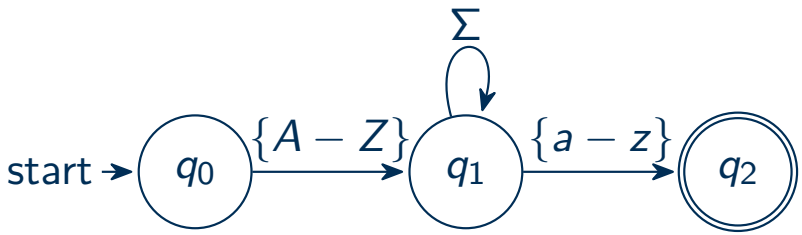
- Star: A^* (includes ε)
- Plus: $A^+ = A \circ A^* = AA^*$

Automaton to Regex

With the alphabet $\Sigma = \{a - z, A - Z, 0 - 9, \$\}$ here are equivalent machines and regular expressions:



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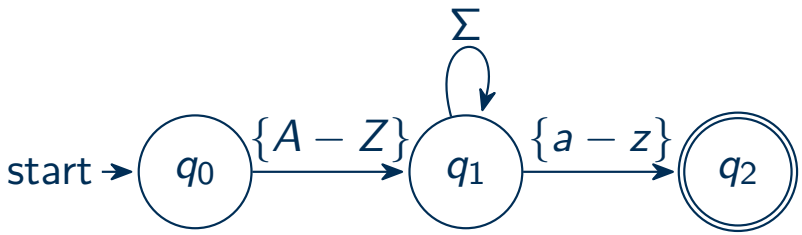

- $\{A-Z\} \circ \Sigma^* \circ \{a-z\}$
- *Python* : $[A-Z][a-zA-Z0-9\backslash\$] * [a-z]$

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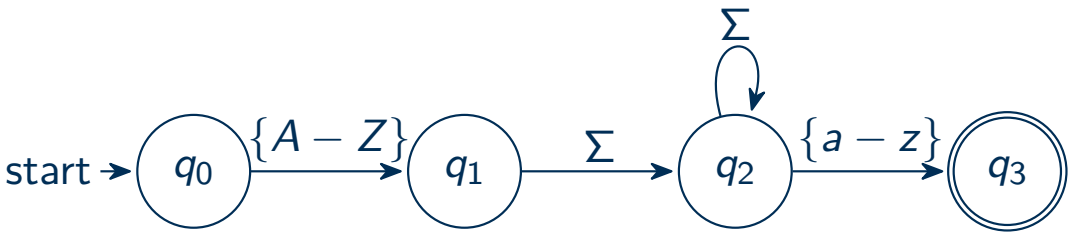

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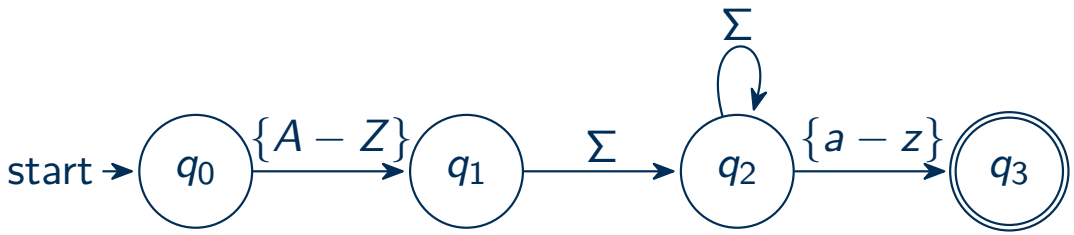

- $\{A-Z\} \circ (\Sigma \circ \Sigma^*) \circ \{a-z\}$
- $\{A-Z\} \circ \Sigma^+ \circ \{a-z\}$
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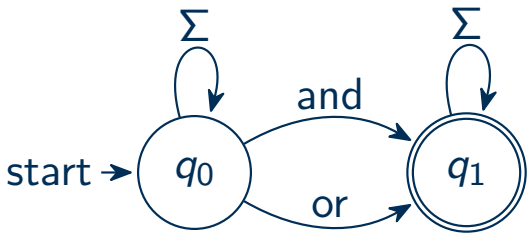

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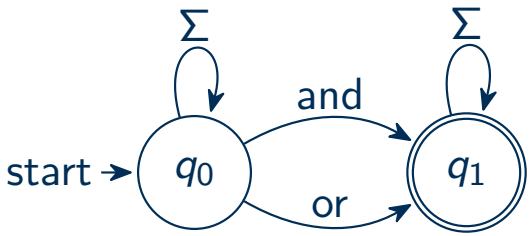
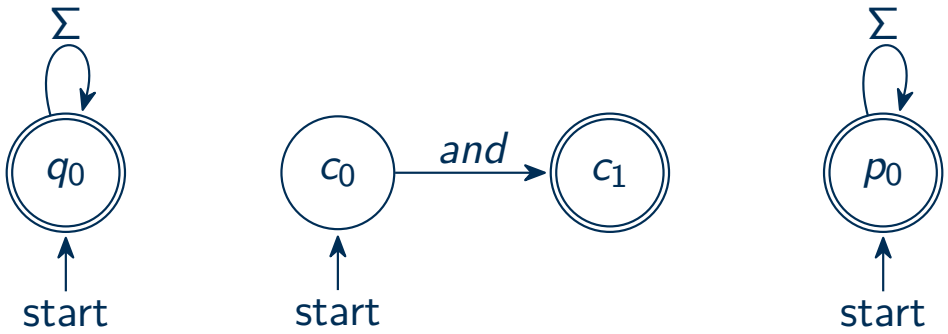


- $\Sigma^* \circ (and \cup or) \circ \Sigma^*$
- $(\Sigma^* \circ (and) \circ \Sigma^*) \cup (\Sigma^* \circ (or) \circ \Sigma^*)$
- *Python* : $[a-zA-Z0-9\backslash\$] * (and|or)[a-zA-Z0-9\backslash\$]*$

Regular Expressions to NFAs

$$(\Sigma^* \circ (and) \circ \Sigma^*) \cup (\Sigma^* \circ (or) \circ \Sigma^*)$$



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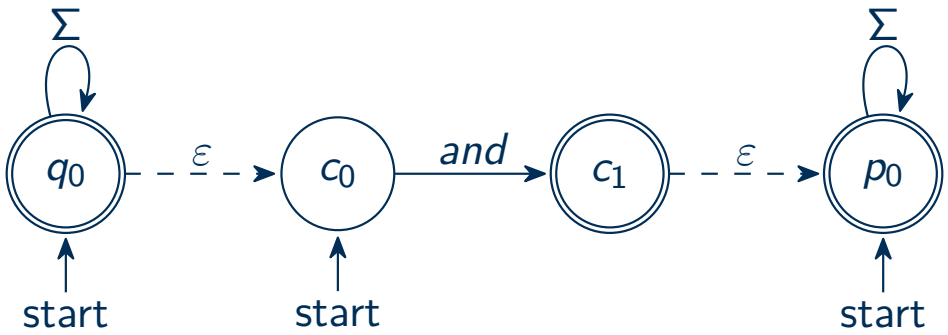

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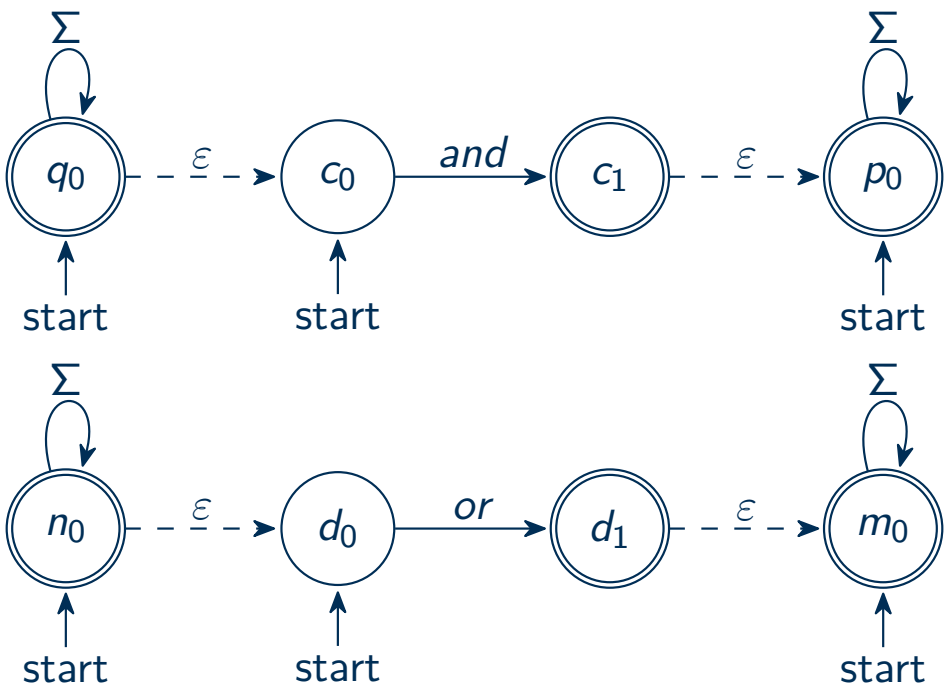

