Number Theory
Summer 2014 Exam

You have two hours for this exam. Answer every question as completely as possible. All the questions are worth the same amount (10 points) so do the problems you are most comfortable with first.

State each definition or theorem, be as precise as possible.

1. State the Principle of Mathematical Induction:

2. Define Divisibility:

3. State the Division Algorithm:

4. Define Modular Equivalence:

5. State Wilson's Theorem:

Prove two of the following three theorems, if there are any lemmas you need to use you must state them clearly and precisely. You may prove the other for extra credit.

1. **Theorem 1 Fundamental Theorem of Arithmetic:** *Every natural number greater than 1 is either prime or may be written uniquely (up to order) as a product of primes.*

2. **Theorem 2 Chinese Remainder Theorem:** *Given a system of linear congruences:*

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_k \pmod{m_k} \end{aligned}$$

in which the moduli are pairwise relatively prime, there exits a unique least positive solution modulo $M = \prod_{i=1}^{k} m_i$.

3. **Theorem 3 Euler's Theorem:** *If the integers a and n are relatively prime and $n > 0$, then*

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

Computations:

1. Use the *Euclidian Algorithm* to show that $d = (21, 13)$ is equal to 1.

2. If possible solve the *Linear Diophantine Equations*:

   (a) $13x + 30y = 19$

   (b) $35x + 77y = 39$

3. Use the *Fundamental Theorem of Arithmetic* to find the $d = (a, b)$ and $l = [a, b]$ when $a = 7064200$ and $b = 1754298$.

4. Solve the linear congruence equation:

   $15x \equiv 33 \pmod{42}$

5. Solve this system of linear congruences using the *Chinese Remainder Theorem*:

$$\begin{aligned} x &\equiv 2 \pmod 5 \\ x &\equiv 1 \pmod 7 \\ x &\equiv 7 \pmod{11} \end{aligned}$$

6. Given $n = 1702701$ find $\phi(n)$, for extra credit also find $\sigma(n)$ and $\tau(n)$.