Probability Notes

Definitions:

- **Sample Space:** A *sample space* is a set or collection of possible outcomes.
  - Flipping a Coin: {Head, Tail}
  - Rolling Two Die: {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
- **Outcome:** An *outcome* is an element of a sample space.
  - A Head
  - A roll of 2
- **Event:** An *event* is any subset of a sample space.
  - Flipping a Head
  - Rolling and even number, {2, 4, 6, 8, 10, 12}
- **Compliment:** The *compliment* of an event is the set of all outcomes from the sample space not in the event.
  - The compliment of getting a Head is getting a Tail.
  - The compliment of getting an even roll is getting an odd roll.
- **Probability:** For each outcome in a sample space there is a *probability*, which is the likelihood of choosing that outcome at random from the sample space.
  - The probability of a Head is $\frac{1}{2}$.
  - The probability of rolling a 2 is $\frac{1}{36}$.

  The *probability* of an event is the likelihood of choosing an outcome from that event.
  - The probability of rolling a prime number is

$$\frac{1}{36} + \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{2}{36} = \frac{15}{36}$$

- **Probability Space:** A *probability space* is a sample space together with a set of probabilities.
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C	H	T
Prob	$\frac{1}{2}$	$\frac{1}{2}$
  -
R	2	3	4	5	6	7	8	9	10	11	12
Prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
  -
X	2	3	4	5	6	7	8	9	10	11	12
Prob	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$
- **And:** If A and B are events in a sample space, the probability of A and B is the sum of probabilities of all the outcomes in both A and B.

$$P(A \text{ and } B) = \sum_{s_i \in A \text{ and } B} P(s_i)$$

  - The probability that a roll is even and prime: probability of a 2.
  - The probability that a roll is odd and greater than 5: probability of a 7, 9, or 11.
- **Or:** If A and B are events in a sample space, the probability of A or B is the sum of the probabilities of all the outcomes in either A or B.

$$P(A \text{ or } B) = \sum_{s_i \in A \text{ or } B} P(s_i)$$

  - The probability that a roll is odd or less than 7: probability of a 2, 3, 4, 5, 6, 7, 9, or 11.

- The probability that a roll is divisible by 4 or 3: probability of 4, 8, 12, 3, 6, or 9.

- **Disjoint:** Two events are disjoint if they contain none of the same outcomes.
  - The event of rolling an even and the event of rolling an odd are disjoint.

- **Independent:** Two events are independent if knowing the outcome of one does not effect the probability of the other.
  - The event that your first flip is a head and the event that your second flip is a tail are independent.
  - The event that one die is a 1 and the event that the other die is a 4 are independent.

- **Conditional Probabilities:** If A and B are events in a sample space, the *conditional probability* of B given A is the probability that B will occur if we know the outcome of A.
  - The probability that the second card dealt is red given that the first card dealt is black.

$$P(C_2 = Red | C_1 = Black) = \frac{26}{51}$$

  - The probability that a roll is prime given that the roll is odd.

$$P(R = prime | R = odd) = \frac{14}{18}$$

- **True Odds:** If A is an event in a sample space with probability $P(A)$, then the *true odds* of A are written $X : Y$ against (read "X to Y against"), where

$$P(A) = \frac{Y}{(X+Y)} \text{ and } (1 - P(A)) = \frac{X}{(X+Y)}.$$

  - The proability of rolling a 7 or an 11 on two dice is 2/9 so the *true odds* of winning on the first roll in craps is 7:2 against.
  - The probability of being dealt two cards totaling 9 in Baccarat is 256/2652, so the odds of winning on a deal of 9 is 2396:256 against.
  - The odds of successfully navigating an asteroid field are 3720 : 1 against, so the probability that you could do so is 1/3721 and the probability you couldn't is 3720/3721.

- **House Odds:** The *house odds* of an event are $X : Y$ if a bet of $\$Y$ pays a return of $\$X$ if you win.
  - The *house odds* of winning on a single number bet in roulette are 35 : 1, so on a one dollar bet you win \$35, plus you get your one dollar back.

- **House Advantage (Expected Value):** If the house odds of an event A are $X : Y$ and the probability of A is $P(A)$ then the *house advantage* or *expected value* is

$$X \cdot P(A) + (-Y) \cdot (1 - P(A)).$$

  - The house odds of winning on a single number bet in roulette are 35 : 1 and the probability of winning that bet is 1/38, so the *house advantage* is

$$35 \cdot \frac{1}{38} + (-1) \cdot \frac{37}{38} = -\frac{2}{38} \approx -0.0526 \text{ or } -5.26\%.$$

  This means that on average, over time, the house wins 5¢ for every dollar bet, 52¢ for a ten dollar bet, or \$5.26 for a hundred dollar bet.

Rules:

Let S be a sample space, s_i be outcomes in S, A and B be events in S, and A^c and B^c be the compliments of A and B in S. The notation $P(B|A)$ is read,“The probability of B given A.”

