CHAPTER **1**

Sets and functions

Everything starts somewhere, although many physicists disagree.
Terry Pratchett, *Hogfather*, 1996

To think like a mathematician requires some mathematics to think about. I wish to keep the number of prerequisites for this book low so that any gaps in your knowledge are not a drag on understanding. Just so that we have some mathematics to play with, this chapter introduces sets and functions. These are very basic mathematical objects but have sufficient abstraction for our purposes.

A set is a collection of objects, and a function is an association of members of one set to members of another. Most high-level mathematics is about sets and functions between them. For example, calculus is the study of functions from the set of real numbers to the set of real numbers that have the property that we can differentiate them. In effect, we can view sets and functions as the mathematician's building blocks.

While you read and study this chapter, think about *how* you are studying. Do you read every word? Which exercises do you do? Do you, in fact, do the exercises? We shall discuss this further in the next chapter on reading mathematics.

Sets

The set is the fundamental object in mathematics. Mathematicians take a set and do wonderful things with it.

Definition 1.1

A ***set*** *is a well-defined collection of objects.*[1]

The objects in the set are called the ***elements*** *or* ***members*** *of the set.*

We usually define a particular set by making a list of its elements between brackets. (We don't care about the ordering of the list.)

[1] The proper mathematical definition of set is much more complicated; see almost any text book on set theory. This definition is intuitive and will not lead us into many problems. Of course, a pedant would ask what does well-defined mean?

If x is a member of the set X, then we write $x \in X$. We read this as 'x is an element (or member) of X' or 'x is in X'.[2] If x is not a member, then we write $x \notin X$.

Examples 1.2

(i) The set containing the numbers 1, 2, 3, 4 and 5 is written $\{1, 2, 3, 4, 5\}$. The number 3 is an element of the set, i.e. $3 \in \{1, 2, 3, 4, 5\}$, but $6 \notin \{1, 2, 3, 4, 5\}$. Note that we could have written the set as $\{3, 2, 5, 4, 1\}$ as the order of the elements is unimportant.

(ii) The set {dog, cat, mouse} is a set with three elements: dog, cat and mouse.

(iii) The set {1, 5, 12, {dog, cat}, {5, 72}} is the set containing the numbers 1, 5, 12 and the sets {dog, cat} and {5, 72}. Note that sets can contain sets as members. Realizing this now can avoid a lot of confusion later.

It is vitally important to note that $\{5\}$ and 5 are not the same. That is, we must distinguish between being a set and being an element of a set. Confusion is possible since in the last example we have $\{5, 72\}$, which is a set in its own right but can also be thought of as an element of a set, i.e. $\{5, 72\} \in \{1, 5, 12, \{\text{dog}, \text{cat}\}, \{5, 72\}\}$.

Let's have another example of a set created using sets.

Example 1.3

The set $X = \{1, 2, \text{dog}, \{3, 4\}, \text{mouse}\}$ has five elements. It has the the four elements, 1, 2, dog, mouse; and the other element is the set $\{3, 4\}$. We can write $1 \in X$, and $\{3, 4\} \in X$. It is vitally important to note that $3 \notin X$ and $4 \notin X$, i.e. the numbers 3 and 4 are not members of X, the set $\{3, 4\}$ is.

Some interesting sets of numbers

Let's look at different types of numbers that we can have in our sets.

Natural numbers

The set of **natural numbers** is $\{1, 2, 3, 4, \ldots\}$ and is denoted by $\mathbb{N}$. The dots mean that we go on forever and can be read as 'and so on'.

Some mathematicians, particularly logicians, like to include 0 as a natural number. Others say that the natural numbers are the counting numbers and you don't start counting with zero (unless you are a computer programmer). Furthermore, how natural is a number that was not invented until recently?

On the other hand, some theorems have a better statement if we take $0 \in \mathbb{N}$. One can get round the argument by specifying that we are dealing with non-negative integers or positive integers, which we now define.

[2] Of course, to distinguish the x and X we read it out loud as 'little x is an element of capital X.'