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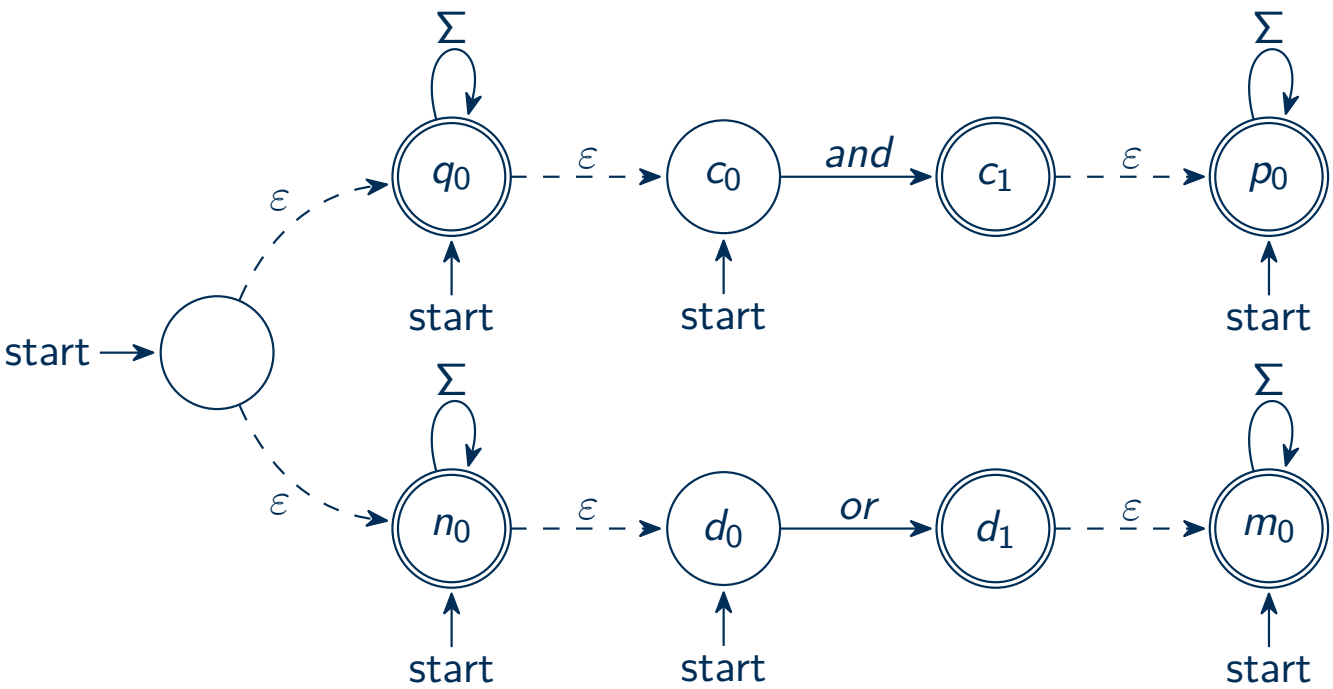

Identities

- Order of Operations: (), $*$, $\circ$, $\cup$
- Distribution: $(A \cup B) \circ C = A \circ C \cup B \circ C$
- Commutative: $A \cup B = B \cup A$
- "Additive" Identity: $A \cup \emptyset = A$ and $A \circ \emptyset = \emptyset$
- "Multiplicative" Identity: $A \circ \varepsilon = A$

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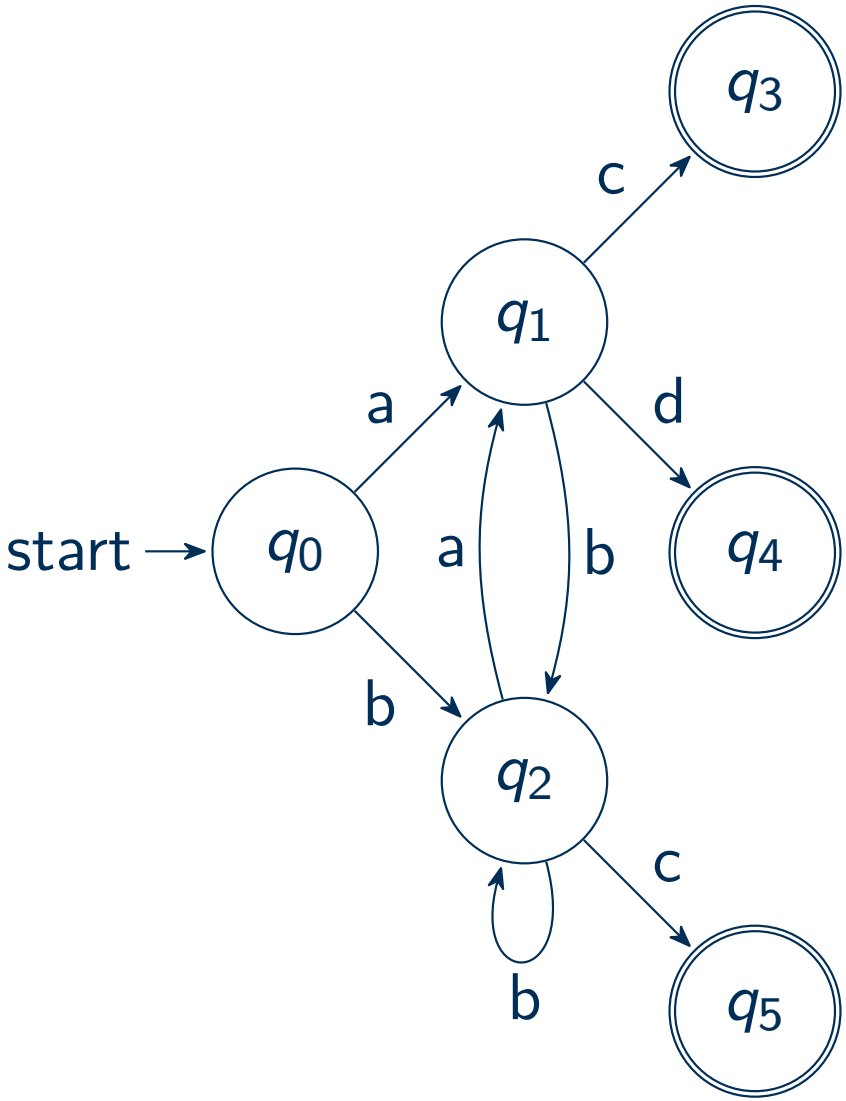
Generalized NFA

Definition (Generalized NFA)

A *Generalized nondeterministic finite automaton* is a 5-tuple $(Q, \Sigma, \delta, q_{start}, q_{accept})$, where

- Q is a finite set of states
- Σ is the input alphabet
- $\mathcal{R}$ is the set of all regular expressions using Σ
- $\delta : (Q \setminus \{q_{accept}\}) \times (Q \setminus \{q_{start}\}) \longrightarrow \mathcal{R}$ is the transition function
- q_{start} is the start state
- q_{accept} is the unique accept state

Converting to a GNFA and Reducing (NFA to Regex)



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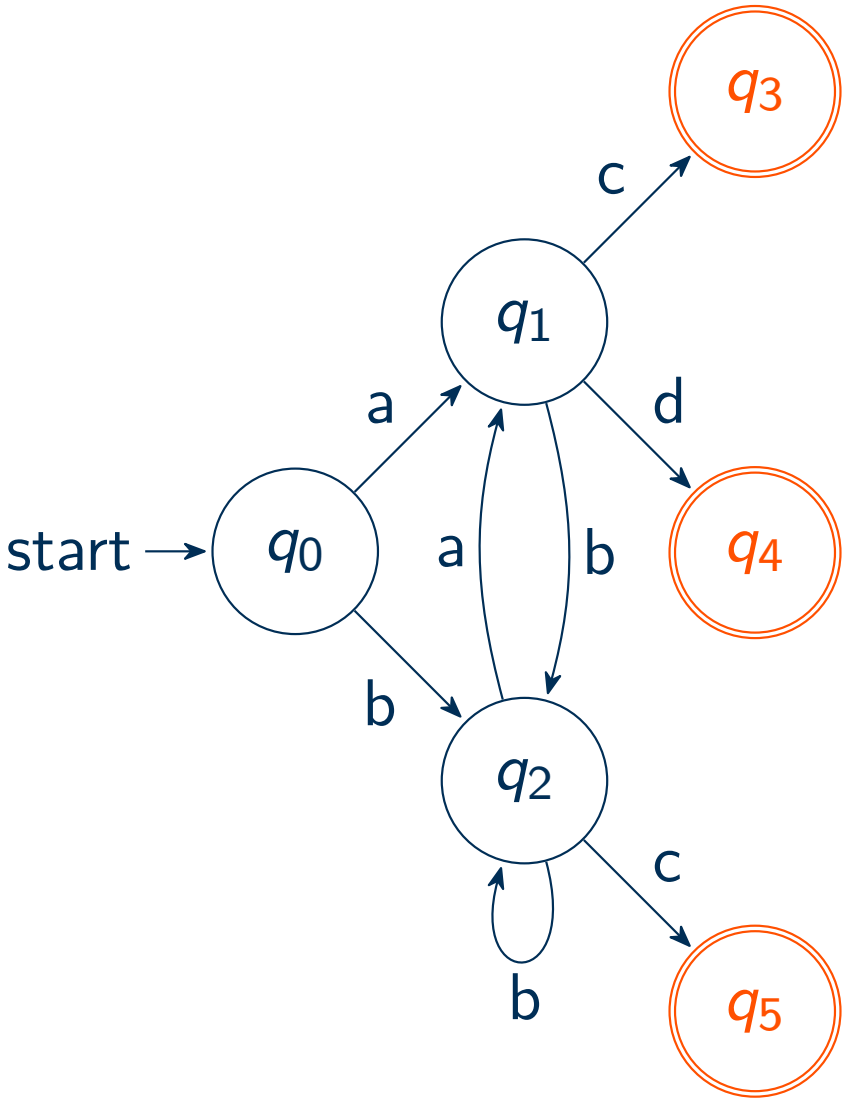

