Theorem 1 *If $p(z) \in \mathbb{C}[z]$ of degree $n \geq 1$ then it has a root in $\mathbb{C}$.*

Proof: If the constant term of $p(z)$ is zero then 0 is the desired root and we are done. If $p(z)$ is not monic then we may divide it by the leading coefficient to get a monic polynomial with the same roots as $p(z)$. So, suppose that

$$p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$$

is a monic polynomial of degree $n \geq 1$ with a non-zero constant term. By the triangle inequality we know

$$|p(z)| \geq |z^n| - |a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0|.$$

Now

$$\lim_{|z|\to\infty} \frac{|z^n|}{|z^n|} - \frac{|a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\cdots+a_1 z+a_0|}{|z^n|} = 1$$

and so

$$\lim_{|z|\to\infty} |z^n| - |a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0| = \infty.$$

Therefore

$$\lim_{|z|\to\infty} |p(z)| = \infty^1$$

and given any real number $Q > 0$ there exists a real number $R > 0$ such that if $|z| > R$ then $|p(z)| > Q$. Let us choose $Q_0 = 1 + |a_0|$ and let R_0 be the corresponding R value. If we let $D_{R_0} = \{z : |z| \leq R_0\}$ then since D_{R_0} is closed and bounded $|p(z)|$ achieves a minimum on this set. That is $\exists\, z_0 \in D_{R_0}$ such that $|p(z_0)| \leq |p(z)|\ \forall\, z \in D_{R_0}$. Since $0 \in D_{R_0}$ this implies that $|p(z_0)| \leq |p(0)| = |a_0|$. However $|p(z)| > 1 + |a_0|$ for all $z \notin D_{R_0}$, thus $|p(z_0)|$ is a global minimum.[2]

Next let

$$f(z) = p(z + z_0) = z^n + b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_1 z + b_0$$

so that $|f(z)|$ will have a global minimum at $z = 0$.[3] If $f(0) = 0$ then we are done, so let's suppose that it does not, that is assume that $f(0) = b_0 \neq 0$. Now we define a new function

$$g(z) = \frac{1}{b_0} f(z) = c_n z^n + c_{n-1}z^{n-1} + c_{n-2}z^{n-2} + \cdots + c_1 z + 1$$

[1] Why do these limits follow one from another?

[2] How do we know that $|p(z_0)| \leq |p(z)|\ \forall\, z \in \mathbb{C}$?

[3] Why is the minimum now at zero and how do we know that the minimum did not change when we added the z_0 inside the parentheses?

where $c_i = b_i/b_0$. Thus $g(z)$ achieves a minimum value of 1 at $z = 0$. Now suppose that $k \in \mathbb{Z}^+$ is the least integer such that $c_k \neq 0$, i.e.

$$c_i = 0,\ \forall\, 0 < i < k.$$

Then we can define $r = \sqrt[k]{\frac{-1}{c_k}}$ and

$$\begin{aligned} h(w) &= g(rw) && (1)\\ &= c_n(rw)^n + \cdots + c_{k+1}(rw)^{k+1} + c_k(rw)^k + 1 && (2)\\ &= 1 - w^k + w^{k+1}(c_{k+1}r^{k+1} + \cdots + c_n r^n w^{n-k-1}) && (3)\\ &= 1 - w^k + w^{k+1}m(w) && (4)\end{aligned}$$

So that $|h(w)|$ has the same minimum as $|g(z)|$.[4] And, if we assume that $0 < w < 1$ is real then by the triangle inequality we can conclude that

$$|h(w)| \leq 1 - w^k(1 - w|m(w)|).$$[5]

Now since $m(w)$[6] is a polynomial we know that $\lim_{w \to 0} w\,|m(w)| = 0$. Therefore, for a sufficiently small $0 < w < 1$, say w_0, we know that $0 < w_0\, m|(w_0)| < 1$ so that $0 < 1 - w_0\,|m(w_0)| < 1$ and thus $0 < 1 - w_0^k(1 - w_0\,|m(w_0)|) < 1$. However this implies that $|h(w_0)| < 1$ which is a contradiction.[7] Thus we may conclude that $p(z_0) = f(0) = 0$[8] and so $p(z)$ has a root in $\mathbb{C}$. $\square$

[4] Why should $h(w)$ have the same minimum value?

[5] What string of equalities or inequalities tells us that this is true?

[6] What is $m(w)$ equal to?

[7] Why is this a contradiction?

[8] Why does this follow from our contradiction?