Integers

The set of **integers** is $\{\ldots, -4, -3, -2, 0, 1, 2, 3, 4, \ldots\}$ and is denoted by $\mathbb{Z}$. The $\mathbb{Z}$ symbol comes from the German word Zahlen, which means number. From this set it is easy to define the **non-negative integers**, $\{0, 1, 2, 3, 4, \ldots\}$, often denoted $\mathbb{Z}^+$. Note that all natural numbers are integers.

Rational numbers

The set of **rational numbers** is denoted by $\mathbb{Q}$ and consists of all fractional numbers, i.e. $x \in \mathbb{Q}$ if x can be written in the form p/q where p and q are integers with $q \neq 0$. For example, $1/2$, $6/1$ and $80/5$. Note that the representation is not unique since, for example, $80/5 = 16/1$. Note also that all integers are rational numbers since we can write $x \in \mathbb{Z}$ as $x/1$.

Real numbers

The **real numbers**, denoted $\mathbb{R}$, are hard to define rigorously. For the moment let us take them to be any number that can be given a decimal representation (including infinitely long representations) or as being represented as a point on an infinitely long number line.

The real numbers include all rational numbers (hence integers, hence natural numbers). Also real are π and e, neither of which is a rational number.[3] The number $\sqrt{2}$ is not rational as we shall see in Chapter 23.

The set of real numbers that are not rational are called **irrational numbers**.

Complex numbers

We can go further and introduce **complex numbers**, denoted $\mathbb{C}$, by pretending that the square root of -1 exists. This is one of the most powerful additions to the mathematician's toolbox as complex numbers can be used in pure and applied mathematics. However, we shall not use them in this book.

More on sets

The empty set

The most fundamental set in mathematics is perhaps the oddest – it is the set with no elements!

[3] The proof of these assertions are beyond the scope of this book. For π see Ian Stewart, *Galois Theory*, 2nd edition, Chapman and Hall 1989, p. 62 and for e see Walter Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill 1976, p. 65.

Definition 1.4

The set with no elements is called the ***empty set*** *and is denoted* $\emptyset$*.*

It may appear to be a strange object to define. The set has no elements so what use can it be? Rather surprisingly this set allows us to build up ideas about counting. We don't have time to explain fully here but this set is vital for the foundations of mathematics. If you are interested, see a high level book on set theory or logic.

Example 1.5

The set $\{\emptyset\}$ is the set that contains the empty set. This set has one element. Note that we can then write $\emptyset \in \{\emptyset\}$, but we *cannot* write $\emptyset \in \emptyset$ as the empty set has, by definition, no elements.

Definition 1.6

Two sets are ***equal*** *if they have the same elements. If set* X *equals set* Y*, then we write* $X = Y$*. If not we write* $X \neq Y$*.*

Examples 1.7

(i) The sets $\{5, 7, 15\}$ and $\{7, 15, 5\}$ are equal, i.e. $\{5, 7, 15\} = \{7, 15, 5\}$.
(ii) The sets $\{1, 2, 3\}$ and $\{2, 3\}$ are not equal, i.e. $\{1, 2, 3\} \neq \{2, 3\}$.
(iii) The sets $\{2, 3\}$ and $\{\{2\}, 3\}$ are not equal.
(iv) The sets $\mathbb{R}$ and $\mathbb{N}$ are not equal.

Note that, as used in the above, if we have a symbol such as $=$ or $\in$, then we can take the opposite by drawing a line through it, such as $\neq$ and $\notin$.

Definition 1.8

If the set X *has a finite number of elements, then we say that* X *is a* ***finite set****. If* X *is finite, then the number of elements is called the* ***cardinality*** *of* X *and is denoted* $|X|$*.*

If X has an infinite number of elements, then it becomes difficult to define the cardinality of X. We shall see why in Chapter 30. Essentially it is because there are different sizes of infinity! For the moment we shall just say that the cardinality is undefined for infinite sets.

Examples 1.9

(i) The set $\{\emptyset, 3, 4, \text{cat}\}$ has cardinality 4.
(ii) The set $\{\emptyset, 3, \{4, \text{cat}\}\}$ has cardinality 3.

Exercises 1.10

What is the cardinality of the following sets?