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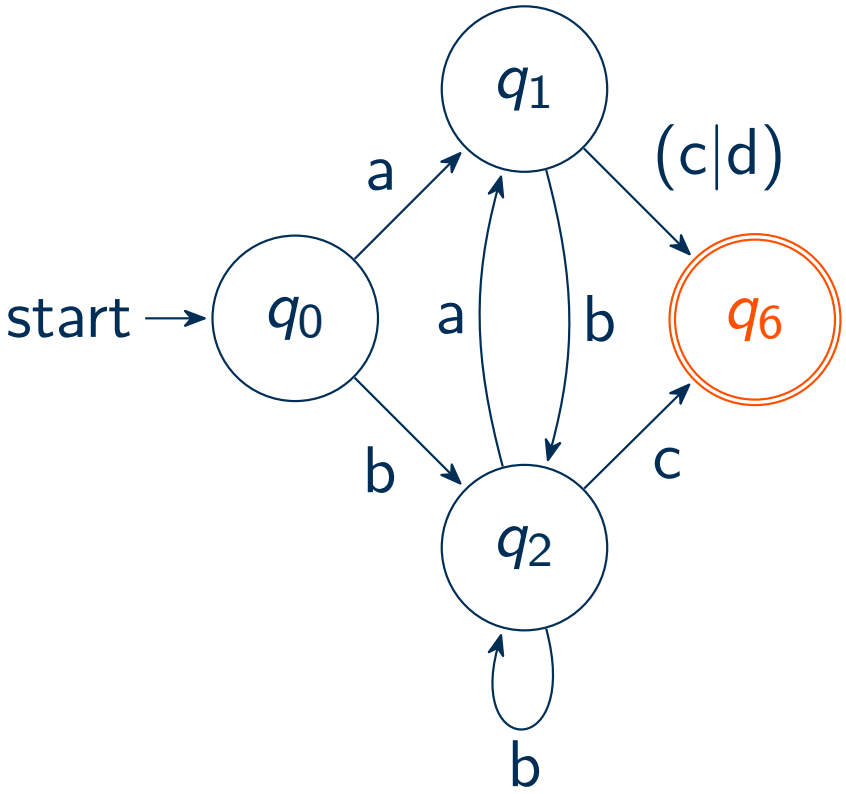

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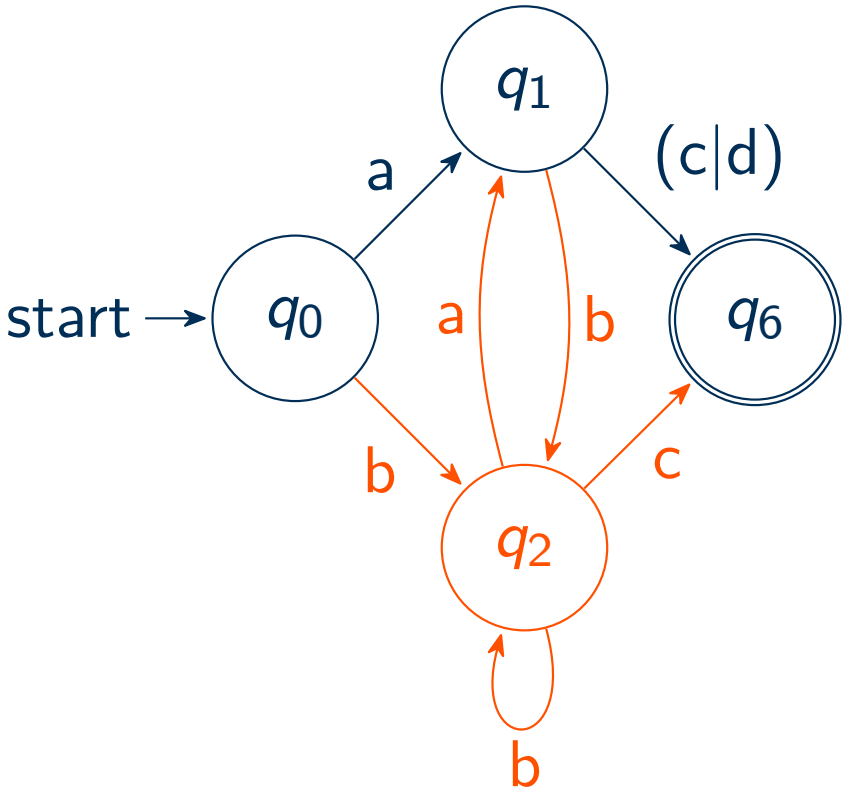

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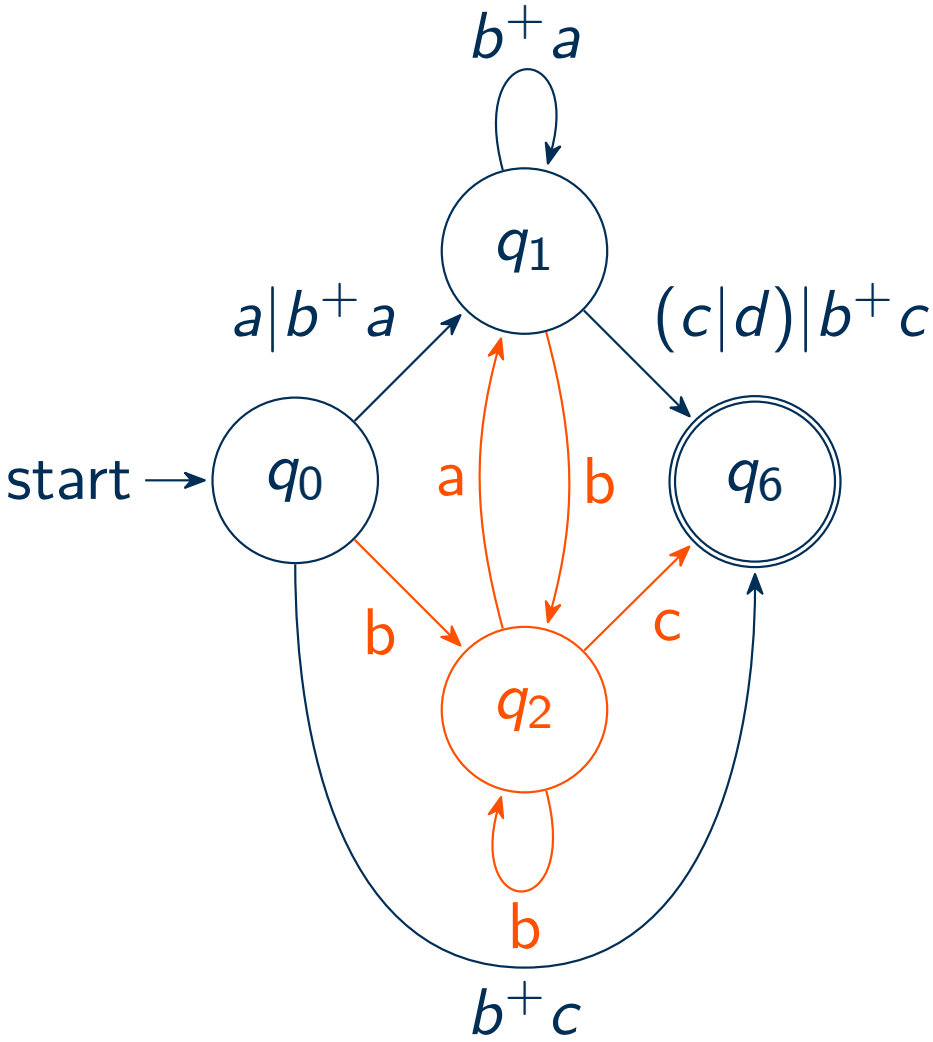

