Polynomial Rings:

F = Field
$F[x]$ = Set of all polynomials with coefficients in x.

All the usual operations

Lemma: $\deg(f\cdot g) = \deg(f) + \deg(g)$ as long as $f, g \neq 0$. Product of the leading coefficients is $\neq 0$.

Cor: $\deg(f) \leq \deg(fg)$, $g \neq 0$

Cor: $F[x]$ is an integral domain. otherwise you could have the degree decrease.

Lemma: $F[x]$ is a Euclidean ring.

→ $\deg(f) \geq 0$

→ $\deg(f) \leq \deg(f\cdot g)$, $g \neq 0$

→ $g = z\cdot f + r$ where $\deg(f) > \deg(r)$ Polynomial long division

where are we using F is a field.

From this we get all the integer like properties of polynomials

→ gcd $(d = \lambda\cdot f + \mu\cdot g)$

→ irreducible $(f = a\cdot b$, one of a, b of degree $0)$

Def: f is primitive if the gcd of its Coefficients is 1.

This is called the content of the polynomial

Def: monic + integer monic

Theorems/Lemmas:

read on your own

→ Product of primitives is primitive

→ Product of monics is monic

→ (Gauss' Lemma) Primitive $\wedge$ product of polynomials with rational Coefficients ⇒ product of poly. with integer coefficients.

→ (The Eisenstein Criterion) $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, p prime, $p \nmid a_n$, $p \mid a_i$ $i < n$, $p^2 \nmid a_0$ ⇒ $f(x)$ is irreducible over the rationals.

Examples :

$$96x^2+20x+1 = \left(8x+\tfrac{2}{3}\right)\left(12x+\tfrac{3}{2}\right) = \tfrac{2}{3}(12x+1)\cdot\tfrac{3}{2}(8x+1) = (12x+1)(8x+1)$$

The proof starts out in a similar fasion then goes on to show this is always what happens.

$x^2+25x+5$, $7x^2+12x+2$, $4x^2+35x+7$, x^2-2 these have real but not rational roots.

Proof of Eisenstein's Criterion:

→ $f(x) = (b_0 + b_1 x + \cdots + b_r x^r)(c_0 + c_1 x + \cdots + c_s x^s)$

→ $a_k = \sum_{i=0}^{k} b_i c_{k-i}$

→ $a_n = b_r c_s \Rightarrow p \nmid b_r$ nor c_s why?

→ $a_0 = b_0 \cdot c_0 \Rightarrow p \mid b_0$ Xor c_0 ← WOLOG $p \mid b_0$ and $p \nmid c_0$.

→ Let k be the least k so that $p \nmid b_k$ (because $p \nmid b_r$ this must exist.)

→ $a_k = b_0 c_k + b_1 c_{k-1} + b_2 c_{k-2} + \cdots + b_k \cdot c_0$

→ $p \mid a_k$, $p \mid b_i$ $(i<k) \Rightarrow p \mid b_k c_0$ $(d \mid a+b \wedge d \mid a \Rightarrow d \mid b)$

→ but $p \nmid b_k$ and $p \nmid c_0$ $\times$.

→ So $f(x)$ could not have been factored.

Def: An integral domain with unit element is a unique factorization domain:

(1) Elements are units or products of irreducible elements

(2) The products in (1) are unique up to order.