(i) $\{1, 2, 5, 4, 6\}$
(ii) $\{\pi, 6, \{\pi, 5, 8, 10\}\}$
(iii) $\{\pi, 6, \{\pi, 5, 8, 10\}, \{\text{dog}, \text{cat}, \{5\}\}\}$
(iv) $\emptyset$
(v) $\mathbb{N}$
(vi) $\{\text{dog}, \emptyset\}$
(vii) $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$
(viii) $\{\emptyset, \{20, \pi, \{\emptyset\}\}, 14\}$

Now we come to another crucial definition, that of being a subset.

Definition 1.11

Suppose X is a set. A set Y is a ***subset*** *of X if every element of Y is an element of X. We write $Y \subseteq X$.*

This is the same as saying that, if $x \in Y$, then $x \in X$.

Examples 1.12

(i) The set $Y = \{1, \{3, 4\}, \text{mouse}\}$ is a subset of $X = \{1, 2, \text{dog}, \{3, 4\}, \text{mouse}\}$.
(ii) The set of even numbers is a subset of $\mathbb{N}$.
(iii) The set $\{1, 2, 3\}$ is not a subset of $\{2, 3, 4\}$ or $\{2, 3\}$.
(iv) For any set X, we have $X \subseteq X$.
(v) For any set X, we have $\emptyset \subset X$.

Remark 1.13

It is vitally important that you distinguish between being an *element* of a set and being a *subset* of a set. These are often confused by students. If $x \in X$, then $\{x\} \subseteq X$. Note the brackets. Usually, and I stress usually, if $x \in X$, then $\{x\} \notin X$, but sometimes $\{x\} \in X$, as the following special example shows.

Example 1.14

Consider the set $X = \{x, \{x\}\}$. Then $x \in X$ and $\{x\} \subseteq X$ (the latter since $x \in X$) but we also have $\{x\} \in X$.

Therefore we *cannot* state any simple rule such as 'if $a \in A$, then it would be wrong to write $a \subset A$', and vice versa.

If you felt a bit confused by that last example, then go back and think about it some more, until you really understand it. This type of precision and the nasty examples that go against intuition, and prevent us from using simple rules, are an important aspect of high-level mathematics.

Definition 1.15

A subset Y of X is called a ***proper subset*** *of X if Y is not equal to X. We denote this by $Y \subset X$. Some people use $Y \subsetneq X$ for this.*

Examples 1.16

(i) $\{1, 2, 5\}$ is a proper subset of $\{-6, 0, 1, 2, 3, 5\}$.
(ii) For any set X, the subset X is not a proper subset of X.
(iii) For any set $X \neq \emptyset$, the empty set $\emptyset$ is a proper subset of X. Note that, if $X = \emptyset$, then the empty set $\emptyset$ is *not* a proper subset of X.
(iv) For numbers, we have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Note that we can use the symbols $\not\subseteq$ to denote 'not a subset of' and $\not\subset$ to denote 'not a proper subset of'.

Now let's consider where the notation came from. It is is obvious that for a finite set the two statements

$$\text{If } X \subset Y, \text{ then } |X| < |Y|,$$

and

$$\text{If } X \subset Y, \text{ then } |X| < |Y|$$

are true. So $\subseteq$ is similar to $\leq$ and $\subset$ is similar to $<$ as concepts and not just as symbols.

An important remark to make here is that not all mathematicians distinguish between $\subseteq$ and $\subset$; some use only $\subset$ and use it to mean 'subset of'. However, I feel the use of $\subseteq$ is far better as it allows us to distinguish between a subset and a proper subset. Imagine what the two statements above would look like if we didn't. They wouldn't be so clear and one wouldn't be true! Or, to see what I mean, imagine what would happen if mathematicians always used $<$ instead of $\leq$.

Defining sets

We can define sets using a different notation: $\{x \mid x \text{ satisfies property } P\}$. The symbol '|' is read as 'such that'. Sometimes the colon ':' is used in place of '|'.

Examples 1.17

(i) The set $\{x \mid x \in \mathbb{N} \text{ and } x < 5\}$ is equal to $\{1, 2, 3, 4\}$. We read the set as 'x such that x is in $\mathbb{N}$ and x is less than 5'.

(ii) The set $\{x \mid 5 \leq x \leq 10\}$ is the set of numbers between 5 and 10. Here we follow the convention that we assume that x is a real number. This is a bad convention as it allows writers to be sloppy, so we should try to avoid using it. Hence, we can also specify some restriction on the x before the | sign, as in the next example.