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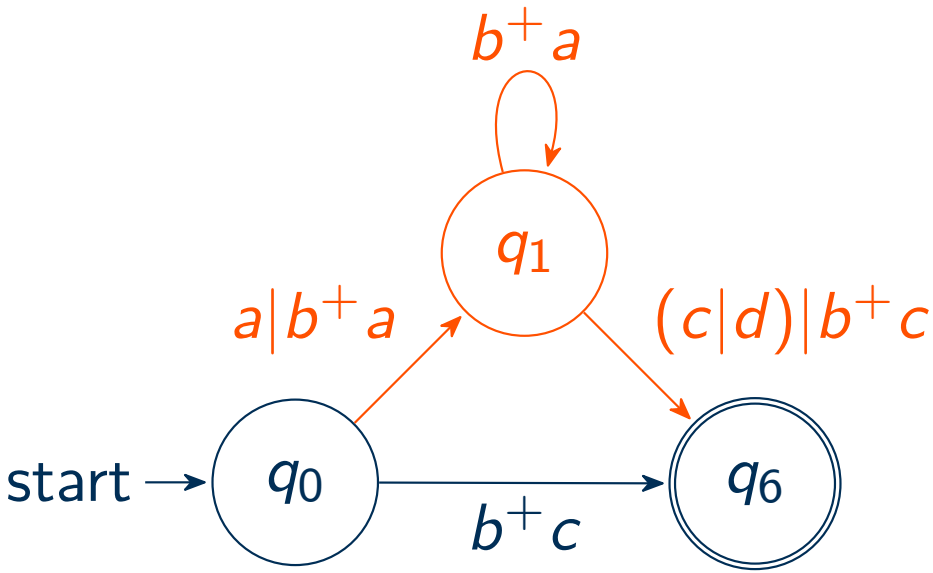

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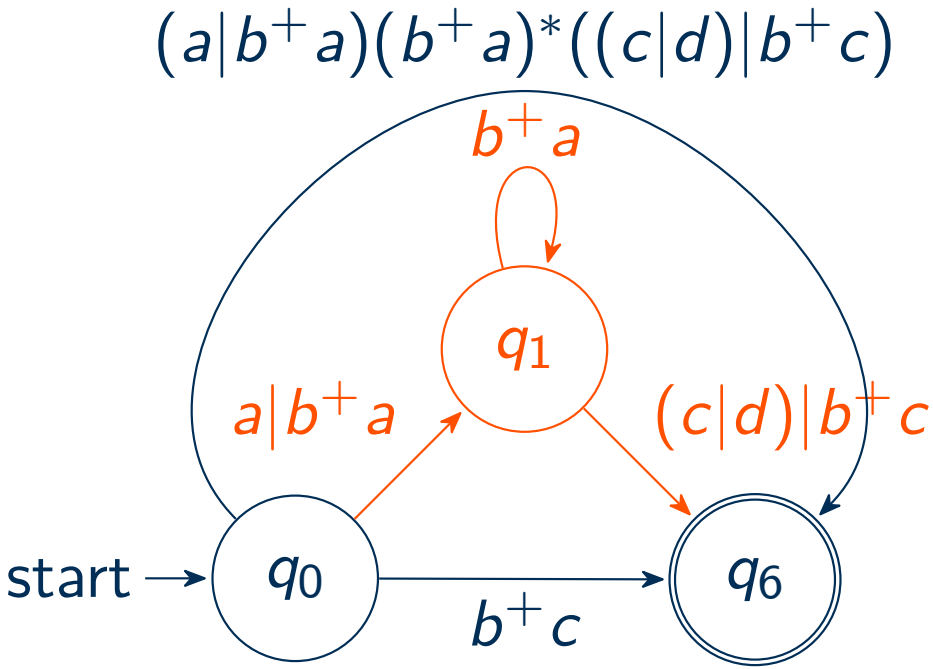

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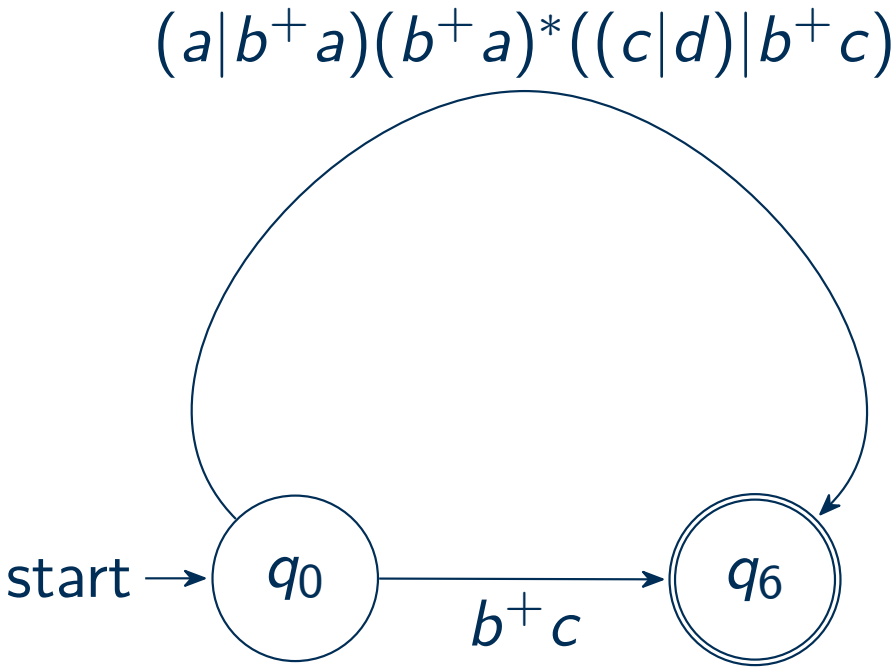

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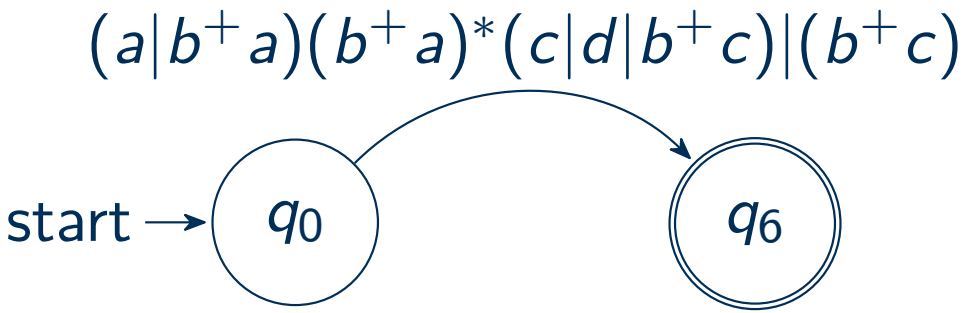

Regular Languages and Regular Expressions

Theorem

A language is regular if and only if some regular expression describes it.

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Statement

Theorem (Pumping Lemma)

If A is a regular language, then there is a number p (the pumping length) such that, if s is any string in A of length at least p, then s may be written in three pieces, $s = xyz$, satisfying the following conditions:

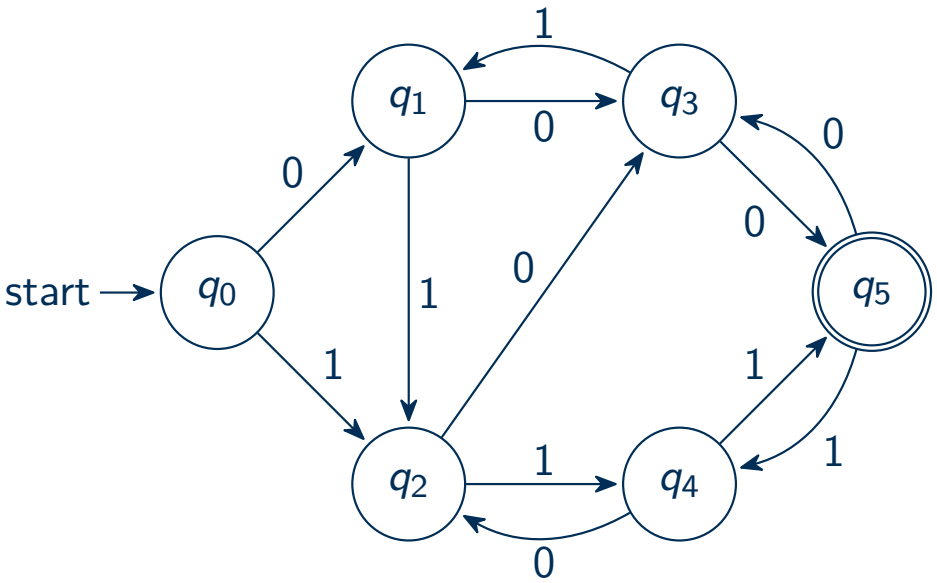
1. *for each $i \geq 0$, $xy^iz \in A$,*
2. *$|y| > 0$, and*
3. *$|xy| \leq p$.*

Pumping Example

Let $\Sigma = \{0, 1\}$ and consider

$$s = 01001001111$$

with $|s| = 11 > 6 = |Q|$.