1. $0 \leq P(s_i) \leq 1$
2. $P(S) = \sum_i P(s_i) = 1$
3. $P(A) = \sum_{s_i \in A} P(s_i)$
4. $P(A) + P(A^c) = 1$
5. $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ (General Addition Rule)
6. If A and B are disjoint, then $P(A \text{ or } B) = P(A) + P(B)$ and $P(A \text{ and } B) = 0$.
7. $P(A \text{ and } B) = P(A)P(B|A)$ (General Multiplication Rule)
8. If A and B are independent, then $P(B|A) = P(B)$ and $P(A \text{ and } B) = P(A)P(B)$.
9. $P(A \text{ and } B) + P(A \text{ and } B^c) = P(A)$
10. $P(A \text{ or } A^c) = P(A) + P(A^c) = 1$
11. $P(A \text{ and } A^c) = 0$
12. $P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A)+P(A^c)P(B|A^c)}$ (Bayes’ Theorem)

Venn Diagram:

	A	A^c	
B	$P(A \text{ and } B)$	$P(A^c \text{ and } B)$	$P(B)$
B^c	$P(A \text{ and } B^c)$	$P(A^c \text{ and } B^c)$	$P(B^c)$
	$P(A)$	$P(A^c)$	

Example:

Let A be the event that I roll an odd number, and let B be the event that I roll a prime number.

- First find the probability of each event.

Roll	2	3	4	5	6	7	8	9	10	11	12
Prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

So the probability of A is:

$$P(A) = P(3, 5, 7, 9, \text{ or } 11) = P(3) + p(5) + P(7) + P(9) + P(11) = \tfrac{18}{36},$$

and the probability of B is:

$$P(B) = P(2, 3, 5, 7, \text{ or } 11) = \tfrac{15}{36}.$$

- Next, find the probability of A and B:

$$P(A \text{ and } B) = P(3, 5, 7, \text{ or } 11) = \frac{14}{36}.$$

- Finally we can use the rules to fill an the entire Venn Diagram.
  - $P(A^c) = 1 - P(A) = \frac{18}{36}$
  - $P(B^c) = 1 - P(B) = \frac{21}{36}$
  - $P(A \text{ and } B^c) = P(A) - P(A \text{ and } B) = \frac{4}{36}$
  - $P(A^c \text{ and } B) = P(B) - P(A \text{ and } B) = \frac{1}{36}$
  - $P(A^c \text{ and } B^c) = P(A^c) - P(A^c \text{ and } B) = \frac{17}{36}$

	A	A^c	
B	$\frac{14}{36}$	$\frac{1}{36}$	$\frac{15}{36}$
B^c	$\frac{4}{36}$	$\frac{17}{36}$	$\frac{21}{36}$
	$\frac{18}{36}$	$\frac{18}{36}$	

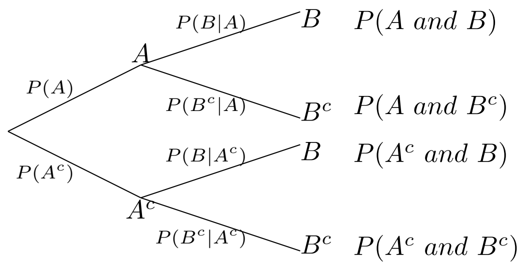
Exercise:

Use the probability rules and definitions in order to fill in a Venn Diagram with the given information.

- $P(A \text{ or } B) = \frac{3}{4}$
- $P(A) = \frac{1}{2}$
- A and B are disjoint

	A	A^c	
B	$P(A \text{ and } B)$	$P(A^c \text{ and } B)$	$P(B)$
B^c	$P(A \text{ and } B^c)$	$P(A^c \text{ and } B^c)$	$P(B^c)$
	$P(A)$	$P(A^c)$	

Decision Tree:



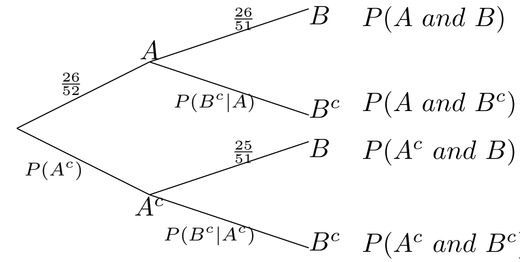
Example:

Make a decision tree were event A is "the first card dealt is red," and event B is "the second card dealt is black."

- First let's calculate a few probabilities:
  - $P(A) = \frac{26}{52}$
  - $P(B|A) = \frac{26}{51}$
  - $P(B|A^c) = \frac{25}{51}$
- Now an observation:

$$P(B|A) + P(B^c|A) = \frac{P(A \ and \ B)}{P(A)} + \frac{P(A \ and \ B^c)}{P(A)} = 1,$$

therefore, we can fill in the rest of the probabilities with what we already have.



Exercise:

Fill in a decision tree using the given information:

- $P(A \ and \ B) = \frac{1}{8}$
- $P(B^c|A) = \frac{1}{4}$
- $P(B|A^c) = \frac{1}{16}$