(iii) The set $\{x \in \mathbb{N} \mid 5 \leq x \leq 10\}$ is the set of natural numbers from 5 to 10 inclusive. That is, the set $\{5, 6, 7, 8, 9, 10\}$.

(iv) It is common to use the notation $[a, b]$ for the set $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ and (a, b) for the set $\{x \in \mathbb{R} \mid a < x < b\}$.

Note that (a, b) can also mean the pair of numbers a and b.

We can also describe sets in the following way $\{x^2 \mid x \in \mathbb{N}\}$ is the set of numbers $\{1, 4, 9, 16, \ldots\}$. There are many possibilities for describing sets so we will not detail them all as it will usually be obvious what is intended.

Operations on sets

In mathematics we often make a definition of some object, for example a set, and then we find ways of creating new ones from old ones, for example we take subsets of sets. We now come to two ways of creating new from old: the union and intersection of sets.

Definition 1.18

Suppose that X and Y are two sets. The ***union*** *of X and Y, denoted $X \cup Y$, is the set consisting of elements that are in X or in Y or in both. We can define the set as $X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$.*

Examples 1.19

(i) The union of $\{1, 2, 3, 4\}$ and $\{2, 4, 6, 8\}$ is $\{1, 2, 3, 4, 6, 8\}$.
(ii) The union of $\{x \in \mathbb{R} \,|\, x < 5\}$ and $\{x \in \mathbb{Z} \,|\, x < 8\}$ is $\{x \in \mathbb{R} \,|\, x \leq 5,\ \text{or } x = 6 \text{ or } x = 7\}$.

Exercises 1.20

(i) Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{-1, 1, 3, 5, 7\}$. Find $X \cup Y$.
(ii) What is $\mathbb{Z} \cup \mathbb{Z}$?

Definition 1.21

Suppose that X and Y are two sets. The ***intersection*** *of X and Y, denoted $X \cap Y$, is the set consisting of elements that are in X and in Y. We can define the set as $X \cap Y = \{x \,|\, x \in X \text{ and } x \in Y\}$.*

Examples 1.22

(i) The intersection of $\{1, 2, 3, 4\}$ and $\{2, 4, 6, 8\}$ is $\{2, 4\}$.
(ii) The intersection of $\{-1, -2, -3, -4, -5\}$ and $\mathbb{N}$ is $\emptyset$.

Exercises 1.23

(i) Find $X \cap Y$ for the following:
  (a) $X = \{x \in \mathbb{R} \,|\, 0 \leq x < 6\}$ and $Y = \{x \in \mathbb{Z} \,|\, -\pi \leq x \leq 7\}$,
  (b) $X = \{0, 2, 4, 6, 8\}$ and $Y = \{1, 3, 5, 7, 9\}$,
  (c) $X = \mathbb{Q}$ and $Y = \{0, 1, \pi, 5\}$.
(ii) Find $\mathbb{Z} \cap \mathbb{Z}$, $\mathbb{Z} \cap \emptyset$, and $\mathbb{Z} \cap \mathbb{R}$.

We will use these definitions in later chapters to give examples of proofs, for example to show statements such as $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ are true.

Exercise 1.24

Find the union and intersection of $\{x \in \mathbb{R} \,|\, x > 7\}$ and $\{x \in \mathbb{N} \,|\, x > 5\}$.

Definition 1.25

The ***difference*** *of X and Y, denoted $X \backslash Y$, is the set of elements that are in X but not in Y. That is, we take elements of X and discard those that are also in Y. We do not require that Y is a subset of X. If Y is defined as a subset of X, then we often call $X \backslash Y$ the* ***complement*** *of Y in X and denote this by Y^{c}.*

Examples 1.26

(i) Let $X = \{1, 2, 3, \text{dog}, \text{cat}\}$ and let $Y = \{3, \text{cat}, \text{mouse}\}$. Then $X \backslash Y = \{1, 2, \text{dog}\}$.
(ii) Let $X = \mathbb{R}$ and $Y = \mathbb{Z}$, then

$$X \backslash Y = \cdots \cup (-3, -2) \cup (-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2) \cup \cdots$$

Products of sets

Here's another example of mathematicians creating new objects from old ones.