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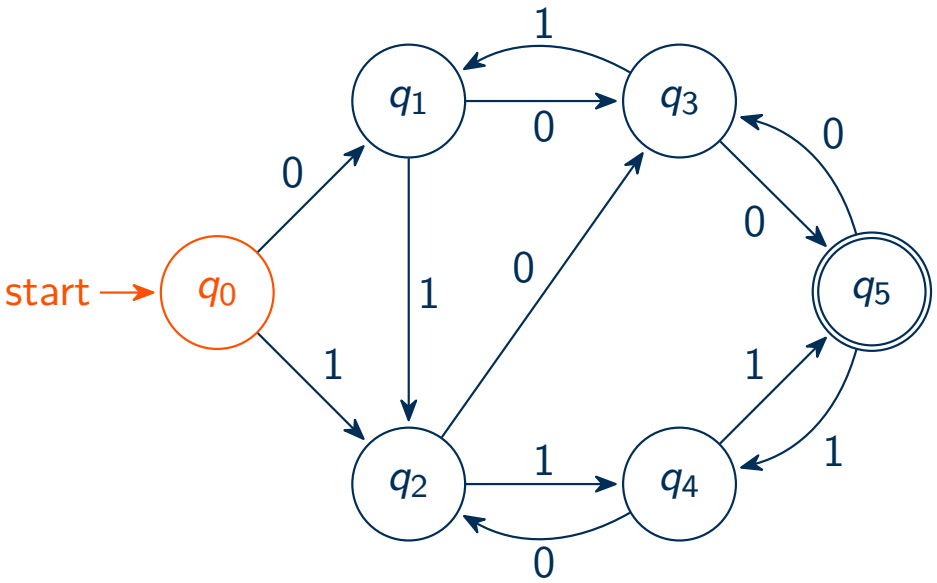

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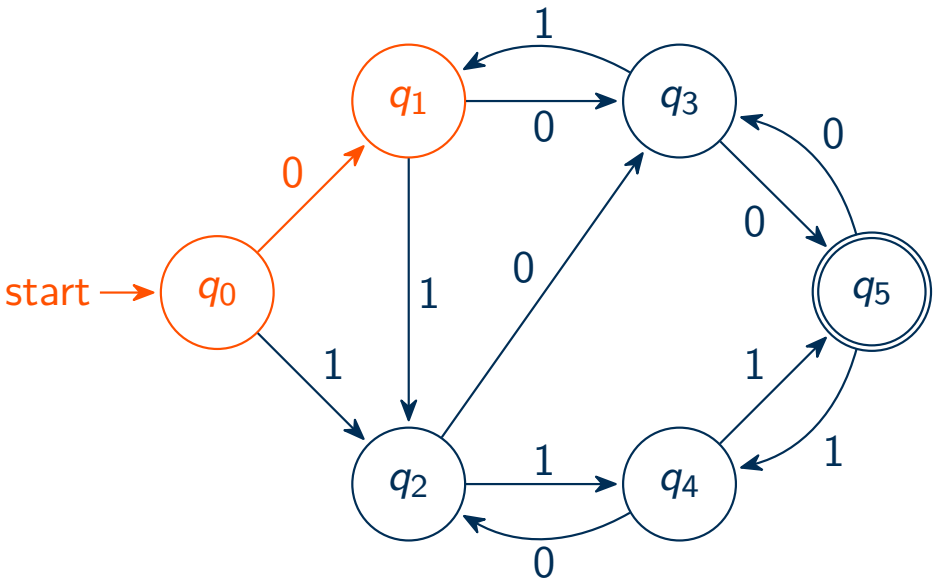

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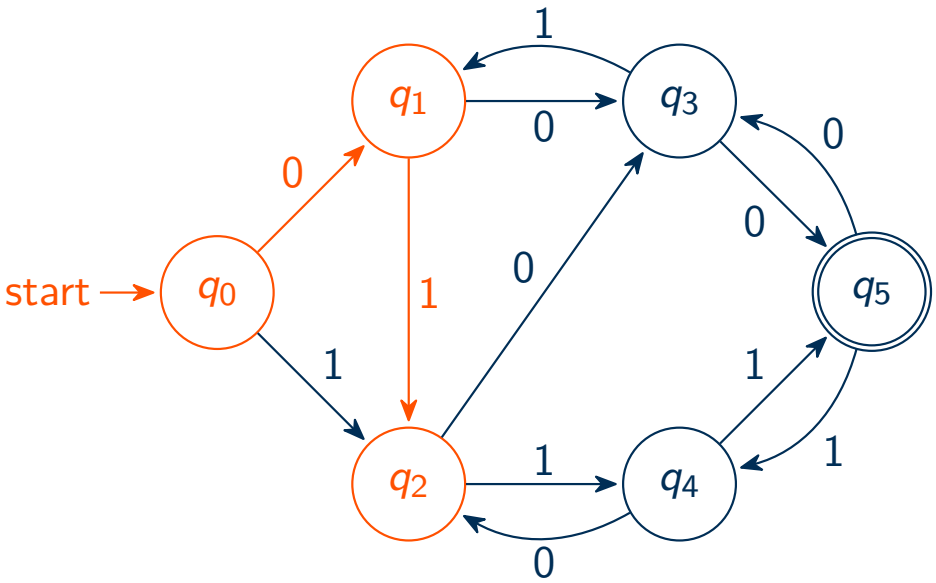

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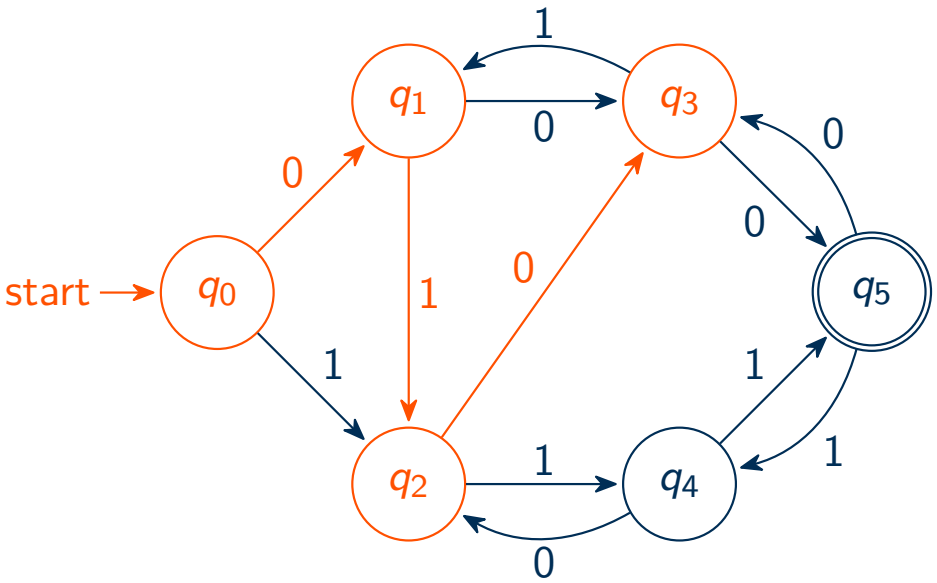

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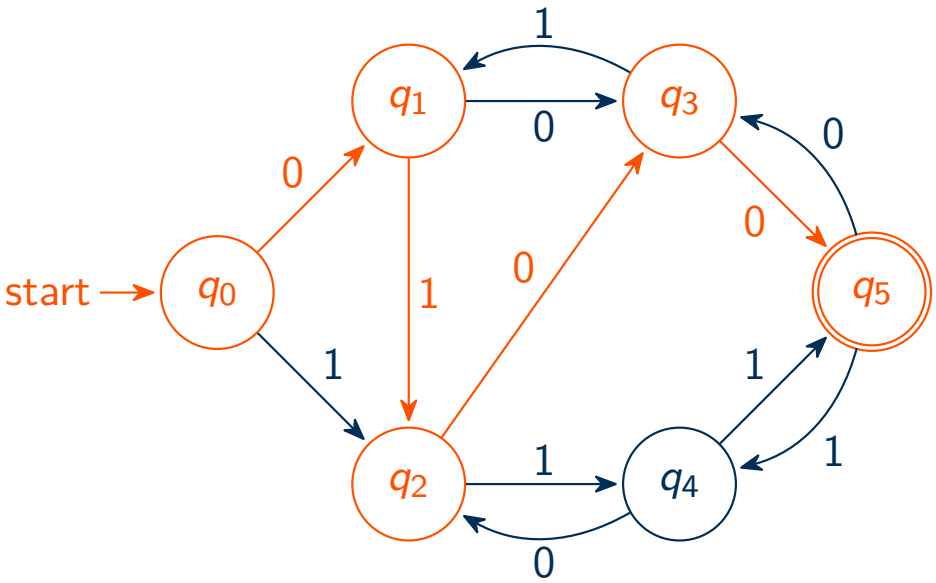

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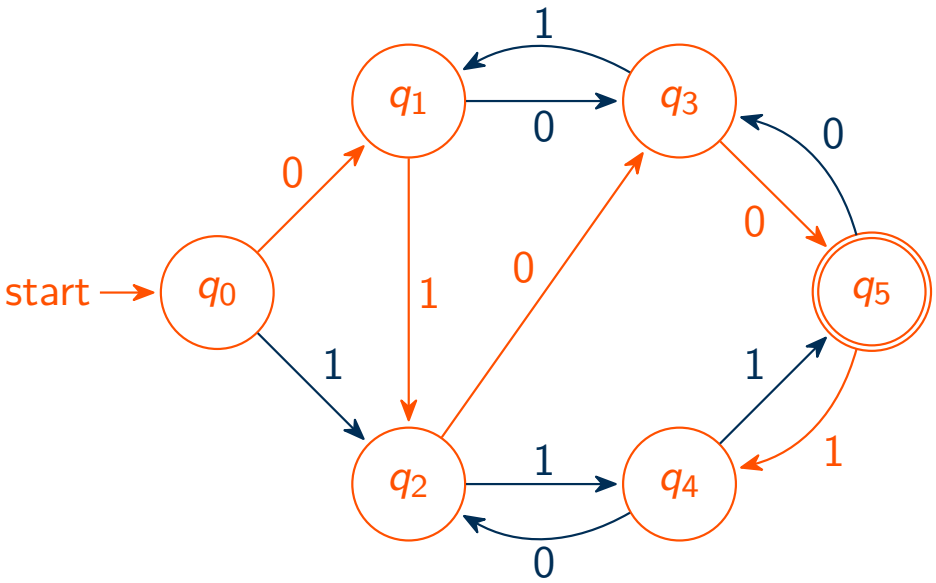