Definition 1.27

Let X and Y be two sets. The ***product*** *of X and Y, denoted $X \times Y$ is the set of all possible pairs (x, y) where $x \in X$ and $y \in Y$, i.e.*

$$X \times Y = \{(x, y) \mid x \in Y \text{ and } y \in Y\}.$$

Note that here (x, y) denotes a pair and has nothing to do with Example 1.17(iv).

Examples 1.28

(i) Let $X = \{0, 1\}$ and $Y = \{1, 2, 3\}$. Then $X \times Y$ has six elements:

$$X \times Y = \{(0, 1),\ (0, 2),\ (0, 3),\ (1, 1),\ (1, 2),\ (1, 3)\}.$$

(ii) The set $\mathbb{R} \times \mathbb{R}$ is denoted $\mathbb{R}^2$. The set $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ is denoted $\mathbb{R}^3$. This is because its elements can be given by triples of real numbers, i.e. its elements are of the form (x, y, z) where x, y and z are real numbers.

Note that $X \times Y$ is not a subset of either X or Y.

Maps and functions

We have defined sets. Now we make a definition for relating elements of sets to elements of other sets.

Definition 1.29

Suppose that X and Y are sets. A ***function*** *or* ***map*** *from X to Y is an association between the members of the sets. More precisely, for every element of X there is a unique element of Y.*

If f is a function from X to Y, then we write $f : X \to Y$, and the unique element in Y associated to x is denoted $f(x)$. This element is called the ***value of*** *x* ***under*** *f or called a* ***value*** *of f. The set X is called the* ***source*** *(or* ***domain****) of f and Y is called the* ***target*** *(or* ***codomain****) of f.*

To describe a function f we usually use a formula to define $f(x)$ for every x and talk about applying f to elements of a set, or to a set.

A schematic picture is shown in Figure 1.1. Note that every element of X has to be associated to one in Y but not vice versa and that two distinct elements of X may map to the same one in Y.

Examples 1.30

(i) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$ for all $x \in \mathbb{Z}$. Then the value of x under f is the square of x. Note that there are elements in the target which are not values of f. For example -1 is not a value since there is no integer x such that $x^2 = -1$.

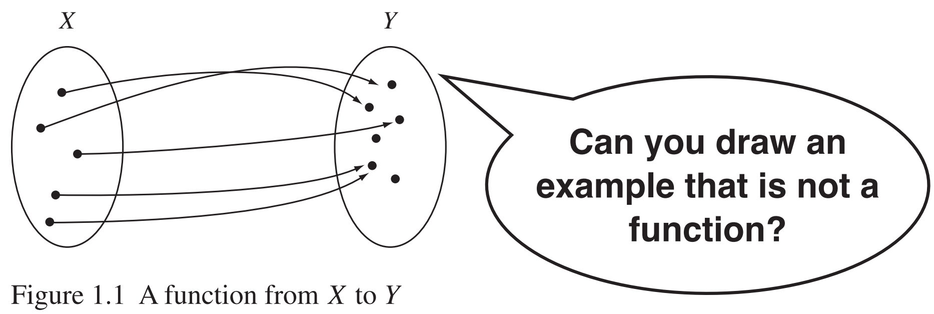


Figure 1.1 A function from X to Y

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 0$. Then the only value of f is 0.
(iii) The cardinality of a set is a function on the set of finite sets. That is $|\,|$: Finite Sets $\to \{0\} \cup \mathbb{N}$. Note that we need 0 in the codomain as the set could be the empty set.
(iv) The **identity map** on X is the map $\mathrm{id} : X \to X$ given by $\mathrm{id}(x) = x$ for all $x \in X$.

Having a formula does not necessarily define a function, as the next example shows.

Example 1.31

The formula $f(x) = 1/(x-1)$ does not define a function from $\mathbb{R}$ to $\mathbb{R}$ as it is not defined for $x = 1$.

We can rescue this example by restricting the source to $\mathbb{R}$ without the element 1. That is, define $X = \{x \in \mathbb{R} \mid x \neq 1\}$, then $f : X \to \mathbb{R}$ defined by $f(x) = 1/(x-1)$ is a function.