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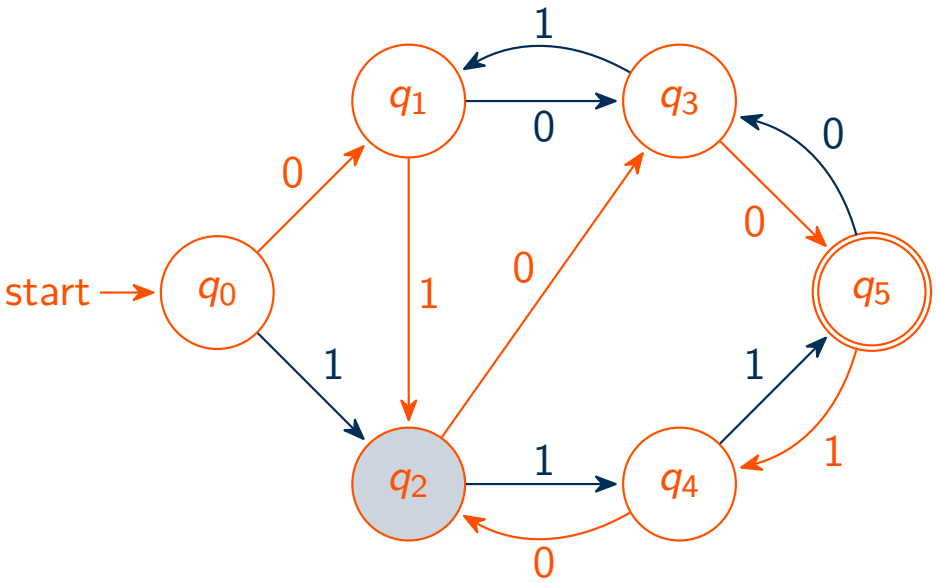

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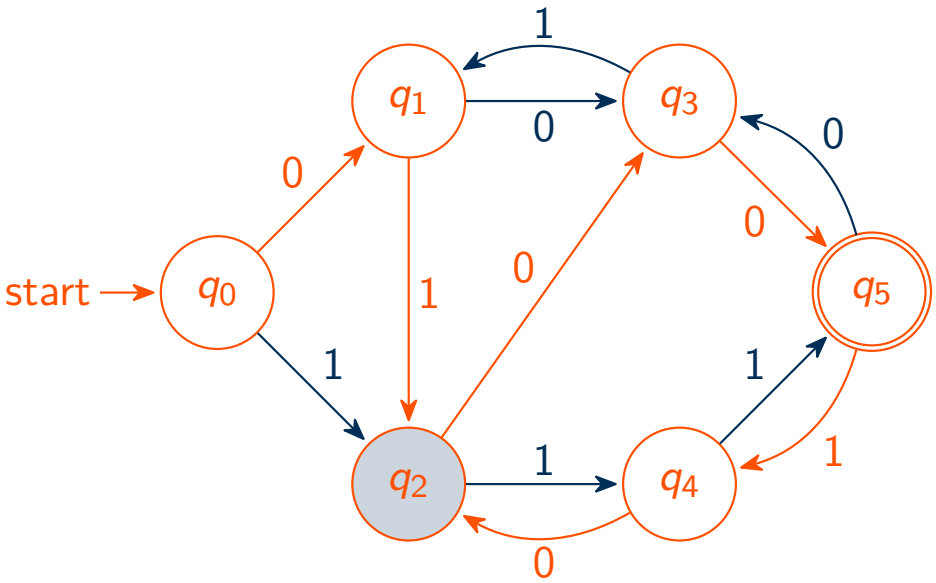
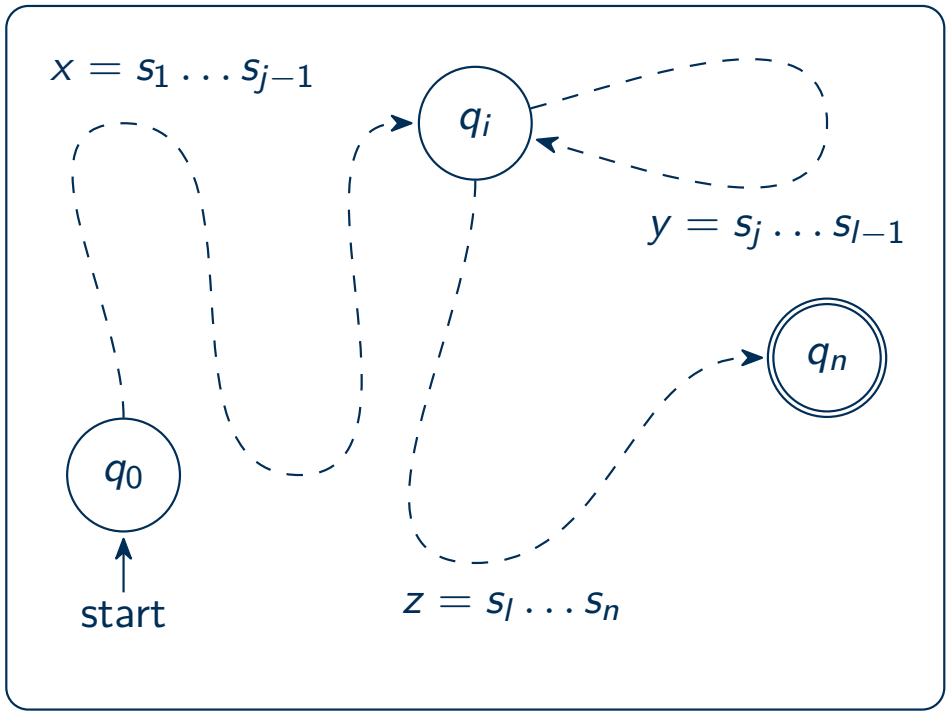


Therefore,

$$x = 01, \ y = 0010, \ \&, \ z = 01111$$

General Idea



- $p = |Q|$

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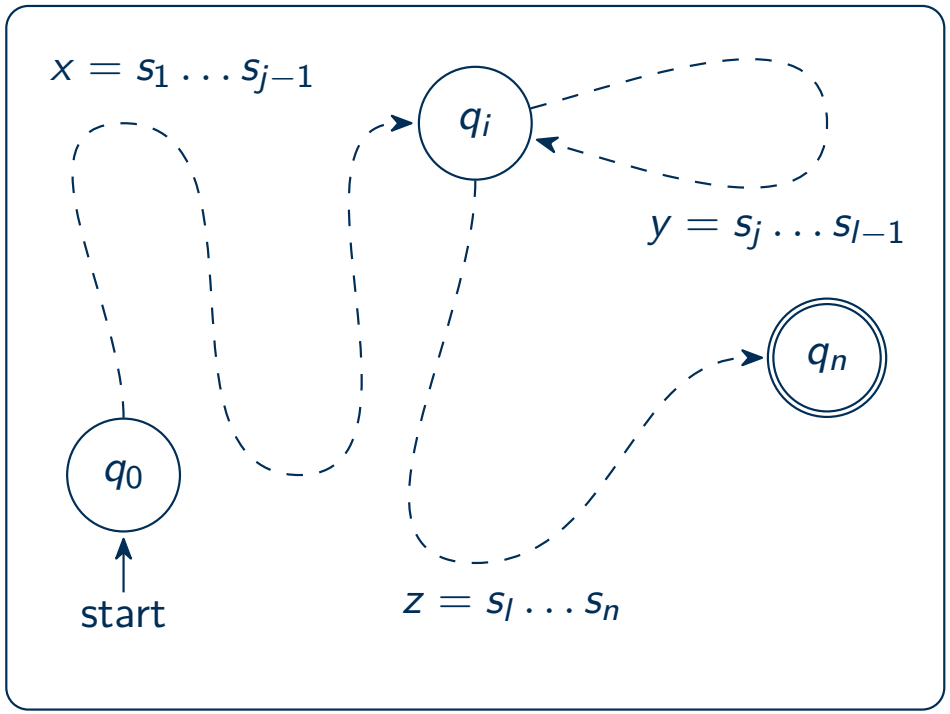

- $p = |Q|$
- $s = s_1 s_2 s_3 \cdots s_n \in A, \ n > p$

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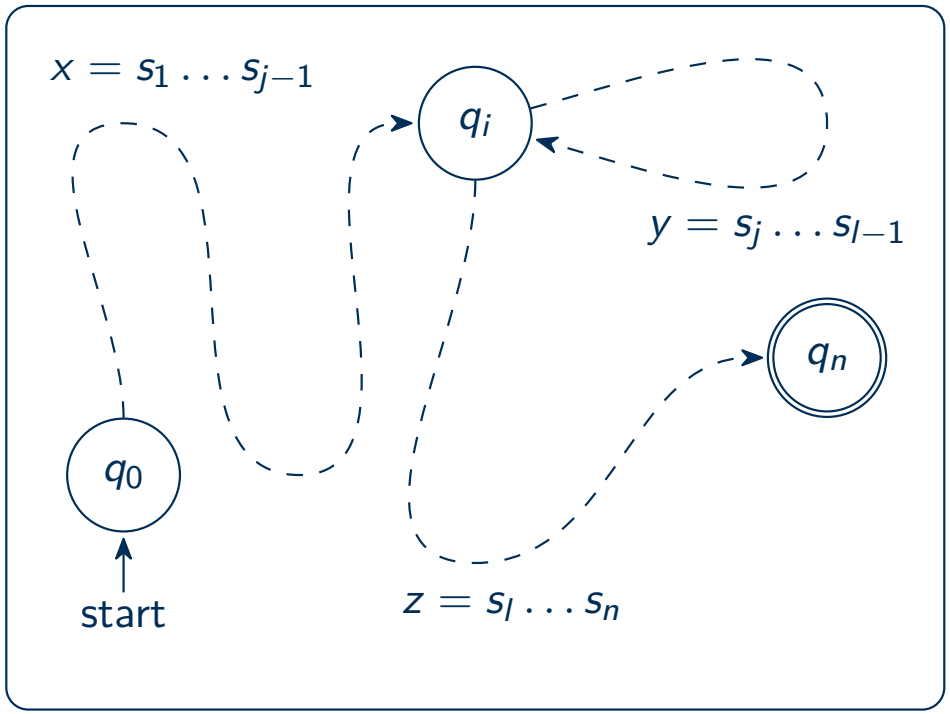

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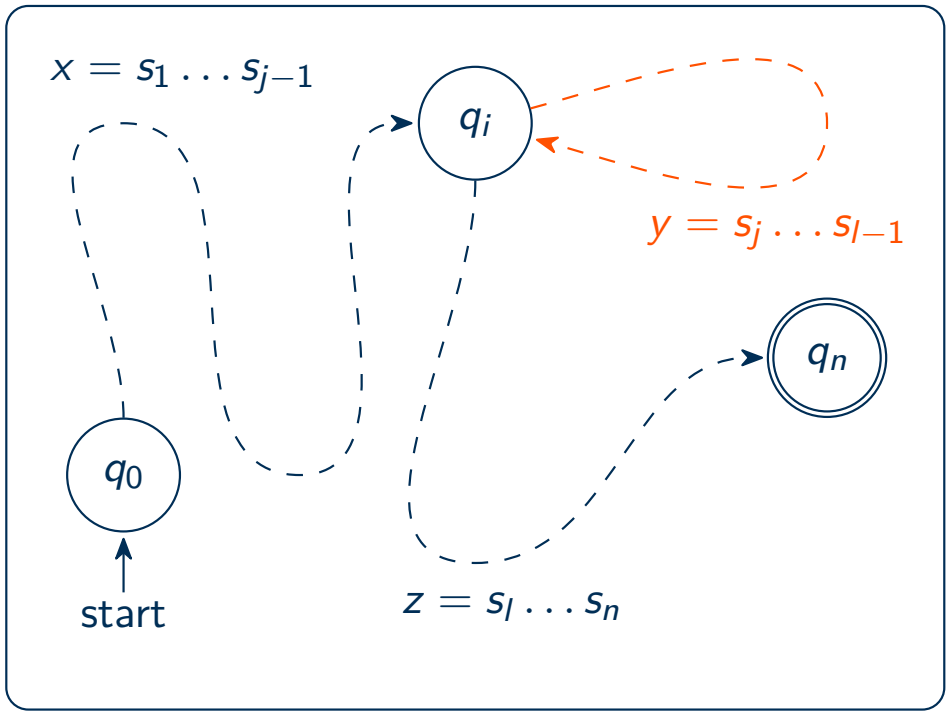

- $p = |Q|$
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- Pigeon holes imply a loop
- Focus on the first loop

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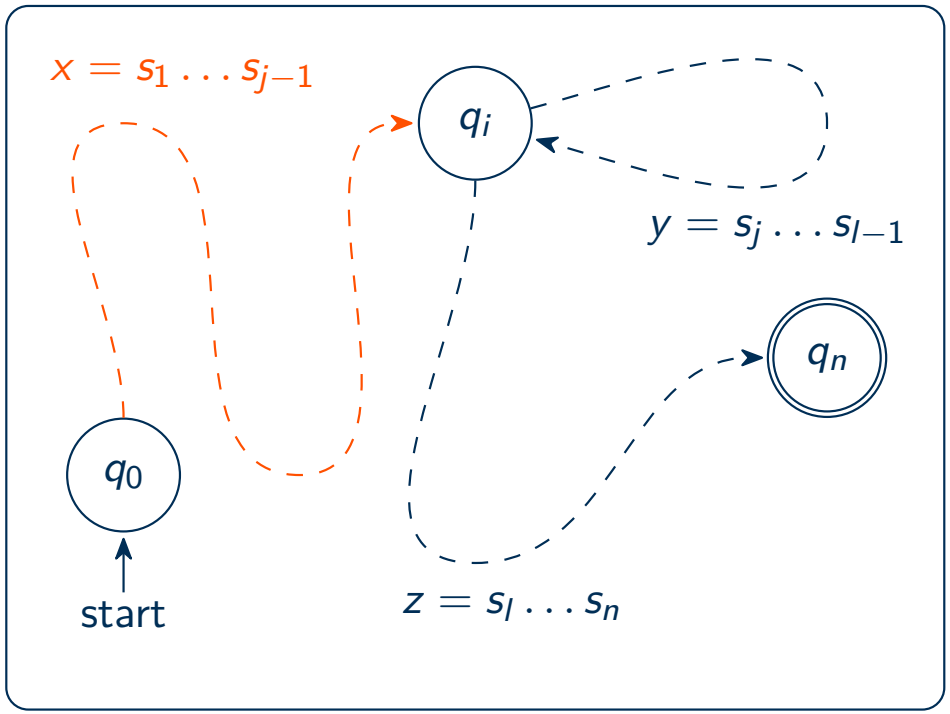

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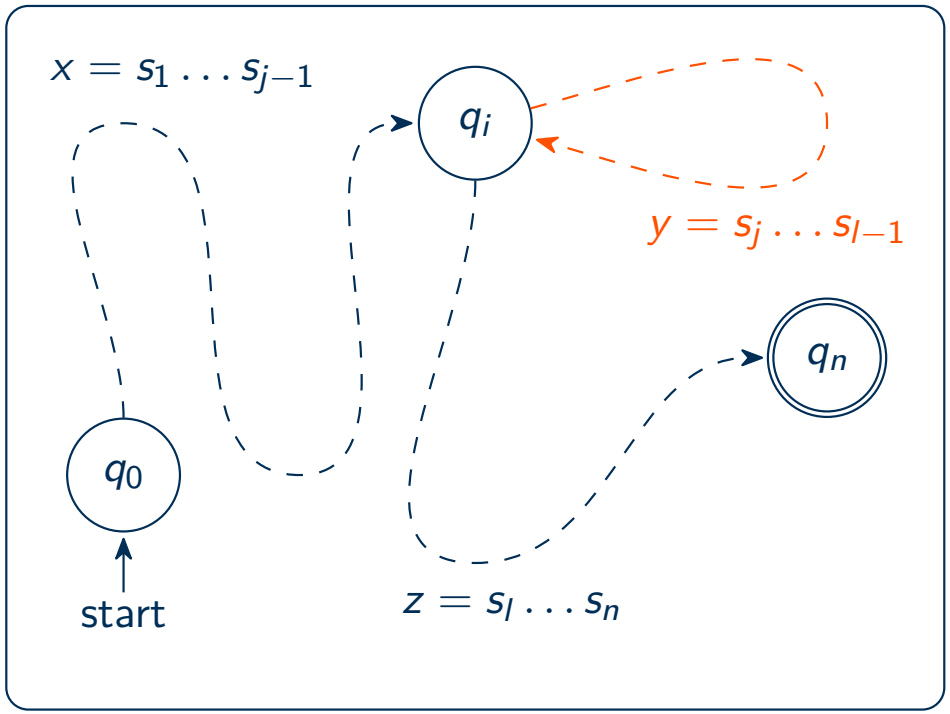