Polynomials provide a good source of examples of functions.

Examples 1.32

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 2x + 3$. Notice again that, although the target is all of $\mathbb{R}$, not every element of the target is a value of f. For example there is no x such that $f(x) = -2$. This is something you can check by attempting to solve $x^2 + 2x + 3 = -2$.
(ii) More generally, from a polynomial we can define a function $f : \mathbb{R} \to \mathbb{R}$ by defining

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some real numbers $a_0, \ldots, a_n$ and a real variable x.
(iii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ can be differentiated, for example a polynomial. Then the derivative, denoted f', is a function.

Exercises 1.33

(i) Find the largest domain that makes $f(x) = x/(x^2 - 5x + 3)$ a function.
(ii) Find the largest domain that makes $f(x) = (x^3 + 2)/(x^2 + x + 2)$ a function.
(iii) Construct an example of a polynomial so that its graph goes through the points $(-1, 5)$ and $(3, -2)$.

Exercises

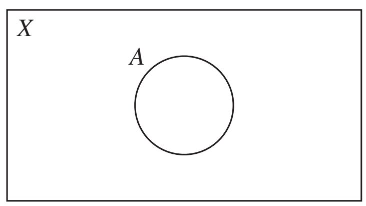
Exercises 1.34

(i) Let $X = \{x \in \mathbb{Z} \mid 0 \leq x \leq 10\}$ and A and B be subsets such that $A = \{0, 2, 4, 6, 8, 10\}$ and $B = \{2, 3, 5, 7\}$. Find $A \cap B$, $A \cup B$, $A \backslash B$, $B \backslash A$, $A \times B$, $X \times A$, A^c, and B^c.

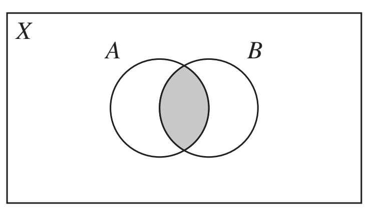
(ii) Find the union and intersection of $\{x \in \mathbb{R} \mid x^2 - 9x + 14 = 0\}$ and $\{y \in \mathbb{Z} \mid 3 \leq y < 10\}$.

(iii) Suppose that A, B and C are subsets of X. Use examples of these sets to investigate the following:
  (a) $(A \cap B) \cup (A \cap C)$ and $A \cap (B \cup C)$,
  (b) $(A \cup B) \cap (A \cup C)$ and $A \cup (B \cap C)$,
  (c) $(A \cup B)^c$ and $A^c \cap B^c$,
  (d) $(A \cup B)^c$ and $A^c \cup B^c$,
  (e) $(A \cap B)^c$ and $A^c \cup B^c$,
  (f) $(A \cap B)^c$ and $A^c \cap B^c$.
  Do you notice anything?

(iv) A **Venn diagram** is useful way of representing sets. If A is a subset of X, then we can draw the following in the plane:



In fact, the precise shape of A is unimportant but we often use a circle. If B is another subset, then we can draw B in the diagram as well. In the following we have shaded the intersection $A \cap B$.



  (a) Draw a Venn diagram for the case that A and B have no intersection.
  (b) Draw Venn diagrams and shade the sets $A \cup B$, A^c, and $(A \cap B)^c$.
  (c) Draw three (intersecting) circles to represent the sets A, B and C. Shade in the intersection $A \cap B \cap C$.
  (d) Using exercise (iii) construct Venn diagrams and shade in the relevant sets.

(v) Analyse how you approached the reading of this chapter.
  (a) If you had not met the material in this chapter before, then did you attempt to understand everything?
  (b) If you had met the material before, did you check to see that I had not made any mistakes?

Summary

- A set is a well-defined collection of objects.
- The empty set has no elements.
- The cardinality of a finite set is the number of elements in the set.
- The set Y is a subset of X if every element of Y is in X.
- A subset Y of X is a proper subset if it is not equal to X.
- The union of X and Y is the collection of elements that are in X or in Y.
- The intersection of X and Y is the collection of elements that are in X and in Y.
- The product of X and Y is the set of all pairs (x, y) where $x \in X$ and $y \in Y$.
- A function assigns elements of one set to another.

For each of these can you find examples? Or, even better, non-examples?