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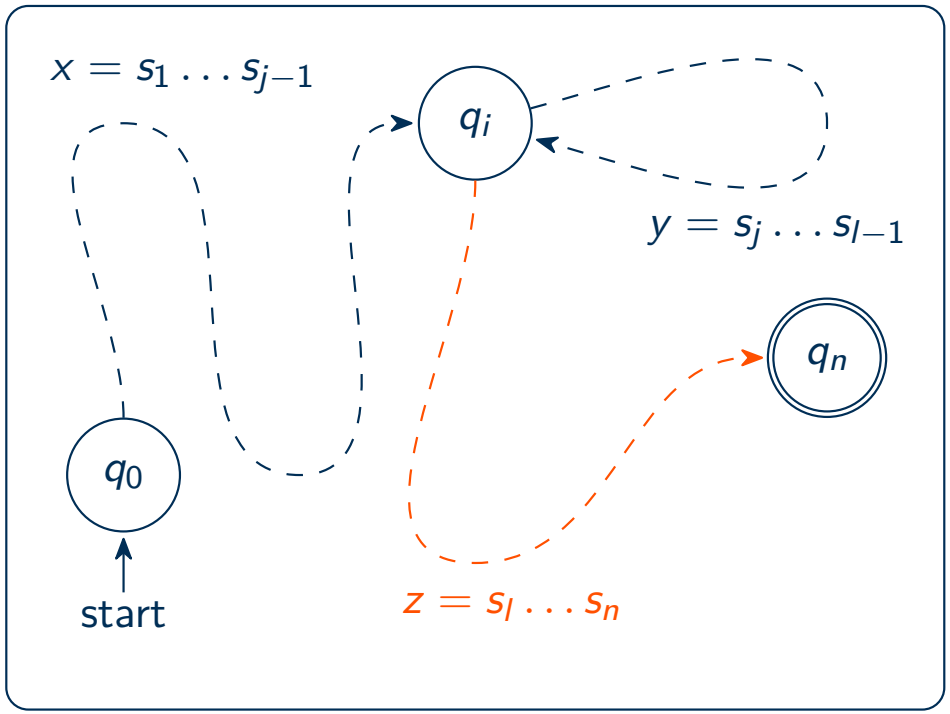

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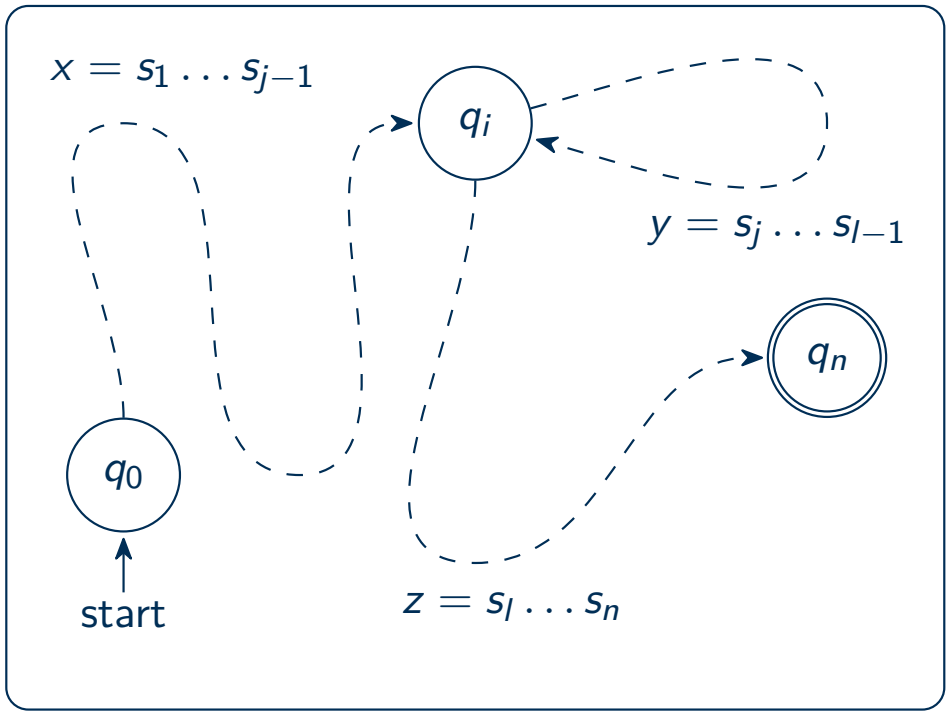
General Idea



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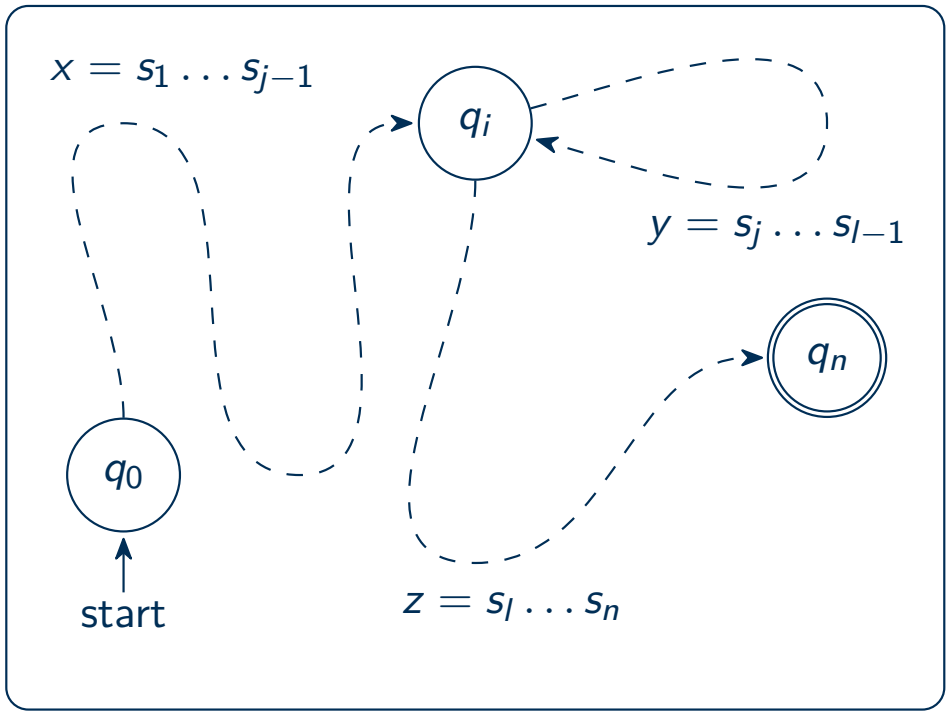
- Pigeon holes imply a loop
- Focus on the first loop
- $x = s_1 \dots s_{j-1}$
- $y = s_j \dots s_{l-1}$
- $z = s_l \dots s_n$
- $j \neq l$ implies $|y| > 0$

General Idea



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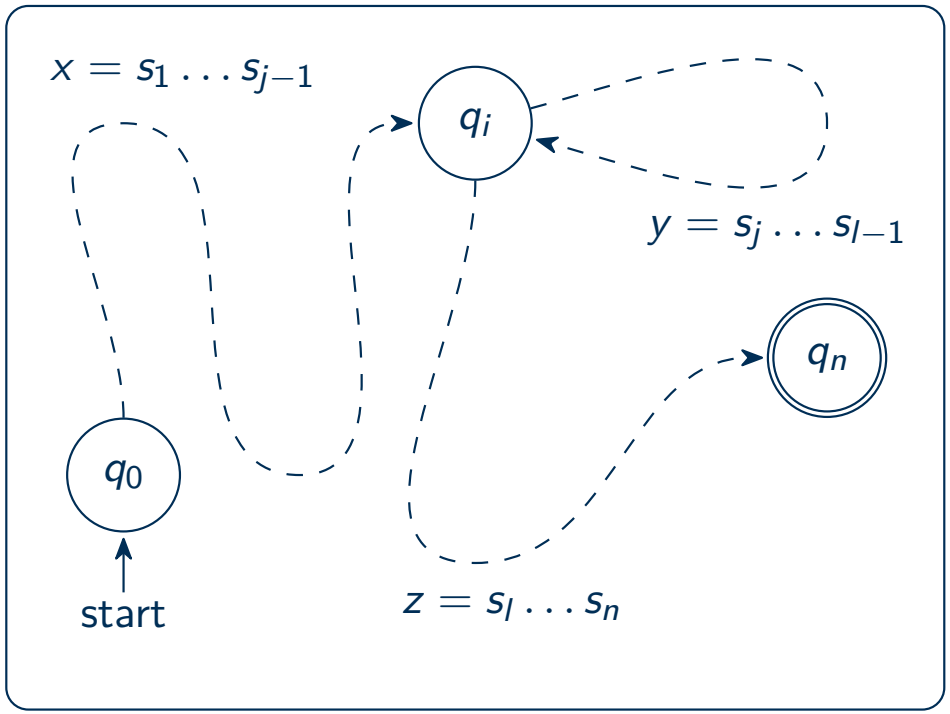
- $j \neq l$ implies $|y| > 0$
- y loops so $xy^i z \in A$

General Idea



- $p = |Q|$
- $s = s_1 s_2 s_3 \cdots s_n \in A, \ n > p$
- Pigeon holes imply a loop
- Focus on the first loop
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- $y = s_j \dots s_{l-1}$
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- $j \neq l$ implies $|y| > 0$
- y loops so $xy^i z \in A$
- y is the first loop so $|xy| \leq p$

Statement

Theorem (Pumping Lemma)

If A is a regular language, then there is a number p (the pumping length) such that, if s is any string in A of length at least p, then s may be written in three pieces, $s = xyz$, satisfying the following conditions:

1. *for each $i \geq 0$, $xy^iz \in A$,*
2. *$|y| > 0$, and*
3. *$|xy| \leq p$.*

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Regular Expressions and the Pumping Lemma

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